Assumption: 1-particle problem is solved

$$\left\{-\frac{\hbar^2}{2m}\Delta + V_1(\vec{x})\right\}\psi_{E_\alpha}(\vec{x}) = E_\alpha\, \psi_{E_\alpha}(\vec{x}) \quad \Rightarrow \quad \alpha: \text{ vector of eigenvalues}$$

orthonormality: $\int d^3x\, \psi^*_{E_\alpha}(\vec{x})\,\psi_{E_{\alpha'}}(\vec{x}) = \delta_{\alpha,\alpha'}$

completeness: $\sum_\alpha \psi^*_{E_\alpha}(\vec{x})\,\psi_{E_\alpha}(\vec{x}') = \delta(\vec{x}-\vec{x}')$

separation ansatz: $\psi_E(\vec{x}_1,\dots,\vec{x}_n) = \psi_{E_{\alpha_1}}(\vec{x}_1)\cdot\ldots\cdot\psi_{E_{\alpha_n}}(\vec{x}_n) = \psi_{E_{\alpha_1}\dots E_{\alpha_n}}(\vec{x}_1,\dots,\vec{x}_n)$

$\Rightarrow E = E_{\alpha_1} + \dots + E_{\alpha_n}$

orthonormality:

$$\int d^3x_1\dots\int d^3x_n\, \psi^*_{E_{\alpha_1}\dots E_{\alpha_n}}(\vec{x}_1,\dots,\vec{x}_n)\,\psi_{E_{\alpha'_1}\dots E_{\alpha'_n}}(\vec{x}_1,\dots,\vec{x}_n) = \delta_{\alpha_1,\alpha'_1}\cdots\delta_{\alpha_n,\alpha'_n}$$

completeness:

$$\sum_{\alpha_1}\dots\sum_{\alpha_n}\psi^*_{E_{\alpha_1}\dots E_{\alpha_n}}(\vec{x}_1,\dots,\vec{x}_n)\,\psi_{E_{\alpha_1}\dots E_{\alpha_n}}(\vec{x}'_1,\dots,\vec{x}'_n) = \delta(\vec{x}_1-\vec{x}'_1)\cdots\delta(\vec{x}_n-\vec{x}'_n)$$

(anti-)symmetrisation due to indistinguishability: $\hat{S}^\varepsilon = \sum_{\hat{P}} \varepsilon^P\, \hat{P}$

$\varepsilon = \pm 1$ number of transpositions for $\hat{P} = \prod \hat{P}_{jk}$

$$\psi^\varepsilon_{\{E_\alpha\}}(\vec{x}_1,\dots,\vec{x}_n) = N^\varepsilon_{\{E_\alpha\}}\,\hat{S}^\varepsilon \prod_{\nu=1}^{n}\psi_{E_{\alpha_\nu}}(\vec{x}_\nu)$$

concrete order of energy eigenvalues unimportant due to (anti-)symmetrisation

check for symmetry:

$$\hat{P}_{jk}\,\hat{S}^\varepsilon = \sum_{\hat{P}} \varepsilon^P \underbrace{\hat{P}_{jk}\hat{P}}_{=\hat{P}'} \underset{P'=P\pm 1}{=} \sum_{\hat{P}'}\varepsilon^{P'\mp 1}\hat{P}' = \varepsilon\sum_{\hat{P}'}\varepsilon^{P'}\hat{P}' = \varepsilon\,\hat{S}^\varepsilon$$

$$\hat{P}_{jk}\,\psi^\varepsilon_{\{E_\alpha\}}(\vec{x}_1,\dots,\vec{x}_n) = \varepsilon\,\psi^\varepsilon_{\{E_\alpha\}}(\vec{x}_1,\dots,\vec{x}_n)$$

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle \Rightarrow \underbrace{\hat{S}^\varepsilon\hat{H}}_{=\hat{H}\hat{S}^\varepsilon}|\psi_E\rangle = E\,\hat{S}^\varepsilon|\psi_E\rangle \Rightarrow \hat{H}\left(\hat{S}^\varepsilon|\psi_E\rangle\right) = E\left(\hat{S}^\varepsilon|\psi_E\rangle\right)$$

$[\hat{H},\hat{P}_{jk}]=0 \Rightarrow$

(anti)-symmetrised wave function solves eigenvalue problem

special case: $\varepsilon = -1$ (fermion)

$$\psi^-_{\{E_\alpha\}}(\vec{x}_1,\dots,\vec{x}_n) = N^-_{\{E_\alpha\}}\sum_{\hat{P}}(-1)^P\,\psi_{E_{\alpha_1}}(\vec{x}_{P(1)})\cdots\psi_{E_{\alpha_n}}(\vec{x}_{P(n)})$$

slater determinant

$$= N^-_{\{E_\alpha\}}\begin{vmatrix}\psi_{E_{\alpha_1}}(\vec{x}_1) & \psi_{E_{\alpha_1}}(\vec{x}_2) & \cdots & \psi_{E_{\alpha_1}}(\vec{x}_n)\\ \psi_{E_{\alpha_2}}(\vec{x}_1) & \psi_{E_{\alpha_2}}(\vec{x}_2) & \cdots & \psi_{E_{\alpha_2}}(\vec{x}_n)\\ \vdots & \vdots & & \vdots\\ \psi_{E_{\alpha_n}}(\vec{x}_1) & \psi_{E_{\alpha_n}}(\vec{x}_2) & \cdots & \psi_{E_{\alpha_n}}(\vec{x}_n)\end{vmatrix}$$

Equality of two rows, i.e. $\alpha_j = \alpha_k$ or of two columns, i.e. $x_j = x_k$, leads to a vanishing wave function $\Rightarrow$ Pauli exclusion principle: two fermions are not allowed either in the same state or at the same point

Note: no corresponding restriction occurs for bosons

But: simplifying the following combinatorial consideration, assume that for bosons not more than one boson is in a state or at one space point

Task: determine normalisation constant $N^\varepsilon_{\{E_\alpha\}}$

$$\hat{P}_{jk}\,\hat{S}^\varepsilon = \varepsilon\,\hat{S}^\varepsilon \qquad \underbrace{\hat{P} = \prod \hat{P}_{jk}}_{} \qquad \hat{P}\,\hat{S}^\varepsilon = \varepsilon^P\,\hat{S}^\varepsilon$$

$$\langle\psi^\varepsilon_{\{E_\alpha\}}|\psi^\varepsilon_{\{E_{\alpha'}\}}\rangle = N^\varepsilon_{\{E_\alpha\}}\sum_{\hat{P}}\varepsilon^P\,\underbrace{\langle\hat{P}\,\psi_{E_{\alpha_1}}\cdots\psi_{E_{\alpha_n}}|\psi^\varepsilon_{\{E_{\alpha'}\}}\rangle}$$

$$= \langle\psi_{E_{\alpha_1}}\cdots\psi_{E_{\alpha_n}}|\underbrace{\hat{P}^+}_{=\hat{P}}\,\psi^\varepsilon_{\{E_{\alpha'}\}}\rangle = \varepsilon^P\,|\psi^\varepsilon_{\{E_{\alpha'}\}}\rangle$$

$$= N^\varepsilon_{\{E_\alpha\}}\underbrace{\sum_{\hat{P}}}_{=n!}\underbrace{\varepsilon^{2P}}_{=1}\langle\psi_{E_{\alpha_1}}\cdots\psi_{E_{\alpha_n}}|\psi^\varepsilon_{\{E_{\alpha'}\}}\rangle$$

$$= n!\,N^\varepsilon_{\{E_\alpha\}}\,\underbrace{N^\varepsilon_{\{E_{\alpha'}\}}}\,\langle\psi_{E_{\alpha_1}}\cdots\psi_{E_{\alpha_n}}|\sum_{\hat{P}'}\varepsilon^{P'}\,\underbrace{\hat{P}'\,\psi_{E_{\alpha'_1}}\cdots\psi_{E_{\alpha'_n}}\rangle}$$

$$\overbrace{}^{N^\varepsilon_{\{E_\lambda\}}} \overset{!}{=} 1 \qquad \overbrace{}^{|\psi_{E_{\alpha'_{p'(1)}}} \cdots \psi_{E_{\alpha'_{p'(n)}}}\rangle}$$

$$\underbrace{\langle \psi_{E_{\alpha_1}} | \psi_{E_{\alpha' p'(1)}} \rangle}_{=\delta_{\alpha_1, \alpha' p'(1)}} \cdot \cdots \cdot \underbrace{\langle \psi_{E_{\alpha_n}} | \psi_{E_{\alpha' p'(n)}} \rangle}_{\delta_{\alpha_n, \alpha' p'(n)}}$$

$$\overset{!}{=} \delta^\varepsilon_{\alpha_1,\dots,\alpha_n;\alpha'_1,\dots,\alpha'_n} = \sum_{\hat{P}} \varepsilon^P \delta_{\alpha_1,\alpha'_{p(1)}} \cdots \delta_{\alpha_n,\alpha'_{p(n)}}$$

$$\Rightarrow N^\varepsilon_{\{E_\lambda\}} \overset{!}{=} \frac{1}{\sqrt{n!}}$$

Remaining task: completeness of (anti-) symmetrised states

$$\sum_{\alpha_1} \cdots \sum_{\alpha_n} \psi^*_{E_{\alpha_1}}(\vec{x}_1) \cdots \psi^*_{E_{\alpha_n}}(\vec{x}_n)\, \psi_{E_{\alpha_1}}(\vec{x}'_1) \cdots \psi_{E_{\alpha_n}}(\vec{x}'_n)$$

$$= \delta(\vec{x}_1 - \vec{x}'_1) \cdots \delta(\vec{x}_n - \vec{x}'_n)$$

apply $\hat{S}^\varepsilon$ both to $\vec{x}_1,\dots,\vec{x}_n$ and $\vec{x}'_1,\dots,\vec{x}'_n$

$$\sum_{\hat{P}} \sum_{\hat{P}'} \varepsilon^{P+P'} \sum_{\alpha_1} \cdots \sum_{\alpha_n} \psi^*_{E_{\alpha_1}}(\vec{x}_{p(1)}) \cdots \psi^*_{E_{\alpha_n}}(\vec{x}_{p(n)})\, \psi_{E_{\alpha_1}}(\vec{x}'_{p'(1)}) \cdots \psi_{E_{\alpha_n}}(\vec{x}'_{p'(n)})$$

$$\alpha_1 \to p(1),\ p(1) \to 1,\ \alpha_n \to p(n),\ p(n) \to n,\ \alpha_1 \to p'(1),\ p'(1) \to 1,\ \alpha_n \to p'(n),\ p'(n) \to n$$

$$= \sum_{\hat{P}} \sum_{\hat{P}'} \varepsilon^{P+P'} \delta(\vec{x}_{p(1)} - \vec{x}'_{p'(1)}) \cdots \delta(\vec{x}_{p(n)} - \vec{x}'_{p'(n)})$$

$$p(1) \to 1,\ p'(1) \to p'(p(1)),\ p(n) \to n,\ p'(n) \to p'(p(n))$$

$$\underbrace{\sum_{\alpha_1} \cdots \sum_{\alpha_n}}_{=n!} \underbrace{\left\{ \sum_{\hat{P}} \varepsilon^P \psi^*_{E_{\alpha p(1)}}(\vec{x}_1) \cdots \psi^*_{E_{\alpha p(n)}}(\vec{x}_n) \right\}}_{=\sqrt{n!}\, \psi^\varepsilon_{\{E_\lambda\}}(\vec{x}_1,\dots,\vec{x}_n)} \underbrace{\left\{ \sum_{\hat{P}'} \varepsilon^{P'} \psi_{E_{\alpha p'(1)}}(\vec{x}'_1) \cdots \psi_{E_{\alpha p'(n)}}(\vec{x}'_n) \right\}}_{=\sqrt{n!}\, \psi^\varepsilon_{\{E_\lambda\}}(\vec{x}'_1,\dots,\vec{x}'_n)}$$

$$= \sum_{\hat{P}} \sum_{\hat{Q}} \varepsilon^q \delta(\vec{x}_1 - \vec{x}_{Q(1)}) \cdots \delta(\vec{x}_n - \vec{x}_{Q(n)})$$

$\uparrow$ $\hat{Q} = \hat{P}\hat{P}'$, $q = P + P'$

$$\sum_{\alpha_1} \cdots \sum_{\alpha_n} \psi^\varepsilon_{\{E_\lambda\}}(\vec{x}_1,\dots,\vec{x}_n)\, \psi^\varepsilon_{\{E_\lambda\}}(\vec{x}'_1,\dots,\vec{x}'_n)$$

$$= \delta^\varepsilon(\vec{x}_1,\dots,\vec{x}_n;\vec{x}'_1,\dots,\vec{x}'_n)$$

$$= \sum_{\hat{P}} \varepsilon^P \delta(\vec{x}_1 - \vec{x}'_{p(1)}) \cdots \delta(\vec{x}_n - \vec{x}'_{p(n)})$$

$\Rightarrow$ basis in the Hilbert space of (anti-) symmetrised particles based on the eigenvalue problem of Hamiltonian

Another basis: eigenvalue problem of coordinate operators

$$|\vec{x}_1,\dots,\vec{x}_n\rangle^\varepsilon = \frac{1}{\sqrt{n!}} \sum_{\hat{P}} \varepsilon^P |\vec{x}_{p(1)}, \vec{x}_{p(2)}, \dots, \vec{x}_{p(n)}\rangle$$

$${}^\varepsilon\langle \vec{x}_1,\dots,\vec{x}_n | \vec{x}'_1,\dots,\vec{x}'_n\rangle^\varepsilon = \delta^\varepsilon(\vec{x}_1,\dots,\vec{x}_n;\vec{x}'_1,\dots,\vec{x}'_n)$$

$$\int d^3x_1 \cdots \int d^3x_n\, |\vec{x}_1,\dots,\vec{x}_n\rangle^\varepsilon\, {}^\varepsilon\langle \vec{x}_1,\dots,\vec{x}_n| = 1$$

concrete example: $n=2$

distinguishable particles

$$\psi_{\vec{x}_1,\vec{x}_2}(\vec{z}_1,\vec{z}_2) = \langle \vec{z}_1,\vec{z}_2 | \vec{x}_1,\vec{x}_2 \rangle = \delta(\vec{z}_1-\vec{x}_1)\,\delta(\vec{z}_2-\vec{x}_2)$$

(anti-) symmetrization:

$$\psi^{\varepsilon}_{\vec{x}_1,\vec{x}_2}(\vec{z}_1,\vec{z}_2) = \frac{1}{\sqrt{2}}\{\delta(\vec{z}_1-\vec{x}_1)\,\delta(\vec{z}_2-\vec{x}_2) + \varepsilon\,\delta(\vec{z}_1-\vec{x}_2)\,\delta(\vec{z}_2-\vec{x}_1)\}$$

$$\int d^3z_1 \int d^3z_2\; \psi^{\varepsilon *}_{x_1,x_1}(\vec{z}_1,\vec{z}_2)\, \psi^{\varepsilon}_{\vec{x}_1',\vec{x}_2'}(\vec{z}_1,\vec{z}_2) = \ldots = \delta^{\varepsilon}(\vec{x}_1,\vec{x}_2;\vec{x}_1',\vec{x}_2')$$

$\rightarrow$ orthonormality

$$\int d^3x_1 \int d^3x_2\; \psi^{\varepsilon *}_{\vec{x}_1,\vec{x}_2}(\vec{z}_1,\vec{z}_2)\, \psi^{\varepsilon}_{\vec{x}_1,\vec{x}_2}(\vec{z}_1',\vec{z}_2') = \ldots = \delta^{\varepsilon}(\vec{z}_1,\vec{z}_2;\vec{z}_1',\vec{z}_2')$$

3. Second Quantization

Motivation:

- (Anti-) symmetrization too cumbersome to perform in concrete applications due to large particle numbers
- Solution: second quantization
  - $\rightarrow$ heuristic introduction: based on defining creation and annihilation operators for particles
  - $\rightarrow$ automatically include (anti-) symmetrization

3.1 Harmonic Oscillator:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2 \hat{x}^2, \quad [\hat{x},\hat{x}]_- = [\hat{p},\hat{p}]_- = 0, \quad [\hat{p},\hat{x}]_- = \frac{\hbar}{i}$$

eigenvalue problem: $\hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle$

here: representation-free solution based on "ladder" operator formalism

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$$

$$\hat{x} = \underbrace{\sqrt{\frac{\hbar}{2m\omega}}}_{\text{oscillator length}}(\hat{a}^+ + \hat{a}), \qquad \hat{p} = \underbrace{\sqrt{\frac{\hbar m\omega}{2}}}_{\text{momentum following from Heisenberg uncertainty}} i(\hat{a}^+ - \hat{a})$$

$$\Rightarrow \hat{H} = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a})$$

commutation relations: $[\hat{a},\hat{a}^+]_- = 1$

$$[\hat{a},\hat{a}]_- = 0 = [\hat{a}^+,\hat{a}^+]_-,$$

$$\hat{H} = \frac{\hbar\omega}{2}(\underbrace{\hat{a}^+\hat{a} + 1}_{=\hat{a}\hat{a}^+} + \hat{a}^+\hat{a}) = \hbar\omega(\underbrace{\hat{a}^+\hat{a}}_{=\hat{n}} + \underbrace{\tfrac{1}{2}}_{\text{zero-point energy}})$$

ABC-rule:

$$[\hat{A}\hat{B},\hat{C}]_- = \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} = (\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B}) + (\hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B}) = \hat{A}[\hat{B},\hat{C}]_- + [\hat{A},\hat{C}]_-\hat{B}$$

$$[\hat{n},\hat{a}^+]_- = [\hat{a}^+\hat{a},\hat{a}^+]_- = \hat{a}^+\underbrace{[\hat{a},\hat{a}^+]_-}_{=1} + \underbrace{[\hat{a}^+,\hat{a}^+]_-}_{=0}\hat{a} = \hat{a}^+$$

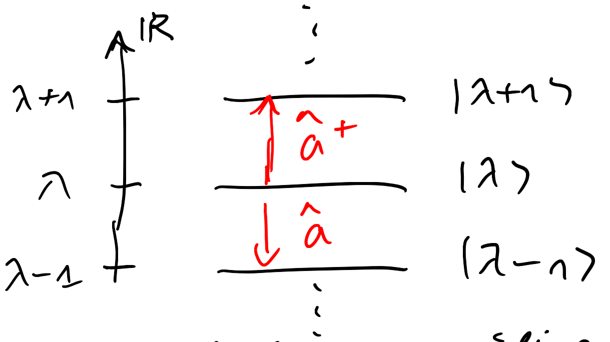
$$[\hat{n},\hat{a}]_- = [\hat{a}^+\hat{a},\hat{a}]_- = \hat{a}^+\underbrace{[\hat{a},\hat{a}]_-}_{=0} + \underbrace{[\hat{a}^+,\hat{a}]_-}_{=-1}\hat{a} = -\hat{a}$$

eigenvalue problem: $\hat{n}|\lambda\rangle = \lambda|\lambda\rangle$

$\hat{n} = \hat{n}^+$: hermitian $\Rightarrow \lambda \in \mathbb{R}$

$$\underline{\hat{n}\{\hat{a}^+|\lambda\rangle\}} = (\hat{a}^+\hat{n} + \hat{a}^+)|\lambda\rangle = \underline{(\lambda+1)\{\hat{a}^+|\lambda\rangle\}} \Rightarrow \hat{a}^+|\lambda\rangle \sim |\lambda+1\rangle$$

$$\hat{n}\{\hat{a}|\lambda\rangle\} = (\hat{a}\hat{n} - \hat{a})|\lambda\rangle = (\lambda-1)\{\hat{a}|\lambda\rangle\} \Rightarrow \hat{a}|\lambda\rangle \sim |\lambda-1\rangle$$



$\hat{a}^+$: raising operator
$\hat{a}$: lowering operator

eigenvalue λ always positive:

$$0 \leq \langle \hat{a}\lambda | \hat{a}\lambda\rangle = \langle\lambda| \underbrace{\underbrace{\hat{a}^+\hat{a}}_{=\hat{n}}\lambda\rangle}_{=\lambda|\lambda\rangle} = \lambda \underbrace{\langle\lambda|\lambda\rangle}_{=1} = \lambda$$

Claim: eigenvalues $\lambda = 0, 1, 2, 3, \ldots, \Rightarrow \lambda \in \mathbb{N}_0$

Assume: $\lambda \notin \mathbb{N}_0$; then one could apply $\hat{a}$ iteratively such that the state $(\hat{a})^k |\lambda\rangle \sim |\lambda - k\rangle$ can become negative

$\Rightarrow \lambda \in \mathbb{N}_0$: $\lambda = n = 0, 1, 2, 3, \ldots$ (lowest state: 0)

ground state: $\hat{a}|0\rangle = 0 \Leftrightarrow \langle 0|\hat{a}^+ = 0$

Aim: construct normalized eigenfunction $|n\rangle$

observation: $\langle \hat{a}^+ n | \hat{a}^+ n\rangle = \langle n|\hat{a}\hat{a}^+ n\rangle = \langle n|\hat{a}^+\hat{a}+1|n\rangle = (n+1)\underbrace{\langle n|n\rangle}_{=1}$

$\hat{a}^+|n\rangle \sim |n+1\rangle \Rightarrow \hat{a}^+|n\rangle = c_n|n+1\rangle$

$c_n^2 = \langle \hat{a}^+ n|\hat{a}^+ n\rangle = n+1 \Rightarrow c_n = \sqrt{n+1}$

$\Big\}\ \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\underline{\underline{|n\rangle}} = \frac{1}{\sqrt{n}}\hat{a}^+|n-1\rangle = \frac{1}{\sqrt{n}}\frac{1}{\sqrt{n-1}}(\hat{a}^+)^2|n-2\rangle = \ldots = \underline{\frac{1}{\sqrt{n!}}(\hat{a}^+)^n}\,\underline{\underline{|0\rangle}}$$

Note: $\hat{a}|n\rangle \sim |n-1\rangle \Rightarrow \hat{a}|n\rangle = D_n|n-1\rangle$

$$\langle \hat{a}n|\hat{a}n\rangle = \langle n|\hat{a}^+\hat{a}|n\rangle = n\cdot\underbrace{\langle n|n\rangle}_{=1} = D_n^2\underbrace{\langle n-1|n-1\rangle}_{=1}$$

$\Rightarrow D_n = \sqrt{n} \Rightarrow \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\Rightarrow E_n = \hbar\omega(n+\frac{1}{2})$; $n = 0, 1, 2, \ldots$

3.2 Creation and Annihilation Operators for Bosons:

ladder formalism for harmonic oscillator $\xrightarrow{\text{analogy}}$	second quantization for bosons
n: quantum number of 1-particle system	$n_{\vec{x}}$: number of bosons at site $\vec{x}$
$\hat{a}^+$: raising operator $\hat{a}$: lowering operator	$\hat{a}^+_{\vec{x}}$: creation operator $\hat{=}$ create boson $\hat{a}_{\vec{x}}$: annihilation operator $\hat{=}$ annih. boson at site $\vec{x}$
$[\hat{a},\hat{a}]_- = 0 = [\hat{a}^+,\hat{a}^+]_- = 0$ $[\hat{a},\hat{a}^+]_- = 1$	$[\hat{a}_{\vec{x}},\hat{a}_{\vec{x}'}]_- = 0 = [\hat{a}^+_{\vec{x}},\hat{a}^+_{\vec{x}'}]_-$ $[\hat{a}_{\vec{x}},\hat{a}^+_{\vec{x}'}]_- = \delta(\vec{x}-\vec{x}')$
$\hat{n} = \hat{a}^+\hat{a}$ $[\hat{n},\hat{a}^+]_- = \hat{a}^+$ $[\hat{n},\hat{a}]_- = -\hat{a}$	$\hat{N} = \int d^3x'\ \hat{a}^+_{\vec{x}'}\hat{a}_{\vec{x}'}$ particle number operator $[\hat{N},\hat{a}^+_{\vec{x}}]_- = \int d^3x'\{\hat{a}^+_{\vec{x}'}\underbrace{[\hat{a}_{\vec{x}'},\hat{a}^+_{\vec{x}}]_-}_{=\delta(\vec{x}-\vec{x}')} + \underbrace{[\hat{a}^+_{\vec{x}'},\hat{a}^+_{\vec{x}}]_-}_{=0}\hat{a}_{\vec{x}'}\} = \hat{a}^+_{\vec{x}}$ $[\hat{N},\hat{a}_{\vec{x}}]_- = -\hat{a}_{\vec{x}}$

ground state $\lvert 0\rangle$ $\hat{a}\lvert 0\rangle = 0 \iff \langle 0\rvert \hat{a}^+ = 0$	vacuum state $\lvert 0\rangle$ $\hat{a}_{\vec{x}}\lvert 0\rangle = 0 \iff \langle 0\rvert \hat{a}^+_{\vec{x}} = 0$
$\lvert n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n \lvert 0\rangle$ basis states of Hilbert space	$\lvert \vec{x}_1, \dots, \vec{x}_n\rangle^{+1} = \hat{a}^+_{\vec{x}_1} \cdots \hat{a}^+_{\vec{x}_n}\lvert 0\rangle$ $\vec{x}_i \neq \vec{x}_j$ for $i \neq j$ [Note: $\lvert \vec{x}_1, \vec{x}_1\rangle^{+1} = \frac{1}{\sqrt{2}} (\hat{a}^+_{\vec{x}_1})^2 \lvert 0\rangle$]

<u>n=1 boson:</u> $\lvert \vec{x}_1\rangle^{+1} = \hat{a}^+_{\vec{x}_1}\lvert 0\rangle$

$${}^{+1}\langle \vec{x}_1 \vert \vec{x}_1'\rangle^{+1} = \langle \hat{a}^+_{\vec{x}_1} 0 \vert \hat{a}^+_{\vec{x}_1'} 0\rangle = \langle 0 \vert \underbrace{\hat{a}_{\vec{x}_1} \hat{a}^+_{\vec{x}_1'}}_{= \cancel{\hat{a}^+_{\vec{x}_1'} \hat{a}_{\vec{x}_1}} + \delta(\vec{x}_1 - \vec{x}_1')} \vert 0\rangle \quad \to 0 \quad \to 0$$

$$= \delta(\vec{x}_1 - \vec{x}_1'), \quad \langle 0 \vert 0\rangle = 1$$

<u>n=2 bosons:</u> $\lvert \vec{x}_1, \vec{x}_2\rangle^{+1} = \hat{a}^+_{\vec{x}_1} \hat{a}^+_{\vec{x}_2}\lvert 0\rangle$; $\vec{x}_1 \neq \vec{x}_2$

$${}^{+1}\langle \vec{x}_1, \vec{x}_2 \vert \vec{x}_1', \vec{x}_2'\rangle^{+1} = \langle \hat{a}^+_{\vec{x}_1} \hat{a}^+_{\vec{x}_2} 0 \vert \hat{a}^+_{\vec{x}_1'} \hat{a}^+_{\vec{x}_2'} 0\rangle$$

$$= \langle 0 \vert \hat{a}_{\vec{x}_2} \underbrace{\hat{a}_{\vec{x}_1} \hat{a}^+_{\vec{x}_1'}}_{= \hat{a}^+_{\vec{x}_1'} \hat{a}_{\vec{x}_1} + \delta(\vec{x}_1 - \vec{x}_1')} \hat{a}^+_{\vec{x}_2'} \vert 0\rangle$$

$\uparrow$ $(\hat{A}\hat{B})^+ = \hat{B}^+\hat{A}^+$

$$= \delta(\vec{x}_1 - \vec{x}_1') \langle 0 \vert \underbrace{\hat{a}_{\vec{x}_2} \hat{a}^+_{\vec{x}_2'}}_{= \delta(\vec{x}_2 - \vec{x}_2') + \cancel{\hat{a}^+_{\vec{x}_2'} \hat{a}_{\vec{x}_2}}} \vert 0\rangle + \langle 0 \vert \underbrace{\hat{a}_{\vec{x}_2} \hat{a}^+_{\vec{x}_1'}}_{= \cancel{\hat{a}^+_{\vec{x}_1'} \hat{a}_{\vec{x}_2}} + \delta(\vec{x}_1' - \vec{x}_2)} \underbrace{\hat{a}_{\vec{x}_1} \hat{a}^+_{\vec{x}_2'}}_{= \cancel{\hat{a}^+_{\vec{x}_2'} \hat{a}_{\vec{x}_1}} + \delta(\vec{x}_1 - \vec{x}_2')} \vert 0\rangle$$

$$= \delta(\vec{x}_1 - \vec{x}_1') \, \delta(\vec{x}_2 - \vec{x}_2') + \delta(\vec{x}_1 - \vec{x}_2') \, \delta(\vec{x}_2 - \vec{x}_1')$$

$$= \delta^{+1}(\vec{x}_1, \vec{x}_2; \vec{x}_1', \vec{x}_2')$$

Conclusion: symmetrisation is automatically taken into account by using creation/annihilation operators.