TU KAISERSLAUTERN
Department of Physics

WS 2020/2021
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Quantum Field Theory — Problem Sheet 5

Problem 9: Energy-Momentum Tensor

a) Prove with the Noether theorem the conservation law for energy and momentum. Provided that the action is invariant with respect to an infinitesimal translation of space-time

$$x'^{\lambda} = x^{\lambda} + \delta x^{\lambda} \,, \quad \psi'^{\sigma}(x'^{\lambda}) = \psi^{\sigma}(x^{\lambda}) \,, \tag{1}$$

then the continuity equation $\partial_\mu \Theta^{\mu\nu}(x^\lambda) = 0$ follows with the canonical energy-momentum tensor

$$\Theta^{\mu}{}_{\nu}(x^{\lambda}) = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^\sigma(x^\lambda)\right)} \partial_\nu \psi^\sigma(x^\lambda) - \mathcal{L}\, \delta^{\mu}{}_{\nu} \,. \tag{2}$$

(2 points)

b) Determine the canonical energy-momentum tensor $\Theta^{\mu\nu}$ of the Maxwell field, see Problem **7 b)**, and prove that it is neither symmetric nor gauge invariant. (4 points)

Problem 10: Symmetrization of Energy-Momentum Tensor

a) Prove with the Noether theorem the conservation law for angular momentum. Provided that the action is invariant with respect to an infinitesimal Lorentz transformation

$$x'^{\lambda} = x^{\lambda} - \frac{i}{2} w_{\nu\kappa} (L^{\nu\kappa})^{\lambda}{}_{\mu} x^{\mu} \,, \quad \psi'^{\sigma}(x'^{\lambda}) = \psi^{\sigma}(x^{\lambda}) - \frac{i}{2} w_{\nu\kappa} (N^{\nu\kappa})^{\sigma}{}_{\tau} \psi^{\tau}(x^{\lambda}) \,, \tag{3}$$

then the continuity equation $\partial_\mu J^{\mu\nu\kappa}(x^\lambda) = 0$ follows with the angular momentum tensor $J^{\mu\nu\kappa}$. Prove that $J^{\mu\nu\kappa}$ decomposes into an orbital angular momentum tensor $L^{\mu\nu\kappa}$ and a spin angular momentum tensor $S^{\mu\nu\kappa}$, which depend on the respective representation $L^{\nu\kappa}$ and $N^{\nu\kappa}$ of the Lorentz group in space-time and the field space. (2 points)

b) Demonstrate that the canonical momentum tensor $\Theta^{\mu\nu}(x^\lambda)$, which is in general not symmetric, can be extended according to the Belifante construction to a symmetric energy-momentum tensor

$$T^{\mu\nu}(x^{\lambda}) = \Theta^{\mu\nu}(x^{\lambda}) + t^{\mu\nu}(x^{\lambda}) \,, \tag{4}$$

where $t^{\mu\nu}(x^\lambda)$ is uniquely determined by the spin angular momentum tensor. Show that (4) does not change neither the energy-momentum conservation nor the conserved energy and momentum. (8 points)

c) Perform the Belifante construction for the example of the Maxwell field. Determine the constant α such that energy density $u = cT^{00}$ reduces to the usual expression in SI units:

$$u = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \,. \tag{5}$$

Is the symmetric energy-momentum tensor of the Maxwell field gauge invariant?

(6 points)

Note: Use the book W. Greiner and J. Reinhardt, *Field Quantization*, Springer (1996)

Drop the solutions in the post box on the 5th floor of building 46 or send them via email to radonjic@physik.uni-kl.de until December 3, 2020 at 12.00!