$$\frac{\partial L'}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L'}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} + \partial_\mu \partial_\nu \chi\, \dot{x}^\nu - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} - \underbrace{\frac{d}{d\sigma}\,\partial_\mu \chi(x^\lambda)}_{=\partial_\nu \partial_\mu \chi\, \dot{x}^\nu} = 0$$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\,\chi = 0$$ (theorem of Schwarz)

- Apart from obvious Lorentz invariance: reparametrization invariance of trajectory parameter σ
- Transformation: $\sigma = \sigma(\sigma')$
- Reparametrization invariance is guaranteed by demanding that L is a homogeneous function of velocities of first order:

$$L(x^\lambda; \alpha \dot{x}^\lambda) = \alpha\, L(x^\lambda; \dot{x}^\lambda)$$

- Proof:

$$A = \int_{\sigma_i}^{\sigma_f} d\sigma\, L(x^\lambda(\sigma); \dot{x}^\lambda(\sigma)) = \int_{\sigma_i'}^{\sigma_f'} d\sigma'\, L\Big(\underbrace{x^\lambda(\sigma(\sigma'))}_{=x^\lambda(\sigma')};\ \frac{d\sigma'}{d\sigma}\underbrace{\dot{x}^\lambda(\sigma(\sigma'))}_{=\dot{x}^\lambda(\sigma')}\Big)\frac{d\sigma}{d\sigma'}$$

$$= \int_{\sigma_i'}^{\sigma_f'} d\sigma'\, L(x^\lambda(\sigma'); \dot{x}^\lambda(\sigma'))\underbrace{\frac{d\sigma'}{d\sigma}\frac{d\sigma}{d\sigma'}}_{=1} = \int_{\sigma_i'}^{\sigma_f'} d\sigma'\, L(x^\lambda(\sigma'); \dot{x}^\lambda(\sigma'))$$

- Later on: two physical choices
  - proper time (time of a co-moving observer in rest frame of particle)
  - laboratory time

10.1.2 Free Particle:

- Minkowski metric $g_{\mu\nu}$: distance between two adjacent space-time points x^μ, $x^\mu + dx^\mu$

$$ds = \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu} = \sqrt{c^2 dt^2 - d\vec{x}^2}$$

$\rightarrow$ manifestly Lorentz invariance

- Momentary rest frame: $d\vec{x}_R = \vec{0} \Rightarrow ds = c\, d\tau$, $d\tau = dt_R$ proper time
- Length of trajectory:

$$\int_{s_i}^{s_f} ds = \int_{\sigma_i}^{\sigma_f} d\sigma\, \frac{ds}{d\sigma} = \int_{\sigma_i}^{\sigma_f} d\sigma\, \sqrt{g_{\mu\nu}\, \dot{x}^\mu(\sigma)\, \dot{x}^\nu(\sigma)}$$

  - Lorentz invariant
  - Integrand is homogeneous in velocities of order one: reparametrization invariance

$\rightarrow$ candidate for an action in relativistic mechanics

- Action for a relativistic particle of mass M:

$$A^{(0)} = -Mc \int_{\sigma_i}^{\sigma_f} d\sigma \sqrt{g_{\mu\nu}\, \dot{x}^\mu(\sigma)\, \dot{x}^\nu(\sigma)}$$

- Proof: correct non-relativistic limit
  trajectory parameter σ: laboratory time t

$$A^{(0)} = -Mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \underset{|\dot{\vec{x}}| \ll c}{\overset{c \to \infty}{=}} -Mc^2 \int_{t_i}^{t_f} dt \left\{1 - \frac{1}{2}\frac{\dot{\vec{x}}^2(t)}{c^2} + \dots\right\}$$

$$= \int_{t_i}^{t_f} dt \Big\{\underbrace{-Mc^2}_{\text{shift due to rest energy}} + \frac{M}{2}\dot{\vec{x}}^2 + \dots\Big\}$$ non-relativistic action

10.1.3 Charged Particle:

- Non-relativistic charged particle: $A^{(int)} = -q \int_{t_i}^{t_f} dt\ \varphi(\vec{x}(t), t)$

$$(\dot{x}^\mu(t)) = \begin{pmatrix} c \\ \dot{\vec{x}}(t) \end{pmatrix}, \quad (A^\mu(x)) = \begin{pmatrix} \varphi(\vec{x},t)/c \\ \vec{A}(\vec{x},t) \end{pmatrix} \Rightarrow A^{(int)} = -q \int_{t_i}^{t_f} dt\ \dot{x}^0(t)\, A^0(\vec{x}(t), t)$$

- Generalization in a relativistic covariant way:

$$A^{(int)} = -q \int_{\sigma_i}^{\sigma_f} dt\ \dot{x}^\mu(\sigma)\, A_\mu(x^\lambda(\sigma))$$

Lorentz invariant, reparametrization invariant

- Adding both contributions: $A = A^{(0)} + A^{(int)} = \int_{\sigma_i}^{\sigma_f} d\sigma \Big\{-Mc \sqrt{g_{\mu\nu}\, \dot{x}^\mu(\sigma)\, \dot{x}^\nu(\sigma)} - q\, \dot{x}^\mu(\sigma) A_\mu(x(\sigma))\Big\}$
- Gauge transformation: $A'_\mu = A_\mu + \partial_\mu \Lambda$, Λ: gauge function

$\Rightarrow$ mechanical gauge with $\boxed{\chi = -q\,\Lambda}$

- Euler-Lagrange equation:

$$\frac{\partial L}{\partial x^\mu} = -q\, \partial_\mu A_\nu \dot{x}^\nu, \qquad \frac{\partial L}{\partial \dot{x}^\mu} = -Mc\, \frac{g_{\mu\nu}\, \dot{x}^\nu}{\sqrt{g_{\kappa\lambda}\, \dot{x}^\kappa \dot{x}^\lambda}} - q A_\mu$$

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} = 0 \quad \text{with} \quad \dot{s}(\sigma) = \sqrt{g_{\mu\nu}\, \dot{x}^\mu(\sigma)\, \dot{x}^\nu(\sigma)}$$

$$M\, \ddot{x}^\mu = \frac{\dot{s}}{c}\, q\, g^{\mu\kappa} \underbrace{F_{\kappa\nu}}_{=\partial_\kappa A_\nu - \partial_\nu A_\kappa} \dot{x}^\nu + M \frac{\ddot{s}}{\dot{s}}\, \dot{x}^\mu$$

$= F^\mu{}_\nu$ electromagnetic field strength tensor

$= $ Lorentz force

- Reparametrization invariance in relativistic mechanics:
proper time τ: $\dot{s}(\tau) = c \quad \ddot{s}(\tau) = 0$

$\Rightarrow \quad m \ddot{x}^\mu = q \, F^\mu{}_\nu \dot{x}^\nu$ (Einstein eq. = relat. extens. of Newton eq.)

10.1.4 Minimal coupling:

- choose trajectory parameter to be laboratory time: $\sigma = t$

$$A = A[\vec{x}(\cdot)] = \int_{t_i}^{t_f} dt \, L(\vec{x}(t); \dot{\vec{x}}(t); t), \quad L = -mc^2\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}} - q\varphi(\vec{x},t) + q\dot{\vec{x}}\cdot\vec{A}(\vec{x},t)$$

- canonical momentum:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \underbrace{\frac{m\dot{\vec{x}}}{\sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}}}}_{= \vec{p}_{kin}} + q\vec{A}(\vec{x}(t)) \quad \Rightarrow \quad \vec{p}_{kin} = \vec{p} - q\vec{A}(\vec{x},t)$$

Inverted relation: $\dot{\vec{x}} = \dfrac{c(\vec{p} - q\vec{A}(\vec{x},t))}{\sqrt{(\vec{p} - q\vec{A}(\vec{x},t))^2 + m^2c^2}}$

- Legendre transformation:

$$\underbrace{H}_{\text{total energy}} = \dot{\vec{x}}\frac{\partial L}{\partial \dot{\vec{x}}} - L = \dots = \underbrace{c\sqrt{[\vec{p} - q\vec{A}(\vec{x},t)]^2 + m^2c^2}}_{= H_{kin}} \underbrace{+ q\varphi(\vec{x},t)}_{\text{pot. energy}}$$

Remark: $c \to \infty$

$$H = mc^2 + \frac{[\vec{p} - q\vec{A}(\vec{x},t)]^2}{2m} + q\varphi(\vec{x},t) + \dots$$

- summary:

free theory $(q=0)$ $\xRightarrow{\text{formal}}$ interacting theory $(q \neq 0)$

$H \longrightarrow H - q\varphi$

$\vec{p} \longrightarrow \vec{p} - q\vec{A}$

- covariant formulation: minimal coupling

$(p^\mu) = \begin{pmatrix} H/c \\ \vec{p} \end{pmatrix} \longrightarrow (p^\mu - qA^\mu), \quad (A^\mu) = \begin{pmatrix} \varphi/c \\ \vec{A} \end{pmatrix}$

$p_\mu \longrightarrow p_\mu - qA_\mu$

- Now: combine minimal coupling with Jordan rule $p_\mu \to \hat{p}_\mu = i\hbar\partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}$

10.2 QED Action:

- scalar QED: charged spin 0 particles (e.g. $\pi^\pm$) + photons
- spinor QED: charged spin 1/2 " (e.g. $e^\pm$) + photons

$\Rightarrow$ derive interaction terms: strength depends on charge q as coupling constant

10.2.1 Scalar QED:

- Free Klein-Gordon theory: $A[\Psi(\cdot), \Psi^*(\cdot)] = \frac{1}{c}\int d^4x \, \mathcal{L}$

$$\mathcal{L} = \mathcal{L}(\Psi(x^\lambda), \partial_\mu\Psi(x^\lambda); \Psi^*(x^\lambda), \partial_\mu\Psi^*(x^\lambda))$$
$$= \frac{\hbar^2}{2m} g^{\mu\nu}\partial_\mu\Psi^*(x^\lambda)\partial_\nu\Psi(x^\lambda) - \frac{mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda)$$

- Minimal coupling: $p_\mu \to p_\mu - qA_\mu$
Jordan rule $p_\mu \to i\hbar\partial_\mu$
combination: $i\hbar\partial_\mu \to i\hbar\partial_\mu - qA_\mu \quad |: i\hbar$

$$\partial_\mu\Psi \to \left(\partial_\mu + \frac{iq}{\hbar}A_\mu\right)\Psi = \mathcal{D}_\mu\Psi$$

gauge covariant derivative

$$\partial_\mu\Psi^* \to \left(\partial_\mu - \frac{iq}{\hbar}A_\mu\right)\Psi^* = \mathcal{D}^*_\mu\Psi^*$$

- Application: interaction theory formally resembles a free theory

$$\mathcal{L} = \frac{\hbar^2}{2m}g^{\mu\nu}\mathcal{D}^*_\mu\Psi^*(x^\lambda)\mathcal{D}_\nu\Psi(x^\lambda) - \frac{mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda)$$
$$= \frac{\hbar^2}{2m}g^{\mu\nu}\left[\partial_\mu - \frac{iq}{\hbar}A_\mu(x^\lambda)\right)\Psi^*(x^\lambda)\left[\partial_\nu + \frac{iq}{\hbar}A_\nu(x^\lambda)\right]\Psi(x^\lambda) - \frac{mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda)$$

- Electromagnetic gauge transformation:

$A_\mu(x^\lambda) \longrightarrow A'_\mu(x^\lambda) = A_\mu(x^\lambda) + \partial_\mu\Lambda(x^\lambda)$ gauge function

$\Psi(x^\lambda) \longrightarrow \Psi'(x^\lambda) = e^{-\frac{iq}{\hbar}\Lambda(x^\lambda)}\Psi(x^\lambda)$

local phase depending on gauge function

$$D'_\mu \Psi'(x^\lambda) = \left\{ \partial_\mu + \frac{iq}{\hbar} A'_\mu(x^\lambda) \right\} \Psi'(x^\lambda) = \left\{ \partial_\mu + \frac{iq}{\hbar} A_\mu(x^\lambda) + \frac{iq}{\hbar} \partial_\mu \Lambda(x^\lambda) \right\} e^{-\frac{i}{\hbar} q \Lambda(x^\lambda)} \Psi(x^\lambda)$$

$$= e^{-\frac{i}{\hbar} q \Lambda(x^\lambda)} \left\{ \partial_\mu + \frac{iq}{\hbar} A_\mu(x^\lambda) \right\} \Psi(x^\lambda) = e^{-\frac{i}{\hbar} q \Lambda(x^\lambda)} D_\mu \Psi(x^\lambda)$$

Result: gauge covariant derivative transforms like the wave function

Note: $\Psi^*(x^\lambda) \to \Psi'^*(x^\lambda) = e^{+\frac{i}{\hbar} q \Lambda(x^\lambda)} \Psi^*(x^\lambda)$

$D_\mu^* \Psi^*(x^\lambda) \to D'^*_\mu \Psi'^*(x^\lambda) = e^{\frac{i}{\hbar} q \Lambda(x^\lambda)} D^*_\mu \Psi^*(x^\lambda)$

- Lagrange density manifestly gauge invariant:

$$\mathcal{L}' = \frac{\hbar^2}{2M} g^{\mu\nu} D'^*_\mu \Psi'^*(x^\lambda) D'_\nu \Psi'(x^\lambda) - \frac{Mc^2}{2} \Psi'^*(x^\lambda) \Psi'(x^\lambda) = \mathcal{L}$$

- Four-vector potential not given but dynamic: add free Maxwell theory

$$S = S[\Psi(\cdot); \Psi^*(\cdot); A_\nu(\cdot)] = \frac{1}{c} \int d^4x \ \mathcal{L}$$

$$\mathcal{L} = \mathcal{L}(\Psi(x^\lambda), \partial_\mu \Psi(x^\lambda); \Psi^*(x^\lambda), \partial_\mu \Psi^*(x^\lambda); A_\nu(x^\lambda), \partial_\mu A_\nu(x^\lambda))$$

$$= \frac{\hbar^2}{2M} g^{\mu\nu} \left(\partial_\mu - \frac{iq}{\hbar} A_\mu\right) \Psi^* \left(\partial_\nu + \frac{iq}{\hbar} A_\nu\right) \Psi - \frac{Mc^2}{2} \Psi^* \Psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} = \mathcal{L}^{(0)} + \mathcal{L}^{(int)}$$

$$\mathcal{L}^{(0)} = \frac{\hbar^2}{2M} g^{\mu\nu} \partial_\mu \Psi^* \partial_\nu \Psi - \frac{Mc^2}{2} \Psi^* \Psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}^{(int)} = - \underset{\sim}{j^\mu} A_\mu$$

four current density

free: $j^\mu = \frac{i\hbar q}{2M} g^{\mu\nu} \{ \Psi^*(x^\lambda) \partial_\nu \Psi(x^\lambda) - \Psi(x^\lambda) \partial_\nu \Psi^*(x^\lambda) \}$

interacting: $j^\mu = \frac{i\hbar q}{2M} g^{\mu\nu} \{ \Psi^*(x^\lambda) \underbrace{D_\nu \Psi(x^\lambda)}_{=\left(\partial_\nu + \frac{iq}{\hbar} A_\nu(x^\lambda)\right)\Psi(x^\lambda)} - \Psi(x^\lambda) \underbrace{D^*_\nu \Psi^*(x^\lambda)}_{=\left(\partial_\nu - \frac{iq}{\hbar} A_\nu(x^\lambda)\right)\Psi^*(x^\lambda)} \}$

$$j^\mu = \frac{i\hbar q}{2M} g^{\mu\nu} \{ \Psi^*(x^\lambda) \partial_\nu \Psi(x^\lambda) - \Psi(x^\lambda) \partial_\nu \Psi^*(x^\lambda) \} - \frac{q^2}{M} g^{\mu\nu} A_\mu(x^\lambda) A_\nu(x^\lambda) \Psi^*(x^\lambda) \Psi(x^\lambda)$$

- Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = \mu_0 \underbrace{j^\nu}_{\text{see above}}$$

$$\frac{\partial \mathcal{L}}{\partial \Psi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} = 0 \Rightarrow \underbrace{g^{\mu\nu} D_\mu D_\nu}_{\text{gauge covariant Laplace}} \Psi + \frac{M^2 c^2}{\hbar^2} \Psi = 0$$

10.2.2 Spinor QED:

- Starting point: Lagrangian of free Dirac theory

$$\mathcal{L} = \overline{\psi}(x) \left(i\hbar c \gamma^\mu \partial_\mu - Mc^2 \right) \psi(x)$$

- global U(1) phase transformation invariance:

$\psi'(x) = e^{-\frac{i}{\hbar} \Lambda q} \psi(x)$, $\quad \overline{\psi}'(x) = e^{+\frac{i}{\hbar} \Lambda q} \overline{\psi}(x)$

Noether theorem: continuity equation for charge conservation, see Appendix

- Local gauge invariance?

$\psi'(x) = e^{-\frac{i}{\hbar} \Lambda(x) q} \psi(x)$, $\quad \overline{\psi}'(x) = e^{\frac{i}{\hbar} q \Lambda(x)} \overline{\psi}(x)$

Lagrangian is no longer invariant

$$\partial_\mu \psi'(x) = e^{-\frac{i}{\hbar} q \Lambda(x)} \left\{ \partial_\mu \psi(x) - \frac{iq}{\hbar} \partial_\mu \Lambda(x) \psi(x) \right\}$$

spoils gauge invariance

- In order to establish local gauge invariance additional fields have to be introduced such that the additional derivative of gauge function is compensated

$\Rightarrow$ Lorentz vector $A_\mu(x)$ as gauge field to compensate for $\partial_\mu \Lambda(x)$

$$\partial_\mu \psi(x) \to D_\mu \psi(x) = \left\{ \partial_\mu + \frac{iq}{\hbar} A_\mu(x) \right\} \psi(x)$$

- Determine transformation property of gauge field by demanding that gauge covariant derivative transforms like spinor

$$D'_\mu \psi'(x) \stackrel{!}{=} e^{-\frac{i}{\hbar} \Lambda(x) q} D_\mu \psi(x)$$

$$\left\{ \partial_\mu + \frac{iq}{\hbar} A'_\mu(x) \right\} e^{-\frac{i}{\hbar} \Lambda(x) q} \psi(x)$$

$$= e^{-\frac{i}{\hbar} \Lambda(x) q} \left\{ \partial_\mu + \frac{iq}{\hbar} A'_\mu(x) - \frac{iq}{\hbar} \partial_\mu \Lambda(x) \right\} \psi(x) \stackrel{!}{=} e^{-\frac{i}{\hbar} \Lambda(x) q} \left\{ \partial_\mu + \frac{iq}{\hbar} A_\mu(x) \right\} \psi(x)$$

$\Rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$ transforms like a four-vector potential

→ minimal coupling of Dirac field to Maxwell field

Note: $\partial_\mu^* \, \overline{\Psi}'(x) = e^{\frac{i}{\hbar} q\Lambda(x)} \partial_\mu^* \, \overline{\Psi}(x)$

– Application:

$\mathcal{L} = \overline{\Psi}(x)\left\{ i\hbar c\gamma^\mu \partial_\mu - mc^2 \right\}\Psi(x) = \overline{\Psi}(x)\left\{ i\hbar c\gamma^\mu\left[\partial_\mu + \frac{iq}{\hbar}A_\mu(x)\right] - mc^2\right\}\Psi(x)$

– Add free Maxwell theory:

$\mathcal{L} = \overline{\Psi}\left[i\hbar c\gamma^\mu\left(\partial_\mu + \frac{iq}{\hbar}A_\mu\right) - mc^2\right]\Psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = \mathcal{L}^{(0)} + \mathcal{L}^{(int)}$

$\mathcal{L}^{(0)} = \overline{\Psi}(x)\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\Psi(x) - \underbrace{\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}}_{= \frac{\varepsilon_0}{2}\vec{E}^2 - \frac{1}{2\mu_0}\vec{B}^2}$

$\vec{E} = -\vec\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$

$\mathcal{L}^{(int)} = - \underbrace{j^\mu}\cdot A_\mu$

$= cq\,\overline{\Psi}(x)\gamma^\mu\Psi(x)$ four-current density

– Euler-Lagrange equations:

$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = \mu_0 \underbrace{j^\nu}_{\text{see above}}$

$\frac{\partial\mathcal{L}}{\partial\overline{\Psi}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\Psi})} = 0 \Rightarrow i\hbar\gamma^\mu \underbrace{D_\mu}_{= \partial_\mu + \frac{iq}{\hbar}A_\mu(x)}\Psi - \frac{mc}{\hbar}\Psi = 0$

10.3 QED Hamilton Function:

– Lagrange density of spinor QED in terms of vector potential:

$\mathcal{L} = \overline{\Psi}(\vec x,t)\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\Psi(\vec x,t) + \frac{\varepsilon_0}{2}\left(\vec\nabla\varphi(\vec x,t)\right)^2 + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec{A}(\vec x,t)}{\partial t}\right)^2 + \varepsilon_0\vec\nabla\varphi(\vec x,t)\cdot\frac{\partial\vec A(\vec x,t)}{\partial t}$

$- \frac{1}{2\mu_0}\left(\vec\nabla\times\vec A(\vec x,t)\right)^2 - qc\,\overline{\Psi}(\vec x,t)\gamma^\mu\Psi(\vec x,t)A_\mu(\vec x,t)$

– Coulomb gauge: $\operatorname{div}\vec A(\vec x,t) = 0$

chapter 8 → $\varphi(\vec x,t)$ is no longer a dynamical degree of freedom

$\varphi(\vec x,t) = \int d^3x' \frac{\rho(\vec x',t)}{4\pi\varepsilon_0|\vec x - \vec x'|} \overset{\text{Dirac}}{=} \int d^3x' \frac{q\,\Psi^\dagger(\vec x',t)\Psi(\vec x',t)}{4\pi\varepsilon_0|\vec x - \vec x'|}$

– canonical momenta:

$\pi(\vec x,t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\Psi(\vec x,t)}{\partial t}\right)} = i\hbar\underbrace{\overline{\Psi}(\vec x,t)\gamma^0}_{\Psi^\dagger(\vec x,t)\gamma^0} \overset{(\gamma^0)^2=1}{=} i\hbar\,\Psi^\dagger(\vec x,t)$

$\overline{\pi}(\vec x,t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\overline{\Psi}(\vec x,t)}{\partial t}\right)} = 0$

$\vec\pi(\vec x,t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)} = \varepsilon_0\left\{\frac{\partial\vec A(\vec x,t)}{\partial t} + \underline{\vec\nabla\varphi(\vec x,t)}\right)$

additional term as charge is non-zero

– Legendre transformation:

$\mathcal{H} = \pi(\vec x,t)\frac{\partial\Psi(\vec x,t)}{\partial t} + \cancel{\frac{\partial\overline{\Psi}(\vec x,t)}{\partial t}\overline{\pi}(\vec x,t)} + \vec\pi(\vec x,t)\frac{\partial\vec A(\vec x,t)}{\partial t} - \mathcal{L}$

$= \cancel{i\hbar\Psi^\dagger(\vec x,t)\frac{\partial\Psi(\vec x,t)}{\partial t}} + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2 + \cancel{\varepsilon_0\frac{\partial\vec A(\vec x,t)}{\partial t}\cdot\vec\nabla\varphi(\vec x,t)}$

$\cancel{- i\hbar c\Psi^\dagger(\vec x,t)\gamma^0\gamma^0\frac{1}{c}\frac{\partial\Psi(\vec x,t)}{\partial t}} + \Psi^\dagger(\vec x,t)\left(-i\hbar c\underbrace{\gamma^0\vec\gamma}_{=\vec\alpha}\vec\nabla + mc^2\underbrace{\gamma^0}_{=\beta}\right)\Psi(\vec x,t)$

$\cancel{- \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2} - \frac{\varepsilon_0}{2}\left(\vec\nabla\varphi(\vec x,t)\right)^2 - \cancel{\varepsilon_0\frac{\partial\vec A(\vec x,t)}{\partial t}\vec\nabla\varphi(\vec x,t)} + \frac{1}{2\mu_0}\left[\vec\nabla\times\vec A(\vec x,t)\right]^2$

$+ q\,\Psi^\dagger(\vec x,t)\underbrace{\gamma^0\gamma^0}_{=1}\Psi(\vec x,t)\,\varphi(\vec x,t) - qc\,\Psi^\dagger(\vec x,t)\gamma^0\vec\gamma\,\Psi(\vec x,t)\vec A(\vec x,t)$

$\mathcal{H} = \Psi^\dagger(\vec x,t)\left(-i\hbar c\vec\alpha\cdot\vec\nabla + mc^2\beta\right)\Psi(\vec x,t) + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2 - \frac{\varepsilon_0}{2}\left(\vec\nabla\varphi(\vec x,t)\right)^2 + \frac{1}{2\mu_0}\left[\vec\nabla\times\vec A(\vec x,t)\right]^2$

$+ q\,\Psi^\dagger(\vec x,t)\Psi(\vec x,t)\varphi(\vec x,t) - qc\,\Psi^\dagger(\vec x,t)\vec\alpha\cdot\Psi(\vec x,t)\vec A(\vec x,t)$

– Hamilton function (partial integration + Coulomb gauge, see chapter 8):

$H = \int d^3x\,\mathcal{H} = \int d^3x\left\{\Psi^\dagger(\vec x,t)\left(-i\hbar c\vec\alpha\vec\nabla + mc^2\beta\right)\Psi(\vec x,t) + \frac{\varepsilon_0}{2}\left(\frac{\partial\vec A(\vec x,t)}{\partial t}\right)^2 \cancel{- \frac{\varepsilon_0}{2}\left(\vec\nabla\varphi(\vec x,t)\right)^2}\right.$

$\left. + \frac{1}{2\mu_0}\partial_k A_l(\vec x,t)\partial_k A_l(\vec x,t) + \underline{\frac{q}{2}\Psi^\dagger(\vec x,t)\Psi(\vec x,t)\varphi(\vec x,t)} - qc\,\Psi^\dagger(\vec x,t)\vec\alpha\,\Psi(\vec x,t)\vec A(\vec x,t)\right\}$

– Auxiliary calculation:

$\Delta\frac{q}{4\pi\varepsilon_0|\vec x - \vec x'|} = -\frac{q}{\varepsilon_0}\delta(\vec x - \vec x') \Rightarrow \Delta\frac{1}{|\vec x - \vec x'|} = -4\pi\,\delta(\vec x - \vec x')$

$$\int d^3x \, \frac{-\varepsilon_0}{2} \left(\vec{\nabla} \varphi(\vec{x},t)\right)^2 = \frac{\varepsilon_0}{2} \int d^3x \, \varphi(\vec{x},t) \, \Delta \varphi(\vec{x},t) = \frac{\varepsilon_0}{2} \int d^3x \int d^3x' \, \varphi(\vec{x},t) \, \frac{q \, \psi^+(\vec{x}',t) \, \psi(\vec{x}',t)}{4\pi\varepsilon_0}$$

- $\Delta \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi \, \delta(\vec{x}-\vec{x}')$

$$= -\frac{q}{2} \int d^3x \, \varphi(\vec{x},t) \, \psi^+(\vec{x},t) \, \psi(\vec{x},t)$$

— Hamilton function of QED: $H = H^{(0)} + H^{(int)}$

$$H^{(0)} = \int d^3x \Big\{ \psi^+(\vec{x},t) \left(-i\hbar c \, \vec{\alpha} \, \vec{\nabla} + mc^2 \beta \right) \psi(\vec{x},t) + \frac{\varepsilon_0}{2} \frac{\partial A_k(\vec{x},t)}{\partial t} \frac{\partial A_k(\vec{x},t)}{\partial t} + \frac{1}{2\mu_0} \partial_k A_\ell(\vec{x},t) \, \partial_k A_\ell(\vec{x},t) \Big\}$$

$$H^{(int)} = -qc \int d^3x \, \overline{\psi}(\vec{x},t) \, \vec{\gamma} \, \psi(\vec{x},t) \, \vec{A}(\vec{x},t) \rightarrow$$

stems from free Dirac Hamiltonian by minimal coupling $\vec{\nabla} \to \vec{\nabla} - \frac{iq}{\hbar} \vec{A}$

$$\underbrace{+ \frac{q^2}{8\pi\varepsilon_0} \int d^3x \int d^3x' \, \frac{\overline{\psi}(\vec{x},t) \gamma^0 \psi(\vec{x},t) \, \overline{\psi}(\vec{x}',t) \gamma^0 \psi(\vec{x}',t)}{|\vec{x}-\vec{x}'|}}$$

instantaneous Coulomb self-interaction; does not contradict special relativity; is cancelled in a covariant calculation for cross section by a corresponding term in Maxwell propagator