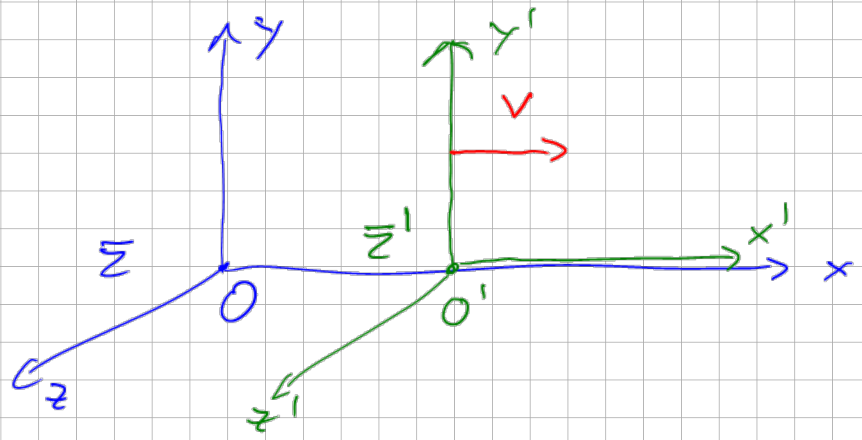
1.6 Lorentz-Transformation:



$$x = \underbrace{v\,t}_{x'=0} + \underbrace{x'\sqrt{1-\frac{v^2}{c^2}}}_{t=0} \qquad (1)$$

$x'=0$ ↓ $\frac{dx}{dt} = v$ velocity of O' in Σ

$t=0$ ↓ length contraction

length contraction of bar of length x' seen in Σ

invert role of Σ and Σ': $x' \leftrightarrow x$, $t' \leftrightarrow t$ and $v \to -v$

$$x' = -v\,t' + x\sqrt{1-\frac{v^2}{c^2}} \qquad (2)$$

Solve now (1) and (2):

$$t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,t' + \frac{v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}\,x'$$

$$x = \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\,t' + \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,x'$$

see problem set 1
→ released on Sunday
→ derive Einstein addition theorem for velocities
$v\,(v_1 = c,\ v_2 = u) = c$
in accordance with (E2)

Notes:

- $v/c \ll 1$: Lorentz transformation give back Galilei transformations
- problem set 2: general Lorentz transformation for a boost in a general

direction

1.7 Fourvectors:

spacetime coordinates of an event:

$$(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) = (x^0, x^i)$$

greek index: $\mu \in \{0, 1, 2, 3\}$, latin index: $i = \{1, 2, 3\}$

compact notation of Lorentz transformation:

$$x^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x'^\nu = \Lambda^\mu{}_\nu \, x'^\nu$$

covariant index contravariant index

Einstein summation convention

automatic summation over identical co- and contravariant indices

Warning: It can never happen that the same index appears three or more times

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & & \ldots & \\ \Lambda^3{}_0 & & & \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

row (μ), column (ν)

first index second index

1.8 Metric:

Light source emits light ray at $O = (0,0,0,0)$ and is detected in $P(tc, x, y, z)$:

$$(ct)^2 - x^2 - y^2 - z^2 = 0$$

As light velocity is largest possible velocity we conclude for physical events

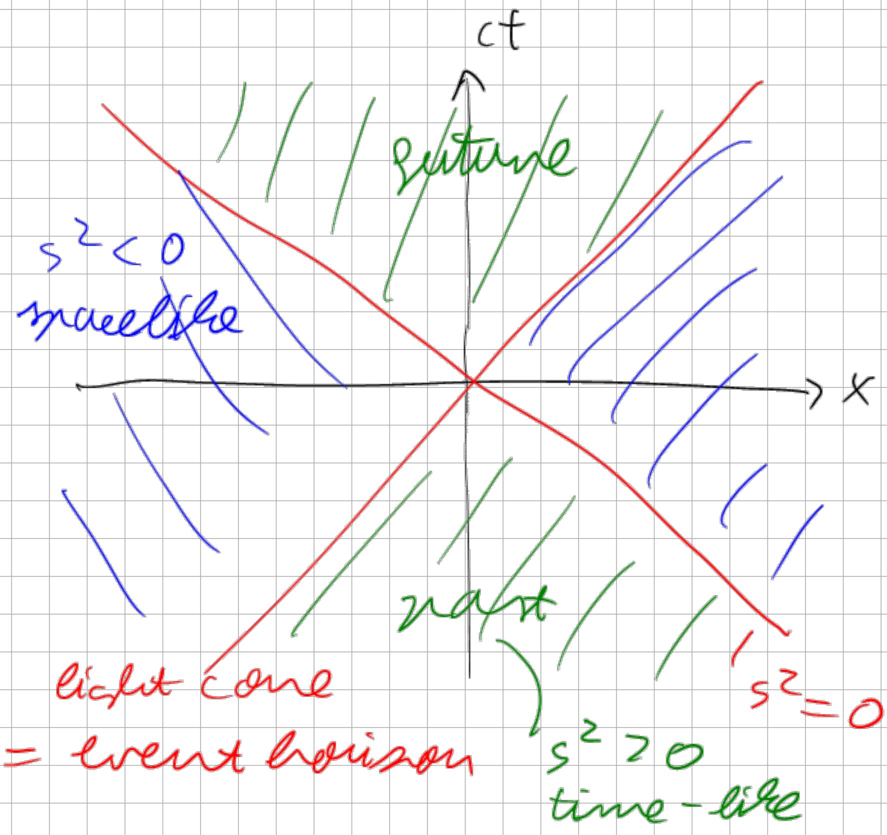
$$(ct)^2 - x^2 - y^2 - z^2 > 0$$

Events with property

$$(ct)^2 - x^2 - y^2 - z^2 < 0$$

do not occur. This motivates introduction of spacetime distance s via

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$



Minkowski metric

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow s^2 = g_{\mu\nu} x^\mu x^\nu$$

in Σ

in Σ': $s'^2 = g_{\mu\nu} x'^\mu x'^\nu = s^2$

$$g_{\mu\nu} \underbrace{x'^\mu}_{\Lambda^\mu{}_{\mu'} x^{\mu'}} \underbrace{x'^\nu}_{\Lambda^\nu{}_{\nu'} x^{\nu'}} = g_{\mu'\nu'} x^{\mu'} x^{\nu'}$$

$$\Lambda^{\mu}{}_{\mu'}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'}\, \underbrace{x^{\mu'} x^{\nu'}} = g_{\mu'\nu'}\, \underbrace{x^{\mu'} x^{\nu'}}$$

$$\Rightarrow \Lambda^{\mu}{}_{\mu'}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\nu'} = g_{\mu'\nu'}$$

row column row

$$\boxed{\underbrace{\Lambda^T \, g \, \Lambda}_{=g'} = g}$$

This identity defines all possible Lorentz transformations

"space"	metric	transformation	identity
3-dim. space	Euclidean metric I	rotations	$R^T I R = I \;\hat{=}\; R^T R = I$
4-dim. space-time	Minkowski metric g	Lorentz transformations (boosts, rotations)	$\Lambda^T g \Lambda = g$
$2n$-dim. phase space	symplectic metric	canonical transformations with Jacobian J	$J^T M J = M$

Note:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \quad \Rightarrow \quad \frac{d}{dt}\underbrace{\begin{pmatrix} q \\ p \end{pmatrix}}_{=\vec{x}} = \underbrace{\left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right)}_{\text{symplectic metric } M} \underbrace{\begin{pmatrix} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{pmatrix}}_{\vec{\partial}_{\vec{x}}} H$$

Scalar product between two fourvectors A^μ and B^ν:

$$(A, B) = g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu$$

Lorentz transformation: $A^\mu = \Lambda^\mu{}_{\mu'} A'^{\mu'}$, $B^\nu = \Lambda^\nu{}_{\nu'} B'^{\nu'}$

$$(A,B) = \underline{g_{\mu\nu}}\, \underline{\Lambda^\mu{}_{\mu'}}\, A'^{\mu'}\, \underline{\Lambda^\nu{}_{\nu'}}\, B'^{\nu'} = g_{\mu'\nu'} A'^{\mu'} B'^{\nu'} = (A', B')$$

1.9 Co- and contravariant components:

contravariant fourvector: A^μ

covariant fourvector: $A_\mu := g_{\mu\nu} A^\nu$

example: $(x_\mu) = (ct, -x, -y, -z)$

scalar product: $(A,B) = A^\mu B_\mu = A_\mu B^\mu$

contravariant Minkowski metric: $g_{\mu\nu} \equiv g^{\mu\nu}$

application: $A^\mu = g^{\mu\nu} A_\nu$

$g^{\mu\nu}$ and $g_{\mu\nu}$ Minkowski metrics allow to raise or lower indices

$$g^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu} \mathrel{\hat{=}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$\Rightarrow g^\mu{}_\nu = \delta^\mu{}_\nu$ 4D Kronecker symbol

Lorentz transformations of covariant fourvectors

$$\underline{A_\alpha} = g_{\alpha\mu} A^\mu = g_{\alpha\mu} \Lambda^\mu{}_\nu A^\nu = \underbrace{\overline{g_{\alpha\mu} \Lambda^\mu{}_\nu\, g^{\nu\beta}}}_{=: \tilde{\Lambda}_\alpha{}^\beta}\, \underline{A'_\beta}$$

$$\tilde{\Lambda} = g \Lambda g \Rightarrow (\tilde{\Lambda}_\alpha{}^\beta) \Rightarrow \tilde{\Lambda}(v) = \Lambda(-v)$$

row column

Property: $\tilde{\Lambda}\,\Lambda = \Lambda\,\tilde{\Lambda} = I_{4\times4}$

generalized tensors: $T^{\overbrace{\alpha_1 \cdots \alpha_m}}{}_{\underbrace{\beta_1 \cdots \beta_n}}$

m contravariant indices; n covariant indices

Tensor of rank (m, n) transforms with m matrices Λ and n matrices $\tilde{\Lambda}$

$$T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n} = \Lambda^{\alpha_1}{}_{\mu_1} \cdots \Lambda^{\alpha_m}{}_{\mu_m} \tilde{\Lambda}_{\beta_1}{}^{\nu_1} \cdots \tilde{\Lambda}_{\beta_n}{}^{\nu_n} T'^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n}$$

A^μ: tensor of rank $(1,0)$; A_μ: tensor of rank $(0,1)$

All physical laws in special relativity will be formulated in terms of tensors of certain ranks:

Chapter 2: classical mechanics in special relativity

Chapter 3: Electrodynamics

Next lecture: Monday, 13.45 - 15.15 in 5.76