General Relativity — Problem Set 1

Problem 1: Lorentz Transformation (14 points)

In Special Relativity the time coordinate t and the three spatial coordinates x, y, z of an inertial system are united to a four-vector x^μ with $\mu = 0, 1, 2, 3$:

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} . \tag{1}$$

Here c denotes the velocity of light. A Lorentz transformation $\Lambda^\mu{}_\nu$ connects the four-vectors x^μ and x'^ν of two inertial systems Σ and Σ' as follows:

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu . \tag{2}$$

According to Einstein we have to postulate that the Lorentz transformation $\Lambda^\mu{}_\nu$ leaves the Minkowski metric

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3}$$

invariant, i.e. we have:

$$g'_{\mu\nu} = \Lambda^\alpha{}_\mu \, \Lambda^\beta{}_\nu \, g_{\alpha\beta} \equiv g_{\mu\nu} . \tag{4}$$

Thus, the Minkowski metrik $g_{\mu\nu}$ has the same form in all inertial systems.

a) For two inertial systems Σ and Σ', which move uniformly relative to each other, we perform the ansatz

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & 0 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5}$$

Determine the equations for the four unknowns $\Lambda^0{}_0$, $\Lambda^0{}_1$, $\Lambda^1{}_0$, and $\Lambda^1{}_1$ from the Eqs. (3)–(5). (4 points)

b) Assume that Σ' moves relative to Σ with velocity v. Show from (1), (2), and (5) that this implies the relation

$$\Lambda^1{}_0 = -\frac{v}{c} \, \Lambda^1{}_1 . \tag{6}$$

(2 points)

c) Determine uniquely the four unknowns $\Lambda^0{}_0$, $\Lambda^0{}_1$, $\Lambda^1{}_0$, and $\Lambda^1{}_1$ from **a)** and **b)** as well as from demanding that the Lorentz transformation reduces to the Galilei transformation in the limit $c \to \infty$. (4 points)

d) Show that performing successively two Lorentz transformations yields again a Lorentz transformation

$$\Lambda^{\mu}{}_{\nu}(v) = \Lambda^{\mu}{}_{\kappa}(v_2)\, \Lambda^{\kappa}{}_{\nu}(v_1) \tag{7}$$

and determine from this the Einstein addition theorem for velocities: $v = v(v_1, v_2)$. Is the addition theorem symmetric $v(v_1, v_2) = v(v_2, v_1)$? Evaluate the addition theorem for the special cases (i) $v_1 \ll c$, $v_2 \ll c$ and (ii) $v_2 \to c$. (4 points)

Problem 2: Doppler Effect (10 points)

Consider a wave propagating in inertial system Σ with light velocity c in x-direction, i.e. $\psi(x,t) = Ae^{i(\omega t - kx)}$ with circular frequency $\omega = 2\pi f$ and frequency f as well as wave vector $k = 2\pi/\lambda$ and wave length λ. Another inertial system Σ' moves with velocity v relative to Σ.

a) Transform the wave with a Galilei transformation from Σ to Σ'. Determine for the wave in Σ' both the frequency f' and the wave length λ'. What result do you get for the light velocity c' in Σ'?
(3 points)

b) Analyze now correspondingly the consequences for a Lorentz transformation from Σ to Σ'. (3 points)

c) The H_α-line of a star (λ=6563 Å) appears red shifted by 30 Å. How fast is the star moving away from Earth?
(2 points)

d) The quasar PC 0910 + 5625 moves away from Earth at the velocity $v = 0.92\,c$. How large is the dimensionless red shift $z = (\lambda' - \lambda)/\lambda$ for an observer on Earth? (2 points)

Drop the solutions in the post box on the 5th floor of building 46 or, in case of illness/quarantine, send them via email to axel.pelster@physik.uni-kl.de until May 6 at 12.00.