Quantum Optics

Problem Sheet 7

Problem 15: Damped Harmonic Oscillator **(15 points)**

In this problem we tackle the equation of motion for the P-representation of a damped harmonic oscillator, as this yields non-singular results compared to **Problem 14**.

a) Argue that the P-representation of a coherent state $|\alpha_0\rangle$ is given by

$$P_0(\alpha,\alpha^*) = \delta\left(\alpha-\alpha_0\right)\delta\left(\alpha^*-\alpha_0^*\right). \tag{1}$$

Consider now the equation of motion for the harmonic oscillator

$$\frac{\partial}{\partial t}P = \left[\left(\frac{\gamma}{2}+i\omega\right)\frac{\partial}{\partial\alpha}\alpha+\left(\frac{\gamma}{2}-i\omega\right)\frac{\partial}{\partial\alpha^*}\alpha^*+\gamma\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P\,, \tag{2}$$

where γ denotes the damping rate and $\bar{n}$ the occupation of the bath to which the oscillator is coupled.

b) Show that with the transformation $\alpha = e^{-i\omega t}\tilde{\alpha}$, $\alpha^* = e^{i\omega t}\tilde{\alpha}^*$ the equation for the transformed P-representation $\tilde{P}(\tilde{\alpha},\tilde{\alpha}^*,t)$ takes the form

$$\frac{\partial}{\partial t}\tilde{P} = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial\tilde{\alpha}}\tilde{a}+\frac{\partial}{\partial\tilde{\alpha}^*}\tilde{\alpha}^*\right)+\gamma\bar{n}\frac{\partial^2}{\partial\tilde{\alpha}\partial\tilde{\alpha}^*}\right]\tilde{P}. \tag{3}$$

c) Expand the complex quantities $\tilde{\alpha}$, $\tilde{\alpha}^*$ by real and imaginary part $\tilde{\alpha} = \tilde{x}+i\tilde{y}$. How does the equation of motion (3) read now? Separate the resulting equation with the ansatz $\tilde{P}(\tilde{x},\tilde{y},t) = X(\tilde{x},t)Y(\tilde{y},t)$. What are the initial conditions for X and Y from (1)?

d) Given the solution

$$X(\tilde{x},t) = \frac{1}{\sqrt{\pi\bar{n}(1-e^{-\gamma t})}}\exp\left\{-\frac{\left(\tilde{x}-\tilde{x}_0\,e^{-\gamma t/2}\right)^2}{\bar{n}(1-e^{-\gamma t})}\right\} \tag{4}$$

and the similar one for $Y(\tilde{y},t)$ how does the complete P-representation $P(\alpha,\alpha^*,t)$ read?

e) Finally, calculate the mean value $\langle\hat{a}\rangle$ and the variance $\left\langle\hat{a}^\dagger\hat{a}\right\rangle-\left\langle\hat{a}^\dagger\right\rangle\langle\hat{a}\rangle$. How do you interpret the results?

f) Illustrate the time evolution of the P-representation in phase space. What is the behaviour of $P(\alpha,\alpha^*,t)$ in the limit $t\to\infty$? Calculate the mean value $\langle\hat{a}\rangle$ as well as the variance $\left\langle\hat{a}^\dagger\hat{a}\right\rangle-\left\langle\hat{a}^\dagger\right\rangle\langle\hat{a}\rangle$ in this long-time limit and interpret your results.

Problem 16: (Anti-)Normal Ordered Displacement Operator **(9 points)**

a) Disentangle the operator $e^{t\hat{Z}}$ with $\hat{Z} = \alpha\hat{a}^\dagger-\alpha^*\hat{a}$ as described in the lecture with the ansatz

$$e^{t\alpha\hat{a}^\dagger-t\alpha^*\hat{a}} = e^{f_1(t)\hat{a}^\dagger}e^{f_2(t)\hat{n}}e^{f_3(t)\hat{a}}e^{f_4(t)}\,. \tag{5}$$

Which differential equations do you find for the functions $f_1(t), f_2(t), f_3(t), f_4(t)$ and which initial conditions $f_1(0), f_2(0), f_3(0), f_4(0)$ do you have to take into account? Show with this that the displacement operator $\hat{D}(\alpha) =$

$e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ reads in normal ordering

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-|\alpha|^2/2} \,. \tag{6}$$

b) Insert a unity factor according to

$$\hat{D}(\alpha) = \left(e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{-\alpha\hat{a}^\dagger} \right) e^{\alpha\hat{a}^\dagger} e^{-|\alpha|^2/2} \tag{7}$$

and determine with this the displacement operator $\hat{D}(\alpha)$ in anti-normal ordered form.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 2 at 12.00.