RPTU Kaiserslautern-Landau | Summer Term 2023
Department of Physics | Priv.-Doz. Dr. Axel Pelster

Quantum Optics — Problem Sheet 6

Problem 12: Coherent State Properties (6 points)

a) Show that the operator $\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$ is unitary.

b) Derive now for any complex number α the similarity transformation

$$\hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha) = \hat{a} - \alpha\,. \tag{1}$$

Hint: Consider to this end the operator $\hat{D}(t\alpha)$ for any real number t and determine its derivative with respect to t.

c) Show that the coherent state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ fulfills the property $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

d) Proof that the unitary operator $\hat{D}(\alpha)$ obeys a group property in the form

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\, e^{i\,\mathrm{Im}(\alpha\beta^*)}\,. \tag{2}$$

Hint: Apply the Baker-Campbell-Hausdorff formula.

Problem 13: Coherent States at Beam Splitter (6 points)

Consider again the beam splitter from **Problem 9** with the incoming photon states $\hat{a}_1$ and $\hat{a}_2$ and the outgoing states $\hat{b}_1$ and $\hat{b}_2$ with the relationships $\hat{b}_1 = (i\hat{a}_1 + \hat{a}_2)/\sqrt{2}$ and $\hat{b}_2 = (\hat{a}_1 + i\hat{a}_2)/\sqrt{2}$. This time, we send coherent states through the beam splitter.

a) Consider one incoming coherent state $|\psi\rangle = \hat{D}_{\hat{a}_1}(\alpha)\,|0\rangle$ with the displacement operator $\hat{D}_{\hat{\eta}}(\alpha) = e^{\alpha\hat{\eta}^\dagger - \alpha^*\hat{\eta}}$ being defined for $\hat{\eta} = \hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2$. Calculate like in **Problem 9c)** the expectation value of having one photon in each output state and having one photon in each output state at the same time. What do you note?

b) Consider now two incoming coherent states, i.e. $|\psi\rangle = \hat{D}_{\hat{a}_1}(\alpha)\hat{D}_{\hat{a}_2}(\beta)\,|0\rangle$, and calculate the same as in **a)**. What do you note here?

Problem 14: P-Representation (12 points)

In this problem we encounter the quantum-classical correspondence, i.e. we map a quantum-mechanical system onto equations for c-numbers. For this purpose we introduce the Glauber-Sudarshan P-representation $P(\alpha, \alpha^*)$.

a) With the annihilation (creation) operator $\hat{a}$ ($\hat{a}^\dagger$) we define the characteristic function, which corresponds to the moment-generating function from **Problem 10**, according to

$$\chi(z, z^*) = \mathrm{tr}\left(\hat{\rho} e^{iz^*\hat{a}^\dagger} e^{iz\hat{a}}\right), \tag{3}$$

where z is some complex number and $\hat{\rho}$ denotes the density matrix. Show that with $P(\alpha, \alpha^*)$ being the inverse Fourier transform of $\chi(z, z^*)$ we can calculate expectation values of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\left\langle \hat{a}^{\dagger p}\hat{a}^q \right\rangle = \int d^2\alpha\; P(\alpha, \alpha^*)\alpha^{*p}\alpha^q, \tag{4}$$

where $\int d^2\alpha$ stands for the integral with respect to $\alpha,\ \alpha^*$ over the whole complex plane.

The next step is now, to find the equation of motion for the distribution $P(\alpha, \alpha^*)$. For this we consider a simple harmonic oscillator with the Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

b) Using the Liouville equation for the density matrix $\hat{\rho}$

$$\dot{\hat{\rho}} = \frac{i}{\hbar}\left[\hat{\rho}, \hat{H}\right]_- \tag{5}$$

show that the equation of motion for the characteristic function $\chi(z, z^*)$ reads

$$\dot{\chi}(z, z^*) = i\omega\left(z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right)\chi(z, z^*). \tag{6}$$

Hints: Do not forget the operator ordering for derivatives of $\chi(z, z^*)$ in the expectation values. You may need the following two identities:

$$e^{-iz^*\hat{a}^\dagger}\hat{a}e^{iz^*\hat{a}^\dagger} = \hat{a} + iz^*, \tag{7}$$

$$e^{iz\hat{a}}\hat{a}^\dagger e^{-iz\hat{a}} = \hat{a}^\dagger + iz. \tag{8}$$

Bonus exercise (2 extra points): Show equations (7) and (8).

c) Using that $P(\alpha, \alpha^*)$ is the inverse Fourier transform of $\chi(z, z^*)$ derive the equation of motion for $P(\alpha, \alpha^*)$.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until May 26 at 12.00.