RPTU Kaiserslautern-Landau
Department of Physics

Summer Term 2023
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Quantum Optics

Problem Sheet 3

Problem 6: Finite Box Potential **(12 points)**

In this exercise we will proceed on working with the Hamiltonian

$$\hat{H} = \frac{1}{2}\int_0^L dx \left[\hat{\Pi}^2 + c^2\left(\partial_x\hat{\phi}\right)^2\right]. \tag{1}$$

At first, we will specify the formalism to Dirichlet boundary conditions.

a) Consider the boundary conditions $\hat{\phi}(0,t) = 0 = \hat{\phi}(L,t)$. What are suitable functions $u_n(x)$ for expanding the field operators in the form

$$\hat{\phi}(x,t) = \sum_n \hat{\Phi}_n(t)u_n(x), \tag{2}$$

$$\hat{\Pi}(x,t) = \sum_n \hat{\Pi}_n(t)u_n(x)? \tag{3}$$

Write down these expansions.

b) Show by using the corresponding procedure from **Problem 4** that the Hamiltonian can again be written as

$$\hat{H} = \sum_{n=1}^{\infty} \hbar\omega_n \left(\hat{a}_n^\dagger(t)\hat{a}_n(t) + \frac{1}{2}\right). \tag{4}$$

c) Solve the dynamical equations for the operators $\hat{a}_n^\dagger(t)$ and $\hat{a}_n(t)$ and write down the full expression for the generalised momentum operator $\hat{\Pi}(x,t)$. Comparing to the lecture and its definition, what is the physical meaning of $\hat{\Pi}(x,t)$?

d) Calculate the vacuum expectation value $\langle 0|\hat{\Pi}(x,t)|0\rangle$. What does this mean? Calculate the expectation value $\langle 0|\hat{\Pi}(x,t)^2|0\rangle$. What do you note about the resulting sum?

Problem 7: 1D Casimir Effect **(12 points)**

In this problem we calculate the Casimir force, whose origin are the fluctuations of the electromagnetic vacuum, between two parallel plates at distance L in a pure 1D setting. In this case, the field is described by the Hamiltonian (4).

a) Calculate the expectation value $E_{\text{disc}} = \langle 0|\hat{H}|0\rangle$ of the Hamiltonian (4). What do you note?

b) In order to *regularise* the expression from **a)** introduce a convergence factor $e^{-\gamma n}$, where n is the summation index. Note, that in the limit $\gamma \downarrow 0$ the original expression is not changed. Expand the result up to order $\mathcal{O}(\gamma^0)$.

c) Now calculate the energy E_{cont} in the case without plates, by going over from the discrete summation to a continuous integration over the energy states. Use also here a convergence factor and show that the Casimir energy $\Delta E = E_{\text{disc}} - E_{\text{cont}}$ yields a finite result in the limit $\gamma \downarrow 0$. What is the physical implication of this result?

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 5 at 12.00.