RPTU Kaiserslautern-Landau
Department of Physics

Summer Term 2023
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Quantum Optics — Problem Sheet 9

Problem 19: Parametric Down Conversion (6 points)

In this exercise we discuss how to generate squeezed states of light by using the mechanism of degenerate parametric down conversion. This process is described by the Hamiltonian

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_p\hat{b}^\dagger\hat{b} + i\hbar g(\hat{a}^2\hat{b}^\dagger - \hat{a}^{\dagger^2}\hat{b}), \tag{1}$$

where $\hat{b}$ denotes some incident pump beam at frequency ω_p and $g \geq 0$ a nonlinear coupling constant.

a) Explain, which kind of microscopic processes the Hamiltonian (1) describes.

b) Assuming that the incident field is in a strong coherent state $\left|\beta e^{-i\omega_p t}\right\rangle$, how can we approximate the operators $\hat{b}$ by c-numbers and, thus, the Hamiltonian (1)?

c) Transform the resulting Hamiltonian from **b)** into the interaction picture. For which frequency ω takes the corresponding time-evolution operator $\hat{U}(t)$ the form of the squeezing operator $\hat{S}(\xi)$? How is the squeezing parameter ξ determined?

Problem 20: Squeezed Coherent States (8 points)

In this exercise we meet a generalisation of squeezed states, namely squeezed coherent states $|\alpha, \xi\rangle$ which are defined by applying the displacement operator $\hat{D}(\alpha)$ at the squeezed state $\hat{S}(\xi)\,|0\rangle$:

$$|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)\,|0\rangle\,. \tag{2}$$

a) Calculate the expectation values for the annihilation operator $\hat{a}$ and for the photon number $\hat{a}^\dagger\hat{a}$. What do you encounter for the limits $\alpha \to 0$ and $\xi \to 0$, respectively?

b) Calculate now the expectation value and the variances for the quadrature operator $\hat{x}_\theta$. Plot and discuss your results in dependency on θ.

c) Calculate the photon statistics of the squeezed coherent state and plot your result for different values of α and ξ.

Problem 21: (Anti-)Normal Ordered Squeezing Operator (10 points)

a) Disentangle the operator $e^{t\hat{Z}}$ with $\hat{Z} = \xi^*\hat{a}^2/2 - \xi\hat{a}^{\dagger 2}/2$ as described in the lecture with the ansatz

$$e^{t\hat{Z}} = e^{f_+(t)\hat{a}^{\dagger 2}/2} e^{f_3(t)(\hat{n}+1/2)/2} e^{f_-(t)\hat{a}^2/2}\,. \tag{3}$$

Which differential equations do you find for the functions $f_\pm(t), f_3(t)$ and which initial conditions $f_\pm(0), f_3(0)$ do you have to take into account?

Note: Use the fact that the generators of the Lie algebra $su(1,1)$ are given by $\hat{K}_+ = \hat{a}^{\dagger 2}/2$, $\hat{K}_3 = (\hat{n}+1/2)/2$, and $\hat{K}_- = \hat{a}^2/2$.

b) Show that $f_+(t)$ obeys the Ricatti differential equation

$$\dot{f}_+(t) - \xi^* f_+^2(t) = -\xi\,. \tag{4}$$

Perform the polar decomposition $\xi = |\xi| e^{i\varphi}$, $\xi^* = |\xi| e^{-i\varphi}$ and make the solution ansatz

$$f_+(t) = c\, e^{i\varphi}\, \frac{\dot{u}(t)}{u(t)}\,. \tag{5}$$

Determine the yet unknown constant c such that the nonlinear differential equation (4) reduces to the linear differential equation

$$\ddot{u}(t) - |\xi|^2 u(t) = 0\,. \tag{6}$$

Show that, finally, that the function $f_+(t)$ is determined by

$$f_+(t) = -e^{i\varphi} \tanh\left(|\xi| t\right)\,. \tag{7}$$

c) Calculate now also $f_3(t)$ and $f_-(t)$ and show that the disentangled exponential operator (3) reads

$$e^{\left(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2}\right)/2} = e^{-e^{i\varphi} \tanh|\xi| \hat{a}^{\dagger 2}/2} \left(\frac{1}{\cosh|\xi|}\right)^{\hat{n}+1/2} e^{e^{-i\varphi} \tanh|\xi| \hat{a}^2/2}\,, \tag{8}$$

which represents the normal ordered form of the squeezing operator.

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to jkrauss@rhrk.uni-kl.de until June 16 at 12.00.