Quantum Optics Problem Sheet 10

Problem 22: Rate Equations and Thermal States of Light (12 points)

In this exercise we investigate the three fundamental Einstein processes of light-matter interaction from the point of view on how to generate thermal states of light. Consider N dye molecules interacting with a single light mode. We assume the molecules to have two electronic degrees of freedom and include the vibronic degrees of freedom later in the exercise in the modelling of the absorption and emission rates. The rate for light absorption of a single molecule is denoted by $B_{12}(\omega)$ and the one for stimulated and spontaneous emission by $B_{21}(\omega)$.

a) Write down the rate equations for the number N_1, N_2 of molecules in the ground and excited state, respectively, as well as for $n(\omega)$ photons in the cavity mode with frequency ω. Explain concisely which term arises from which of the three fundamental Einstein processes of light-matter interaction and illustrate the latter via meaningful sketches.

b) Calculate the steady state of the resulting equations of motion for N_1, N_2 and $n(\omega)$.

c) Now we take into account that the vibronic degrees of freedom of the molecules are thermalised by using the experimentally found Kennard-Stepanov relation

$$B_{12}(\omega) = B_{21}(\omega) \exp\left(\frac{\hbar\omega}{k_{\mathrm{B}}T}\right) , \tag{1}$$

where ω is the frequency of the cavity mode and T the temperature. Show that with this relation the steady-state equation for the photon number n takes the form of the Bose-Einstein distribution. How do you interpret physically the resulting chemical potential?

d) Which conclusion can you draw from the steady-state solution for N_1, N_2? Interpret your finding physically.

Problem 23: Light-Matter Interaction (12 points)

In the lecture notes we derived with Eq. (4.21) the so-called $\mathbf{xE}$ Hamiltonian for the light-matter interaction: $\hat{H}_{\mathbf{xE}} = -q\mathbf{x}\cdot\mathbf{E}$. However, there is a second possibility to describe the light-matter interaction via the $\mathbf{pA}$ Hamiltonian, which we will derive in this exercise. As a starting point, consider Hamiltonian (4.5) in radiation gauge, which is defined by both $\mathrm{div}\mathbf{A} = 0$ and vanishing scalar potential:

$$\hat{H}(\mathbf{x}, t) = \frac{1}{2M}\left[\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}, t)\right]^2 . \tag{2}$$

Note: In this exercise we will use as in the lecture notes the notation $q = -e$ for the electron charge with $e > 0$ denoting the elementary charge.

a) Show within the dipole approximation and for small vector potentials $\mathbf{A}$ that the Hamiltonian (2) can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathbf{pA}}, \tag{3}$$

where $\hat{H}_0 = \hat{\mathbf{p}}^2/(2M)$ and $\hat{H}_{\mathbf{pA}} = -q\hat{\mathbf{p}}\cdot\mathbf{A}/M$.

b) Consider now a monochromatic plane wave $\mathbf{E} = \mathbf{E}_0 \cos(\omega t)$. How do the interaction Hamiltonians $\hat{H}_{\mathbf{pA}}$ and $\hat{H}_{\mathbf{xE}}$ read in this situation, respectively? Show that the ratio of the transition amplitudes for both interaction Hamiltonians

$$c_{fi}^{(1)}(t, \hat{H}_{\mathbf{pA}}) = -\frac{i}{\hbar} \int_0^t dt' e^{i\omega_{fi} t'} \langle f | \hat{H}_{\mathbf{pA}}(t') | i \rangle \,, \tag{4}$$

$$c_{fi}^{(1)}(t, \hat{H}_{\mathbf{xE}}) = -\frac{i}{\hbar} \int_0^t dt' e^{i\omega_{fi} t'} \langle f | \hat{H}_{\mathbf{xE}}(t') | i \rangle \tag{5}$$

are related according to

$$\frac{c_{fi}^{(1)}(t, \hat{H}_{\mathbf{pA}})}{c_{fi}^{(1)}(t, \hat{H}_{\mathbf{xE}})} = \frac{\omega_{fi}}{\omega} \,. \tag{6}$$

Here $|i\rangle$ and $|f\rangle$ denote some eigenstates of the unperturbed Hamiltonian $\hat{H}_0$, ω_f and ω_i stand for the corresponding eigenfrequencies, and $\omega_{fi} = \omega_f - \omega_i$ abbreviates the transition frequency.
Note: Consider the commutator $\left[\hat{\mathbf{x}}, \hat{H}_0\right]_- =?$. Furthermore, restrict yourself to the case $\omega_{fi} > 0$ and apply the rotating wave approximation.

The result (6) implies that you have obtained a difference in measurable quantities like the transition rates. The reason, why this yields a difference, is hidden in the gauge transformations. Operators of physical quantities $\hat{G}(\mathbf{A})$ change under unitary transformations $\hat{U} = e^{iq\chi(\mathbf{x},t)/\hbar}$ involving a gauge function χ according to $\hat{G}(\mathbf{A}') = \hat{U}\hat{G}(\mathbf{A})\hat{U}^\dagger$ with a gauge transformed vector potential $\mathbf{A}'(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) + \boldsymbol{\nabla}\chi(\mathbf{x}, t)$.

c) Show that in this sense the operator of the mechanical momentum $\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} - q\mathbf{A}(\mathbf{x}, t)$ is an operator of a physical quantity and the operator of the canonical momentum $\hat{\mathbf{p}}$ is none. Does in this sense the Hamiltonian (2) in the lecture notes represent the operator of a physical quantity?

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 23 at 12.00.