TU KAISERSLAUTERN
Department of Physics

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Quantum Field Theory — Problem Sheet 3

Problem 4: Lorentz Group I

A coordinate transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$ is a Lorentz transformation if the Minkowski metric g is invariant:

$$g_{\mu\nu} = \Lambda^{\sigma}{}_{\mu} \Lambda^{\rho}{}_{\nu} g_{\sigma\rho} \,. \tag{1}$$

a) Investigate the Lorentz transformations in the vicinity of the unity element. To this end exploit the ansatz $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + w^{\mu}{}_{\nu}$ and show that it solves (1) for small deviations $w^{\mu}{}_{\nu}$ provided that $w^{\mu}{}_{\nu}$ fulfills a condition. (1 point)

b) Decompose $w^{\mu}{}_{\nu} = -\frac{i}{2}(L^{\alpha\beta})^{\mu}{}_{\nu} w_{\alpha\beta}$ into the expansion coefficients $w_{\alpha\beta}$ and the basis generators $(L^{\alpha\beta})^{\mu}{}_{\nu}$. Show that the basis generators $L^{\alpha\beta}$ of the Lorentz group in the Minkowski space-time are given by the following representation matrices:

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} = i\left(g^{\alpha\mu}\, g^{\beta}{}_{\nu} - g^{\beta\mu}\, g^{\alpha}{}_{\nu}\right) \,. \tag{2}$$

(2 points)

c) Prove the symmetry properties

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} + \left(L^{\alpha\beta}\right)_{\nu}{}^{\mu} = 0 \,, \tag{3}$$

$$\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu} + \left(L^{\beta\alpha}\right)^{\mu}{}_{\nu} = 0 \,, \tag{4}$$

and demonstrate that the commutator of the basis generators is given by

$$\left[L^{\alpha\beta}\,,\,L^{\gamma\delta}\right]_{-} = i\left(g^{\alpha\delta}\,L^{\beta\gamma} + g^{\beta\gamma}\,L^{\alpha\delta} - g^{\alpha\gamma}\,L^{\beta\delta} - g^{\beta\delta}\,L^{\alpha\gamma}\right) \,. \tag{5}$$

(3 points)

d) The basis generators $L^{\alpha\beta}$ can be divided into two classes:

$$M_k = L^{0k}\,, \qquad L_k = \frac{1}{2}\,\epsilon_{klm}\,L^{lm}\,, \tag{6}$$

where the Latin indices take the values 1,2,3, whereas the Greek indices can have the values 0,1,2,3. Using (5) determine the commutators

$$[M_k\,,\,M_l]_{-} = ?\,, \qquad [L_k\,,\,L_l]_{-} = ?\,, \qquad [L_k\,,\,M_l]_{-} = ?\,. \tag{7}$$

(2 points)

Problem 5: Lorentz Group II

a) Determine the Lorentz transformation $R(\boldsymbol{\varphi})$ between two inertial systems S and S′, which are rotated with respect to each other. To this end apply the Lie theorem and evaluate

$$R(\boldsymbol{\varphi}) \;=\; \exp\left(-i\,\boldsymbol{L}\,\boldsymbol{\varphi}\right)\,, \tag{8}$$

where the matrix-valued exponential function is defined via its Taylor series. (4 points)

b) Verify the following two properties of $R(\boldsymbol{\varphi})$:

$$\text{Tr}\;R(\boldsymbol{\varphi}) \;=\; 2 + 2\cos|\boldsymbol{\varphi}|\,, \tag{9}$$

$$R(\boldsymbol{\varphi})\begin{pmatrix} 0 \\ \boldsymbol{\varphi} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ \boldsymbol{\varphi} \end{pmatrix}. \tag{10}$$

(2 points)

c) Find the boost transformation $B(\boldsymbol{\xi})$ between two inertial systems S and S′, which move uniformly with respect to each other. Apply to this end the Lie theorem and evaluate explicitly the matrix-valued exponential function

$$B(\boldsymbol{\xi}) \;=\; \exp\left(-i\,\boldsymbol{M}\,\boldsymbol{\xi}\right)\,. \tag{11}$$

(3 points)

d) Demonstrate then the relation

$$\boldsymbol{\xi} \;=\; \boldsymbol{\xi}(\boldsymbol{v}) \tag{12}$$

between the rapidity $\boldsymbol{\xi}$ and the velocity $\boldsymbol{v}$, with which the coordinate system S′ moves with respect to the other coordinate system S. The relation (12) follows from demanding that the origin of S′ being described by the coordinates of S and S′ according to

$$\left(x^\mu\right) \;=\; \begin{pmatrix} c\,t \\ \boldsymbol{v}\,t \end{pmatrix}, \qquad \left(x'^\mu\right) \;=\; \begin{pmatrix} c\,t' \\ \boldsymbol{0} \end{pmatrix} \tag{13}$$

and the boost transformation (11) relating them via $x'^\mu \;=\; \left(B(\boldsymbol{\xi})\right)^\mu{}_\nu\, x^\nu$. What is the resulting relationship between t und t'? With the help of (11) and (12) display that the boost transformation B depends on the velocity $\boldsymbol{v}$:

$$B(\boldsymbol{v}) \;=\; B\left(\boldsymbol{\xi}(\boldsymbol{v})\right)\,. \tag{14}$$

(3 points)

e) For $\boldsymbol{v} = v\boldsymbol{e}$ with an arbitrary unity vector $\boldsymbol{e}$ the boost transformations (14) represent a subgroup of the Lorentz group. Show the corresponding closure relation

$$B(\boldsymbol{v}) = B(\boldsymbol{v}_2)\, B(\boldsymbol{v}_1) \tag{15}$$

and the addition theorem for velocities

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{v}_1, \boldsymbol{v}_2) . \tag{16}$$

Is the result symmetric with respect to the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$? What would one get in the limits $v_1 \ll c, v_2 \ll c$ and $v_2 = c$, respectively? (2 points)

Problem 6: Lorentz Group III

The infinitesimal Lorentz transformation of a space-time four-vector is given by

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} , \tag{17}$$

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} , \tag{18}$$

$$\omega^{\mu}{}_{\nu} = -\frac{i}{2} \left(L^{\alpha\beta}\right)^{\mu}{}_{\nu}\, \omega_{\alpha\beta} . \tag{19}$$

a) Consider a scalar field $\psi(x)$, which is invariant with respect to a Lorentz transformation:

$$\psi'(x) = \psi\left(\Lambda^{-1} x\right) . \tag{20}$$

With the help of (17)–(19) employ (20) to infinitesimal Lorentz transformations. Show

$$\psi'(x) = \left\{1 - \frac{i}{2}\, \omega_{\alpha\beta}\, \hat{L}^{\alpha\beta}\right\} \psi(x) , \tag{21}$$

where the differential operators

$$\hat{L}^{\alpha\beta} = \frac{1}{\hbar} \left\{x^{\alpha}\, \hat{p}^{\beta} - x^{\beta}\, \hat{p}^{\alpha}\right\} \tag{22}$$

contain the momentum operators $\hat{p}^{\alpha} = i\hbar\, \partial^{\alpha}$. (1 points)

b) Determine the commutators of the differential operators (22) with respect to both space-time and momentum operators:

$$\left[\hat{L}^{\alpha\beta}, x^{\gamma}\right]_{-} = ? , \qquad \left[\hat{L}^{\alpha\beta}, \hat{p}^{\gamma}\right]_{-} = ? . \tag{23}$$

With the help of (23) show that the differential operators have the commutator

$$\left[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}\right]_{-} = i\left(g^{\alpha\delta}\, \hat{L}^{\beta\gamma} + g^{\beta\gamma}\, \hat{L}^{\alpha\delta} - g^{\alpha\gamma}\, \hat{L}^{\beta\delta} - g^{\beta\delta}\, \hat{L}^{\alpha\gamma}\right) . \tag{24}$$

Which conclusion can you draw from comparing (5) and (24)? (3 points)

Drop the solutions in the post box on the 5th floor of building 46 or send them via email to radonjic@physik.uni-kl.de until November 19, 2020 at 12.00!