TU KAISERSLAUTERN
Department of Physics

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Quantum Field Theory — Problem Sheet 8

Problem 14: Quantum Fluctuations of Electric Field

a) What is the relation between the respective second-quantized operators of the electric field $\hat{\mathbf{E}}(\mathbf{x},t)$ and the transversal vector potential $\hat{\mathbf{A}}_\perp(\mathbf{x},t)$? Write down the Fourier decomposition of the components $\hat{E}_i(\mathbf{x},t)$ of the second-quantized electric field by taking into account the results of Problem **13 b)**. (2 points)

b) Determine the vacuum expectation value of the electric field operator, i.e. $\langle 0|\hat{E}_i(\mathbf{x},t)|0\rangle$. (1 point)

c) Show that the vacuum correlation function of the components of the electric field operator is of the form

$$\langle 0|\hat{E}_i(\mathbf{x},t)\hat{E}_{i'}(\mathbf{x}',t')|0\rangle = \sum_{\lambda=\pm}\int d^3k\, F_{ii'}(\mathbf{k};\lambda)\, e^{i[\mathbf{k}(\mathbf{x}-\mathbf{x}')-c|\mathbf{k}|(t-t')]} \tag{1}$$

and determine $F_{ii'}(\mathbf{k};\lambda,\lambda')$. (2 points)

d) Prove that the the vacuum correlation function of the electric field operator reads

$$\langle 0|\hat{\mathbf{E}}(\mathbf{x},t)\hat{\mathbf{E}}(\mathbf{x}',t')|0\rangle = \int_0^\infty dk\, F(k) \tag{2}$$

and determine the integrand $F(k)$. Evaluate the remaining k-integral with the help of the integral representation of the Gamma function

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1}\, e^{-t}\,. \tag{3}$$

Hint: Prove the identity

$$\int_0^\infty dt\, t^{x-1}\, e^{-iat} = \frac{\Gamma(x)}{(ia)^x}\,, a>0 \tag{4}$$

with an appropriate integration contour in the complex plane. (4 points)

e) Which result do you get for the vacuum correlation function of the electric field operator with equal spatial arguments, i.e. $\langle 0|\hat{\mathbf{E}}(\mathbf{x},t)\hat{\mathbf{E}}(\mathbf{x},t')|0\rangle$? Which conclusion can you now draw for the vacuum expectation value of the electric field energy? (2 points)

Problem 15: Casimir Effect

Consider two metal plates of area A which have the distance d.

a) Show that the vacuum energy between the two metal plates is given by

$$E_{\mathrm{P}} = {\sum_{n=0}^{\infty}}' A \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \hbar c \sqrt{\left(\frac{\pi n}{d}\right)^2 + k_y^2 + k_z^2} \tag{5}$$

Here the sum carries a prime in order to record the presence of an extra factor 1/2 for the term $n = 0$. **Hint:** Consider the contributions of the electric and the magnetic field separately. (2 points)

b) Show that the vacuum energy in the absence of the two metal plates reads

$$E_{\mathrm{V}} = \frac{d}{\pi} \int_0^{\infty} dk_x A \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \hbar c \sqrt{k_x^2 + k_y^2 + k_z^2} \tag{6}$$

(2 points)

c) Prove that the Casimir energy $E_{\mathrm{C}} = E_{\mathrm{P}} - E_{\mathrm{V}}$ is of the form

$$E_{\mathrm{C}} = B \int_0^{\infty} d\tau\, \tau^{-5/2} \left({\sum_{n=0}^{\infty}}' e^{-\pi^2 n^2 \tau / d^2} - \int_0^{\infty} dn\, e^{-\pi^2 n^2 / d^2} \right) \tag{7}$$

and determine the constant B. **Hint:** Use the Schwinger trick

$$\frac{1}{a^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} d\tau\, \tau^{x-1}\, e^{-a\tau} \tag{8}$$

which follows from (3). (3 points)

d) Prove the distributional identity

$$\sum_{n=-\infty}^{\infty} \delta(x - n) = \sum_{m=-\infty}^{\infty} e^{-2\pi i m x} \tag{9}$$

by determining the Fourier transform of the comb function on the left-hand side. Show with this the Poisson sum formula

$${\sum_{n=0}^{\infty}}' f(n) - \int_0^{\infty} dn\, f(n) = \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \int_0^{\infty} dx\, f(x)\, e^{-2\pi i m x}\,. \tag{10}$$

(4 points)

e) Evaluate the Casimir energy (7) with the Poisson sum formula (10). **Hint:** Use $\Gamma(-1/2) = -2\sqrt{\pi}$ and $\zeta(4) = \pi^4/90$ with the Riemann zeta function $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$. (2 points)

f) Is the resulting Casimir force $F_{\mathrm{C}} = -\partial E_{\mathrm{C}}/\partial d$ between the two plates repulsive or attractive? Which Casimir pressure do you get for the distance $d = 1\ \mu$m? Provided that the area is $A = 1(\mu\mathrm{m})^2$, which value does the Casimir force F_{C} have and is it measurable? (3 points)

Drop the solutions in the post box on the 5th floor of building 46 or send them via email to radonjic@physik.uni-kl.de until January 7, 2021 at 12.00!