TU KAISERSLAUTERN
Department of Physics

WS 2020/2021

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Quantum Field Theory — Problem Sheet 4

Problem 7: Euler-Lagrange Equations

Assume that the action $\mathcal{A}$ of a field theory is given by

$$\mathcal{A} = \int_\Omega d^4x\, \mathcal{L}\left(\psi^\sigma(x^\lambda); \partial_\mu \psi^\sigma(x^\lambda)\right) \,, \tag{1}$$

where $\mathcal{L}$ denotes the Lagrange density.

a) The field-theoretic Hamilton principle states that the action is extremized with the help of the functional derivative

$$\frac{\delta \mathcal{A}}{\delta \psi^\sigma(x^\lambda)} = 0\,. \tag{2}$$

Derive with this the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \psi^\sigma(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^\sigma(x^\lambda)\right)} = 0\,. \tag{3}$$

(4 points)

b) The Lagrange density of the Maxwell field in vacuum reads

$$\mathcal{L} = \alpha\, F_{\mu\nu}\, F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{4}$$

Show that the corresponding Euler-Lagrange equations reduce to the Maxwell equations in the vacuum. (6 points)

Problem 8: Noether Theorem

Prove the Noether theorem as follows. Provided that the action (1) is invariant with respect to an infinitesimal transformation of the space-time coordinates and the fields

$$x'^\lambda = x^\lambda + \delta x^\lambda\,, \quad \psi'^\sigma(x'^\lambda) = \psi^\sigma(x^\lambda) + \delta\psi^\sigma(x^\lambda)\,, \tag{5}$$

then the continuity equation $\partial_\mu \rho^\mu(x^\lambda) = 0$ follows with the four-current density

$$\rho^\mu(x^\lambda) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^\sigma(x^\lambda)\right)} \delta\psi^\sigma(x^\lambda) - \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^\sigma(x^\lambda)\right)} \partial_\nu \psi^\sigma(x^\lambda) - \mathcal{L}\, \delta^\mu{}_\nu \right\} \delta x^\nu\,. \tag{6}$$

Which quantity is conserved due to the continuity equation? (12 points)

Note: Use the book W. Greiner and J. Reinhardt, *Field Quantization*, Springer (1996)

Drop the solutions in the post box on the 5th floor of building 46 or send them via email to radonjic@physik.uni-kl.de until November 26, 2020 at 12.00!