$$\frac{\kappa}{c^2}(-2)\frac{i}{\hbar}mc^2 = \frac{i\hbar}{2mc^2}(-2)\frac{i}{\hbar}mc^2 \doteq 1 \qquad \kappa = \frac{i\hbar}{2m}$$

$$\Rightarrow \rho(\vec{x},t) = \frac{i\hbar}{2mc^2}\left\{\Psi^*(\vec{x},t)\frac{\partial \Psi(\vec{x},t)}{\partial t} - \Psi(\vec{x},t)\frac{\partial \Psi^*(\vec{x},t)}{\partial t}\right\}, \quad \vec{j}(\vec{x},t) = \frac{i\hbar}{2m}\left\{\Psi(\vec{x},t)\vec{\nabla}\Psi^*(\vec{x},t) - \Psi^*(\vec{x},t)\vec{\nabla}\Psi(\vec{x},t)\right\}$$

conserved quantity: $Q = \int d^3x\, \rho(\vec{x},t)$

$$\frac{\partial Q}{\partial t} = \int d^3x \frac{\partial \rho(\vec{x},t)}{\partial t} = -\int d^3x\, \operatorname{div}\vec{j}(\vec{x},t) \overset{\text{Gauß}}{=} -\oint d\vec{f}\cdot\vec{j}(\vec{x},t) \equiv 0$$

scalar product: $\langle \Psi_1, \Psi_2\rangle = \frac{i\hbar}{2mc^2}\int d^3x \left\{\Psi_1^*(\vec{x},t)\frac{\partial \Psi_2(\vec{x},t)}{\partial t} - \Psi_2(\vec{x},t)\frac{\partial \Psi_1^*(\vec{x},t)}{\partial t}\right\}$

not positive definite

e.g. $\Psi_1 = \Psi_2 = N\, e^{-\frac{i}{\hbar}(mc^2)t} \Rightarrow \langle \Psi_1, \Psi_2\rangle = -N^2 < 0$

non-relativistic limit:

$$\langle \Psi_1, \Psi_2\rangle = \frac{i\hbar}{2mc^2}\int d^3x \left\{\psi_1^*\frac{\partial \psi_2}{\partial t} - \psi_2\frac{\partial \psi_1^*}{\partial t} - 2\frac{i}{\hbar}mc^2\psi_1^*\psi_2\right\} \xrightarrow{c\to\infty} \underbrace{\int d^3x\, \psi_1^*\psi_2}_{\text{positive definite scalar product of Schrödinger}}$$

Moral: Each theory has its own "natural" scalar product

Result: $Q = \int d^3x \langle \Psi, \Psi\rangle$ {can be either pos. or neg.}

Q or better eQ ≙ electric charge of Klein-Gordon field

⇒ complex-valued Klein-Gordon field

Conversely: $\Psi^*(\vec{x},t) = \Psi(\vec{x},t)$ ≙ electrically neutral, $Q \equiv 0$

7.3 Canonical Field Quantization:

$$\mathcal{L} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m}\vec{\nabla}\Psi^*\cdot\vec{\nabla}\Psi - \frac{1}{2}mc^2\Psi^*\Psi$$

Canonically conjugated momentum fields:

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi}{\partial t}, \qquad \pi = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}$$

Hamilton density:

$$\mathcal{H} = \pi^*\frac{\partial \Psi^*}{\partial t} + \pi\frac{\partial \Psi}{\partial t} - \mathcal{L} = \frac{2mc^2}{\hbar^2}\pi^*\pi + \frac{\hbar^2}{2m}\vec{\nabla}\Psi^*\vec{\nabla}\Psi + \frac{1}{2}mc^2\Psi^*\Psi$$

Hamilton function: $H = \int d^3x\, \mathcal{H} \longrightarrow \hat{H}$

$\Psi^*(\vec{x},t)$, $\Psi(\vec{x},t)$, $\pi^*(\vec{x},t)$, $\pi(\vec{x},t)$ classical fields

↓ second quantization

$\hat{\Psi}^+(\vec{x},t)$, $\hat{\Psi}(\vec{x},t)$, $\hat{\pi}^+(\vec{x},t)$, $\hat{\pi}(\vec{x},t)$ operators

Bosonic field quantization ≙ equal-time commutation relations

$$[\hat{\Psi}(\vec{x},t), \hat{\Psi}(\vec{x}',t)]_- = 0 = [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- \qquad [\hat{\Psi}(\vec{x},t), \hat{\Psi}^+(\vec{x}',t)]_- = 0 = [\hat{\Psi}(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_-$$

$$[\hat{\Psi}^+(\vec{x},t), \hat{\Psi}(\vec{x}',t)]_- = 0 = [\hat{\pi}^+(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_- \qquad [\hat{\pi}(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_- = 0 = [\hat{\Psi}^+(\vec{x},t), \hat{\pi}(\vec{x}',t)]_-$$

$$[\hat{\Psi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- = i\hbar\,\delta(\vec{x}-\vec{x}') = [\hat{\Psi}^+(\vec{x},t), \hat{\pi}^+(\vec{x}',t)]_-$$

Hamilton operator

$$\hat{H} = \int d^3x \left\{\frac{2mc^2}{\hbar^2}\hat{\pi}^+(\vec{x},t)\hat{\pi}(\vec{x},t) + \frac{\hbar^2}{2m}\vec{\nabla}\hat{\Psi}^+(\vec{x},t)\vec{\nabla}\hat{\Psi}(\vec{x},t) + \frac{1}{2}mc^2\hat{\Psi}^+(\vec{x},t)\hat{\Psi}(\vec{x},t)\right\}$$

Remark: Operator ordering problem here not present due to commutation relations.

Heisenberg equations:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\vec{x},t) = [\hat{\Psi}(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\Psi}(\vec{x},t)}{\partial t} = \frac{2mc^2}{\hbar^2}\hat{\pi}^+(\vec{x},t) \qquad (1)$$

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}^+(\vec{x},t) = [\hat{\Psi}^+(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\Psi}^+(\vec{x},t)}{\partial t} = \frac{2mc^2}{\hbar^2}\hat{\pi}(\vec{x},t) \qquad (2)$$

$$i\hbar\frac{\partial}{\partial t}\hat{\pi}(\vec{x},t) = [\hat{\pi}(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\pi}(\vec{x},t)}{\partial t} = \frac{\hbar^2}{2m}\Delta\hat{\Psi}^+(\vec{x},t) - \frac{mc^2}{2}\hat{\Psi}^+(\vec{x},t) \qquad (3)$$

$$i\hbar\frac{\partial}{\partial t}\hat{\pi}^+(\vec{x},t) = [\hat{\pi}^+(\vec{x},t), \hat{H}]_- \Rightarrow \frac{\partial \hat{\pi}^+(\vec{x},t)}{\partial t} = \frac{\hbar^2}{2m}\Delta\hat{\Psi}(\vec{x},t) - \frac{mc^2}{2}\hat{\Psi}(\vec{x},t) \qquad (4)$$

$$\left.\begin{aligned}(1)+(4) &\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\hat{\Psi}(\vec{x},t) = 0\\ (2)+(3) &\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)\hat{\Psi}^+(\vec{x},t) = 0\end{aligned}\right\}$$ Klein-Gordon equation for field operators

7.4 Plane Waves:

Expansion of field operator $\hat{\Psi}(\vec{x},t)$ into plane waves

$$\hat\Phi(\vec x,t) = \int d^3p\; \underbrace{N_{\vec p}}_{\text{normalization} \Rightarrow \text{drawn later}}\; \underbrace{\hat a_{\vec p}(t)}_{\text{Fourier operators}}\; \underbrace{e^{\frac{i}{\hbar}\vec p\vec x}}_{\text{plane wave}}$$

Inserting into Klein-Gordon-equations: $\frac{\partial^2}{\partial t^2}\hat a_{\vec p}(t) + \frac{\vec p^2c^2+m^2c^4}{\hbar^2}\hat a_{\vec p}(t) = 0$

general solution: $\hat a_{\vec p}(t) = \hat a^{(1)}_{\vec p}\exp\{-\frac{i}{\hbar}E_{\vec p}t\} + \hat a^{(2)}_{\vec p}\exp\{+\frac{i}{\hbar}E_{\vec p}t\}$

relativistic energy-momentum dispersion: $E_{\vec p} = \sqrt{\vec p^2c^2+m^2c^4} = E_{-\vec p}$

$$\hat\Phi(\vec x,t) = \int d^3p\, N_{\vec p}\Big\{\hat a^{(1)}_{\vec p}\, e^{+\frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)} + \underbrace{\hat a^{(2)}_{\vec p}\, e^{\frac{i}{\hbar}(\vec p\vec x + E_{\vec p}t)}}_{\xrightarrow{\vec p\to-\vec p}\ \hat a^{(2)}_{-\vec p}\, e^{-\frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)}}\Big\}$$

$\hat a^{(2)}_{-\vec p} =: \hat a^{(2)}_{\vec p}$

assumption: $N_{\vec p} = N_{-\vec p}$

$$\Rightarrow \hat\Phi(\vec x,t) = \sum_{r=1}^{2}\int d^3p\;\hat a^{(r)}_{\vec p}\;\underbrace{u^{(r)}_{\vec p}(\vec x,t)}_{= N_{\vec p}\exp\{\varepsilon_r\frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)\}}$$

$$\varepsilon_r = \begin{cases} +1 & ;\ r=1 \\ -1 & ;\ r=2\end{cases}$$

Fix normalization $N_{\vec p}$ by:

$$\langle u^{(r)}_{\vec p}, u^{(r')}_{\vec p'}\rangle = \varepsilon_r\cdot\delta_{r,r'}\,\delta(\vec p-\vec p')$$

$$= \frac{i\hbar}{2mc^2}\int d^3x\Big\{u^{(r)*}_{\vec p}(\vec x,t)\frac{\partial u^{(r')}_{\vec p'}(\vec x,t)}{\partial t} - \frac{\partial u^{(r)*}_{\vec p}(\vec x,t)}{\partial t}u^{(r')}_{\vec p'}(\vec x,t)\Big\}$$

$$= \frac{i\hbar}{2mc^2}\frac{-i}{\hbar}\big(\varepsilon_r E_{\vec p} + \varepsilon_{r'}E_{\vec p'}\big)N_{\vec p}N_{\vec p'}\, e^{\frac{i}{\hbar}(\varepsilon_r E_{\vec p} - \varepsilon_{r'}E_{\vec p'})t}\cdot\underbrace{\int d^3x\, e^{\frac{i}{\hbar}(\varepsilon_{r'}\vec p' - \varepsilon_r\vec p)\vec x}}_{=(2\pi\hbar)^3\cdot\delta(\varepsilon_{r'}\vec p' - \varepsilon_r\vec p) = (2\pi\hbar)^3\,\delta(\vec p' - \varepsilon_r\vec p\,\varepsilon_{r'})} \quad (*)$$

$E_{\vec p'} = E_{\varepsilon_r\varepsilon_{r'}\vec p} = E_{\vec p}$; $N_{\vec p'} = N_{\varepsilon_r\varepsilon_{r'}\vec p} = N_{\vec p}$

$$= \frac{(2\pi\hbar)^3E_{\vec p}}{mc^2}\cdot\underbrace{\frac{\varepsilon_r+\varepsilon_{r'}}{2}}_{=\begin{cases}\varepsilon_r & ;\ r=r' \\ 0 & ;\ r\neq r'\end{cases} = \varepsilon_r\cdot\delta_{r,r'}} N^2_{\vec p}\; e^{\frac{i}{\hbar}(\varepsilon_r-\varepsilon_{r'})E_{\vec p}t}\;\delta(\vec p' - \varepsilon_r\varepsilon_{r'}\vec p)$$

$$= \underbrace{\frac{(2\pi\hbar)^3E_{\vec p}}{mc^2}N^2_{\vec p}}_{=1}\;\varepsilon_r\cdot\delta_{r,r'}\,\delta(\vec p'-\vec p) \overset{!}{=} \varepsilon_r\,\delta_{r,r'}\,\delta(\vec p-\vec p') \qquad (\varepsilon_r^2 = 1)$$

$$\Rightarrow N_{\vec p} = \sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec p}}} \overset{!}{=} N_{-\vec p} \quad \text{due to } E_{\vec p} = E_{-\vec p}$$

Remark: $u^{(1)*}_{\vec p}(\vec x,t) = u^{(2)}_{\vec p}(\vec x,t)$, $u^{(2)*}_{\vec p}(\vec x,t) = u^{(1)}_{\vec p}(\vec x,t)$

$$\langle u^{(r)*}_{\vec p}, u^{(r')*}_{\vec p'}\rangle = (-\varepsilon_r)\cdot\delta_{r,r'}\,\delta(\vec p-\vec p')$$

7.5 Fourier Operators:

$$\hat\Phi(\vec x,t) = \sum_{r=1}^{2}\int d^3p\,\hat a^{(r)}_{\vec p}\,u^{(r)}_{\vec p}(\vec x,t), \qquad \hat\Phi^+(\vec x,t) = \sum_{r=1}^{2}\int d^3p\,\hat a^{(r)+}_{\vec p}\,u^{(r)*}_{\vec p}(\vec x,t)$$

Invert these relation with the help of scalar product:

$$\langle u^{(r)}_{\vec p}, \hat\Phi\rangle = \sum_{r'=1}^{2}\int d^3p'\,\hat a^{(r')}_{\vec p'}\underbrace{\langle u^{(r)}_{\vec p}, u^{(r')}_{\vec p'}\rangle}_{=\varepsilon_r\,\delta_{r,r'}\,\delta(\vec p-\vec p')} = \varepsilon_r\,\hat a^{(r)}_{\vec p}$$

$$\hat a^{(r)}_{\vec p} = \varepsilon_r\langle u^{(r)}_{\vec p}, \hat\Phi\rangle \quad\text{and}\quad \hat a^{(r)+}_{\vec p} = -\varepsilon_r\langle u^{(r)*}_{\vec p}, \hat\Phi^+\rangle$$

$$\hat a^{(r)}_{\vec p} = \frac{i\hbar\varepsilon_r}{2mc^2}\int d^3x\Big\{u^{(r)*}_{\vec p}(\vec x,t)\underbrace{\frac{\partial\hat\Phi(\vec x,t)}{\partial t}}_{=\frac{2mc^2}{\hbar^2}\hat\pi^+(\vec x,t)} - \frac{\partial u^{(r)*}_{\vec p}(\vec x,t)}{\partial t}\hat\Phi(\vec x,t)\Big\}$$

$$\hat a^{(r)+}_{\vec p} = \frac{-i\hbar\varepsilon_r}{2mc^2}\int d^3x\Big\{u^{(r)}_{\vec p}(\vec x,t)\frac{2mc^2}{\hbar^2}\hat\pi(\vec x,t) - \frac{\partial u^{(r)}_{\vec p}(\vec x,t)}{\partial t}\hat\Phi^+(\vec x,t)\Big\}$$

$$\Rightarrow \big[\hat a^{(r)}_{\vec p}, \hat a^{(r')}_{\vec p'}\big]_- = 0 = \big[\hat a^{(r)+}_{\vec p}, \hat a^{(r')+}_{\vec p'}\big]_- = 0$$

$$[\hat a^{(r)}_{\vec p}, \hat a^{(r')+}_{\vec p'}]_- = \frac{i\hbar\epsilon_r}{2mc^2}\,\frac{-i\hbar\epsilon_{r'}}{2mc^2}\,\frac{2mc^2}{\hbar^2}\int d^3x\int d^3x'$$

$$\cdot\Big\{-u^{(r)}_{\vec p}(\vec x,t)\frac{\partial u^{(r')*}_{\vec p'}(\vec x',t)}{\partial t}\underbrace{[\hat\Phi(\vec x,t),\hat\pi(\vec x',t)]_-}_{=i\hbar\,\delta(\vec x-\vec x')} - u^{(r)*}_{\vec p}(\vec x,t)\frac{\partial u^{(r')}_{\vec p'}(\vec x',t)}{\partial t}\underbrace{[\hat\pi^+(\vec x,t),\hat\Phi^+(\vec x',t)]_-}_{=-i\hbar\,\delta(\vec x-\vec x')}\Big\}$$

$$= \epsilon_r\epsilon_{r'}\underbrace{\langle u^{(r)}_{\vec p}, u^{(r')}_{\vec p'}\rangle}_{=\epsilon_r\,\delta_{rr'}\,\delta(\vec p'-\vec p)} = \boxed{\epsilon_{r}}\cdot\delta_{r,r'}\,\delta(\vec p'-\vec p)\qquad(\epsilon_r^2=1)$$

minus sign for $r=2$ is a problem

7.6 Hamilton Operator:

$$\hat H=\int d^3x\Big\{\frac{2mc^2}{\hbar^2}\hat\pi^+(\vec x,t)\hat\pi(\vec x,t)+\frac{\hbar^2}{2m}\vec\nabla\hat\Phi^+(\vec x,t)\vec\nabla\hat\Phi(\vec x,t)+\frac{mc^2}{2}\hat\Phi^+(\vec x,t)\hat\Phi(\vec x,t)\Big\}$$

$$\hat\Phi(\vec x,t)=\sum_{r=1}^{2}\int d^3p\,\hat a^{(r)}_{\vec p}\,u^{(r)}_{\vec p}(\vec x,t),\quad u^{(r)}_{\vec p}(\vec x,t)=\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec p}}}\,e^{\frac{i}{\hbar}\epsilon_r(\vec p\vec x-E_{\vec p}t)}$$

$$\hat\Phi^+(\vec x,t)=\sum_{r=1}^{2}\int d^3p\,\hat a^{(r)+}_{\vec p}\,u^{(r)*}_{\vec p}(\vec x,t)$$

$$\vec\nabla\hat\Phi(\vec x,t)=\sum_{r=1}^{2}\int d^3p\,\frac{i}{\hbar}\epsilon_r\vec p\,\hat a^{(r)}_{\vec p}\,u^{(r)}_{\vec p}(\vec x,t)$$

$$\vec\nabla\hat\Phi^+(\vec x,t)=\sum_{r=1}^{2}\int d^3p\,\frac{-i}{\hbar}\epsilon_r\vec p\,\hat a^{(r)+}_{\vec p}\,u^{(r)*}_{\vec p}(\vec x,t)$$

$$\hat\pi(\vec x,t)=\frac{\hbar^2}{2mc^2}\frac{\partial\hat\Phi^+(\vec x,t)}{\partial t}=\frac{\hbar^2}{2mc^2}\int d^3p\,\frac{i}{\hbar}\epsilon_rE_{\vec p}\,\hat a^{(r)+}_{\vec p}\,u^{(r)*}_{\vec p}(\vec x,t)$$

$$\hat\pi^+(\vec x,t)=\quad''\quad=\frac{\hbar^2}{2mc^2}\int d^3p\,\frac{-i}{\hbar}\epsilon_rE_{\vec p}\,\hat a^{(r)}_{\vec p}\,u^{(r)}_{\vec p}(\vec x,t)$$

$$\hat H=\sum_{r=1}^{2}\sum_{r'=1}^{2}\int d^3p\int d^3p'\Big\{\frac{\epsilon_r\epsilon_{r'}E_{\vec p}E_{\vec p'}}{2mc^2}+\frac{\hbar^2}{2m}\frac{1}{\hbar^2}\epsilon_r\epsilon_{r'}\vec p\cdot\vec p'+\frac{mc^2}{2}\Big\}\hat a^{(r)+}_{\vec p}\hat a^{(r')}_{\vec p'}$$

$$\cdot\underbrace{\int d^3x\,u^{(r)*}_{\vec p}(\vec x,t)\,u^{(r')}_{\vec p'}(\vec x,t)}_{=\frac{mc^2}{\cancel{(2\pi\hbar)^3}}\frac{1}{E_{\vec p'}}e^{\frac{i}{\hbar}(\epsilon_rE_{\vec p}-\epsilon_{r'}E_{\vec p})t}\cancel{(2\pi\hbar)^3}\,\delta(\vec p'-\epsilon_r\epsilon_{r'}\vec p)}$$

$$\hat H=\sum_{r=1}^{2}\sum_{r'=1}^{2}\int d^3p\Big(\frac{\epsilon_r\epsilon_{r'}E_{\vec p}^2}{2}+\frac{\vec p^2}{2m}+\frac{mc^2}{2}\Big)\frac{mc^2}{E_{\vec p}}\underbrace{e^{\frac{i}{\hbar}(\epsilon_r-\epsilon_{r'})t}}_{=1}\hat a^{(r)+}_{\vec p}\hat a^{(r')}_{\vec p'}$$

$E_{\vec p}^2=\vec p^2c^2+m^2c^4$

$$=\underbrace{\frac{\epsilon_r\epsilon_{r'}+1}{2}}E_{\vec p}^2$$

$$=\begin{cases}1 & ;\ r=r'\\ 0 & ;\ r\neq r'\end{cases}=\delta_{r,r'}$$

$$\Rightarrow\quad\hat H=\sum_{r=1}^{2}\int d^3p\,E_{\vec p}\,\hat a^{(r)+}_{\vec p}\hat a^{(r)}_{\vec p}$$

7.7 Charge Operator:

$$Q=\int d^3x\,\langle\Phi,\Phi\rangle=\int d^3x\,\frac{i\hbar}{2mc^2}\Big\{\Phi^*(\vec x,t)\frac{\partial\Phi(\vec x,t)}{\partial t}-\frac{\partial\Phi^*(\vec x,t)}{\partial t}\Phi(\vec x,t)\Big\}$$

$$=\frac{i}{\hbar}\int d^3x\,\{\Phi^*(\vec x,t)\,\pi^*(\vec x,t)-\pi(\vec x,t)\,\Phi(\vec x,t)\}$$

↓ second quantization

$$\hat Q=\frac{i}{\hbar}\int d^3x\,\{\hat\Phi^+(\vec x,t)\,\hat\pi^+(\vec x,t)-\hat\pi(\vec x,t)\,\hat\Phi(\vec x,t)\}$$

Remark: Operator ordering does play a role here as $\hat\Phi,\hat\pi$; $\hat\Phi^+,\hat\pi^+$ do not commute

Q conserved at first quantized level ⟶ $\hat Q$ conserved at second quantized level

⇒ $[\hat Q,\hat H]_-=0$ is, indeed, fulfilled

⟶ insert therein plane wave decomposition

⟶ charge operator gets decomposed into Fourier operators

$$\hat Q=\sum_{r=1}^{2}\int d^3p\,\epsilon_r\,\hat a^{(r)+}_{\vec p}\hat a^{(r)}_{\vec p}\qquad\begin{cases}r=1: & \text{charge content: } +1\\ r=2: & \text{charge content: } -1\end{cases}$$

7.8 Redefinition of Fourier operators:

$[\hat{a}^{(1)}_{\vec{p}}, \hat{a}^{(1)+}_{\vec{p}'}]_- = \delta(\vec{p}-\vec{p}')$ $\Rightarrow$ $\left\{\begin{array}{l} \hat{a}^{(1)}_{\vec{p}}: \text{annihilation} \\ \hat{a}^{(1)+}_{\vec{p}}: \text{creation} \end{array}\right\}$ operator for particle sort $1 = \nu$

$\underbrace{[\hat{a}^{(2)}_{\vec{p}}, \hat{a}^{(2)+}_{\vec{p}'}]_- = - \delta(\vec{p}-\vec{p}')}$ $\Rightarrow$ $\left\{\begin{array}{l} \hat{a}^{(2)}_{\vec{p}}: \text{creation} \\ \hat{a}^{(2)+}_{\vec{p}}: \text{annihilation} \end{array}\right\}$ operator for particle sort $\nu = 2$

$[\hat{a}^{(2)+}_{\vec{p}}, \hat{a}^{(2)}_{\vec{p}'}]_- = + \delta(\vec{p}-\vec{p}')$

$\Rightarrow$ This suggests redefinition

particle sort a: $\hat{a}_{\vec{p}} = \hat{a}^{(1)}_{\vec{p}}$, $\hat{a}^+_{\vec{p}} = \hat{a}^{(1)+}_{\vec{p}}$

particle sort b: $\hat{b}_{\vec{p}} = \hat{a}^{(2)+}_{\vec{p}}$, $\hat{b}^+_{\vec{p}} = \hat{a}^{(2)}_{\vec{p}}$

$[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{p}'}]_- = [\hat{b}_{\vec{p}}, \hat{b}^+_{\vec{p}'}]_- = i\hbar\, \delta(\vec{p}-\vec{p}')$; all other commutators vanish

$$\hat{\Psi}(\vec{x},t) = \sum_{\nu=1}^{2} \int d^3p\, \hat{a}^{(\nu)}_{\vec{p}} u^{(\nu)}_{\vec{p}}(\vec{x},t) = \int d^3p \{ \hat{a}_{\vec{p}} \underbrace{u_{\vec{p}}(\vec{x},t)}_{\sim e^{-\frac{i}{\hbar}E_{\vec{p}}t}} + \hat{b}^+_{\vec{p}} \underbrace{u^*_{\vec{p}}(\vec{x},t)}_{\sim e^{\frac{i}{\hbar}E_{\vec{p}}t} = e^{-\frac{i}{\hbar}(E_{\vec{p}})t},\ = E_{\vec{p}}(-t)} \}$$

$$= u^{(1)}_{\vec{p}}(\vec{x},t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec{p}}}}\, e^{\frac{i}{\hbar}(\vec{p}\vec{x} - E_{\vec{p}}t)}$$

$$\hat{\Psi}^+(\vec{x},t) = \int d^3p \{ \hat{a}^+_{\vec{p}} u^*_{\vec{p}}(\vec{x},t) + \hat{b}_{\vec{p}} u_{\vec{p}}(\vec{x},t) \}$$

Hamilton operator:

$$\hat{H} = \int d^3p\, E_{\vec{p}} (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \underbrace{\hat{b}_{\vec{p}} \hat{b}^+_{\vec{p}}}_{= \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}} + \delta(\vec{0})}) = \int d^3p\, E_{\vec{p}} (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}}) + \underbrace{\delta(\vec{0}) \int d^3p\, E_{\vec{p}}}_{\text{twice infinite}}$$

$$\hat{Q} = \int d^3p\, (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}_{\vec{p}} \hat{b}^+_{\vec{p}}) = \int d^3p\, (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}}) - \delta(\vec{0}) \int d^3p\, 1$$

vacuum state $\hat{a}_{\vec{p}} |0\rangle = 0$, $\hat{b}_{\vec{p}} |0\rangle = 0$

$\Rightarrow$ $\langle 0 | \hat{H} | 0 \rangle = \delta(\vec{0}) \int d^3p\, E_{\vec{p}}$ $\quad$ $\langle 0 | \hat{Q} | 0 \rangle = -\delta(\vec{0}) \int d^3p\, 1$

$\Rightarrow$ normal ordering : "get rid of infinite large vacuum contribution"

$$:\hat{H}: = \hat{H} - \langle 0 | \hat{H} | 0 \rangle = \int d^3p\, E_{\vec{p}} (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} + \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}})$$

$$:\hat{Q}: = \hat{Q} - \langle 0 | \hat{Q} | 0 \rangle = \int d^3p\, (\hat{a}^+_{\vec{p}} \hat{a}_{\vec{p}} - \hat{b}^+_{\vec{p}} \hat{b}_{\vec{p}})$$

Physical interpretation:

particle sort a: energy $E_{\vec{p}}$, charge $+1$ $\hat{=}$ π^+ particle

" " b: energy $E_{\vec{p}}$, charge -1 $\hat{=}$ π^- antiparticle