RPTU Kaiserslautern-Landau
Department of Physics

Summer Term 2023

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Quantum Optics — Problem Sheet 1

Problem 1: Canonical Quantisation of Harmonic Oscillator (8 points)

As a start we canonically quantise a 1D harmonic oscillator and solve the resulting equations of motion for the operators in Heisenberg representation. We will guide you step by step throughout this procedure.

a) Write down the Lagrange function for the harmonic oscillator and derive the Hamilton function via a Legendre transformation. How do the classical equations of motion look like?

b) We go now over to the quantum theory and introduce the canonical commutation relation of space and momentum operators by replacing the Poisson bracket $\{\cdot,\cdot\}$ by the commutator $-i[\cdot,\cdot]/\hbar$. How do the Heisenberg equations of motion look like?

c) How are annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ formally defined, respectively? Derive them for the harmonic oscillator and formulate the Hamilton operator in terms of these. Can you solve their equations of motion?

Problem 2: Functional Derivative (8 points)

In this problem we review the derivation of functionals. This technique is a mathematical pillar of field theory. An introduction to the functional derivative can simply be found on Wikipedia. One needs to know only the following two formulas for functional derivatives:

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y), \tag{1}$$

$$\frac{\delta f(\phi(x))}{\delta\phi(y)} = \frac{\partial f(\phi(x))}{\partial\phi(x)}\frac{\delta\phi(x)}{\delta\phi(y)}. \tag{2}$$

Calculate the functional derivatives of the following functionals:

a) With respect to $\phi(x)$:

$$I[\phi(y)] = \frac{1}{2}\int dy\ \phi(y)^2 \tag{3}$$

b) With respect to $x(t)$:

$$S[x(\tau)] = \int d\tau\ \left\{\frac{m[\dot{x}(\tau)]^2}{2} - V(x(\tau))\right\} \tag{4}$$

c) With respect to $\psi^*(\mathbf{x},t)$:

$$S[\psi(\mathbf{x}',t'),\psi^*(\mathbf{x}',t')] = \int dt' \int d^3x'\ \left\{i\hbar\psi^*(\mathbf{x}',t')\frac{\partial\psi(\mathbf{x}',t')}{\partial t'} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*(\mathbf{x}',t')\cdot\boldsymbol{\nabla}\psi(\mathbf{x}',t') - V(\mathbf{x}')|\psi(\mathbf{x}',t')|^2\right\} \tag{5}$$

What is the physical meaning of S?

Problem 3: Heisenberg Uncertainty Relation **(8 points)**

a) How are the eigenstates $|n\rangle$ of the harmonic oscillator defined and characterised? How can they be constructed from the ground state $|0\rangle$?

b) Calculate the expectation values $\langle\cdot\rangle = \langle n| \cdot |n\rangle$ for the variances $\left\langle(\Delta\hat{x})^2\right\rangle$ and $\left\langle(\Delta\hat{p})^2\right\rangle$ with $(\Delta\hat{x})^2 = \hat{x}^2 - \langle\hat{x}\rangle^2$ and $(\Delta\hat{p})^2 = \hat{p}^2 - \langle\hat{p}\rangle^2$.

c) Check the validity of the Heisenberg uncertainty relation

$$\left\langle(\Delta\hat{x})^2\right\rangle\left\langle(\Delta\hat{p})^2\right\rangle \geq \frac{1}{4}\left|\langle[\hat{x},\hat{p}]\rangle\right|^2 .$$

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until April 24 at 10.00.