Quantum Optics — Problem Sheet 2

Problem 4: Canonical Field Quantisation (12 points)

In this exercise we generalise the canonical quantisation procedure from **Problem 1** to fields. At first we perform the continuum limit of a chain of coupled oscillators and then quantise the resulting field-theoretic expressions.

a) Consider a chain of N point masses with mass m. The chain has the length $L = Na$, where a is the distance between two oscillators. The oscillators are coupled by springs with spring constants K. Write down the Lagrange function for this problem.

b) We now go over to the field-theoretic formulation of the problem. For this purpose, we have to perform the continuum limit, meaning we consider the limiting process $a \to 0$, where the density $\mu = m/a$ remains constant. Introduce the velocity $c = \sqrt{\lim_{a\to 0} Ka/\mu}$ and the continuous field $\phi(na,t) := x_n(t)$. Which form does the Lagrangian take in this respective case? What is the physical meaning of this procedure?

c) How is the field-theoretic generalised momentum Π defined? Show, that the Hamilton function derived from the Lagrange function in **b)** takes the form

$$H = \int_0^L dx \, \left[\frac{\Pi^2}{2\mu} + \frac{\mu c^2}{2} \left(\partial_x \phi\right)^2 \right] . \tag{1}$$

d) What is an appropriate generalisation of the quantisation condition for the fields Π and ϕ? Quantise the Hamiltonian and derive the equations of motion for the fields. What kind of equation do you find? Are you surprised?

Problem 5: Field Operators (12 points)

Our aim in this exercise is to investigate the Hamilton operator from **Problem 4** in greater detail.

a) We expand the field operators in discrete Fourier series as

$$\hat{\phi}(x,t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\Phi}_n(t) e^{ik_n x}, \tag{2}$$

$$\hat{\Pi}(x,t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\Pi}_n(t) e^{ik_n x}. \tag{3}$$

What are suitable values for k_n? Which commutation relations do the new operators $\hat{\Phi}_n$ and $\hat{\Pi}_n$ satisfy?
Hint: Use the identity

$$\delta_{nm} = \frac{1}{2\pi} \int_0^{2\pi} dx \; e^{i(n-m)x}. \tag{4}$$

b) Show that in this new basis the Hamilton operator from **Problem 4d)** takes the form

$$\hat{H} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\mu} \hat{\Pi}_n^\dagger \hat{\Pi}_n + \frac{\mu \omega_n^2}{2} \hat{\Phi}_n^\dagger \hat{\Phi}_n \right) . \tag{5}$$

How are the frequencies ω_n defined? What is the physical interpretation of this Hamiltoni operator?

c) Diagonalise the Hamilton operator (5) by introducing creation (annihilation) operators $\hat{a}_n^\dagger$ ($\hat{a}_n$). Check the commutation relations for creation and annihilation operators explicitly from the corresponding ones of the operators $\hat{\Phi}_n$ and $\hat{\Pi}_n$.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until April 28 at 12.00.