RPTU Kaiserslautern-Landau
Department of Physics

Summer Term 2023

Priv.-Doz. Dr. Axel Pelster

Quantum Optics

Problem Sheet 5

Problem 10: Generating Functions (12 points)

In this problem we will learn how to work with generating functions in the realm of statistics. Don't be afraid, most of the calculations here are just one liners.

We start with the moment-generating function $M_x(s)$. It is defined by the expectation value

$$M_x(s) = \langle e^{sx} \rangle \,, \tag{1}$$

where $\langle \bullet \rangle = \int dx \;\; \bullet \, P(x)$ denotes the expectation value with respect to the probability density $P(x)$.

a) Show that with this definition the moments $\langle x^n \rangle$ can be calculated by

$$\langle x^n \rangle = \left. \frac{d^n M_x(s)}{ds^n} \right|_{s=0} . \tag{2}$$

b) Calculate the moment-generating function $M_x(s)$ for the Gauß distribution

$$P(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-(x-x_0)^2/\sigma^2} . \tag{3}$$

Determine the first three moments of the Gauß distribution (3) by using $M_x(s)$.

c) The so-called cumulant generating function W_x is defined by the logarithm $W_x(s) = \ln M_x(s)$ and the corresponding cumulants as

$$\kappa_n(x) = \left. \frac{d^n W_x(s)}{ds^n} \right|_{s=0} . \tag{4}$$

How are the first two cumulants expressed by the moments?

d) Calculate all cumulants for the Gauß distribution (3). What do you note?

Lastly, we proof two nice identities for the cumulants.

e) Show that for arbitrary constant q the equations

$$\kappa_1(x+q) = \kappa_1(x) + q \tag{5}$$

$$\kappa_{n>1}(x+q) = \kappa_{n>1}(x) . \tag{6}$$

f) Show that for $y = \sum_i x_i$, where the x_i are stochastically independent variables,

$$\kappa_n(y) = \sum_i \kappa_n(x_i) \tag{7}$$

holds.

Problem 11: Statistics of Coherent States **(12 points)**

In this exercise we take a closer look at the moments of coherent states and their dynamics. To this end we use the moment-generating function

$$M_{\hat{O}}(s) = \left\langle e^{s\hat{O}} \right\rangle , \tag{8}$$

where $\langle \bullet \rangle = \langle \psi | \bullet | \psi \rangle$ denotes the expectation value with respect to some wave function $|\psi\rangle$.

a) Calculate the moment-generating function $M_{\hat{a}}(s)$ for the annihilation operator $\hat{a}$ and the corresponding one $M_{\hat{n}}(s)$ for the particle number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ in a coherent state $|\alpha\rangle$, which is defined by the eigenvalue problem $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Determine the first two moments for both operators. Compare these two with the corresponding expressions in a Fock state $|n\rangle$.

b) Consider the Hamilton operator for a harmonic oscillator: $\hat{H} = \hbar\omega(\hat{n} + 1/2)$. What is the dynamics of the annihilation operator in the Heisenberg picture $\hat{a}(t)$? Determine from this the corresponding dynamics of a coherent state $|\alpha(t)\rangle$ following from $\hat{a}(t)|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$.

c) For the case of **b)** calculate the expectation value of the momentum and position operator as well as their standard deviation. Confirm that the Heisenberg uncertainty relation holds for all times.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 19 at 12.00.