$$R_y(\Theta) = e^{-iL_y\Theta} = \begin{pmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R(\Theta,\phi) = \begin{pmatrix} \cos\Theta\cos\phi & -\sin\phi & \sin\Theta\cos\phi \\ \cos\Theta\sin\phi & \cos\phi & \sin\Theta\sin\phi \\ -\sin\Theta & 0 & \cos\Theta \end{pmatrix}$$

$$\vec{k} = R(\Theta,\phi)\underbrace{k\begin{pmatrix}0\\0\\1\end{pmatrix}}_{k\vec{e}_z} = k\begin{pmatrix} \sin\Theta\cos\phi \\ \sin\Theta\sin\phi \\ \cos\Theta \end{pmatrix} \quad ✓$$

$$\vec{\varepsilon}(\vec{k},\lambda) = R(\Theta,\phi)\,\vec{\varepsilon}(k\vec{e}_z,\lambda) = \ldots = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\Theta\cos\phi - \lambda i\sin\phi \\ \cos\Theta\sin\phi + \lambda i\cos\phi \\ -\sin\Theta \end{pmatrix}$$

$$\hat{h}(\vec{k})\,\vec{\varepsilon}(\vec{k},\lambda) = \ldots = \lambda\,\vec{\varepsilon}(\vec{k},\lambda) \quad ✓$$

$$\vec{\varepsilon}(\vec{k},\lambda)\Big|_{\substack{\Theta=0\\ \phi=0}} = \vec{\varepsilon}(k\vec{e}_z,\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \lambda i\\ 0\end{pmatrix} \quad ✓$$

8.11 Properties of Polarization Vectors:

- Solution of wave equation:

$$\vec{A}(\vec{x},t) = \int d^3k\,\left\{ \vec{A}(\vec{k})\,e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} + \vec{A}^+(\vec{k})\,e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} \right\}$$

but: $\operatorname{div}\vec{A}(\vec{x},t) = 0$ Coulomb gauge $\Leftarrow$ $\vec{k}\cdot\hat{\vec{A}}(\vec{k}) = 0$ transversality condition

- Fourier operator vector has two transversal degrees of freedom:

$$\hat{\vec{A}}(\vec{k}) = \underbrace{N_{\vec{k}}}_{\text{normalization constants}} \sum_{\lambda=\pm1} \underbrace{\vec{\varepsilon}(\vec{k},\lambda)}\,\hat{a}_{\vec{k},\lambda}$$

transversality fulfilled provided $\vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = 0$

check:
$$k\begin{pmatrix} \cos\Theta\cos\phi \\ \sin\Theta\sin\phi \\ \cos\Theta \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\Theta\cos\phi - i\lambda\sin\phi \\ \cos\Theta\sin\phi + \lambda i\cos\phi \\ -\sin\Theta \end{pmatrix} = 0$$

$$\vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = 0$$

- Orthonormalization relations:

$$\left.\begin{aligned} \vec{\varepsilon}(\vec{k},\lambda)\,\vec{\varepsilon}(\vec{k},\lambda)^* &= \ldots = 1 \\ \vec{\varepsilon}(\vec{k},\lambda)\,\vec{\varepsilon}(\vec{k},-\lambda)^* &= \ldots = 0 \end{aligned}\right\} \quad \vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda') = \delta_{\lambda,\lambda'}$$

- Behaviour under inversion $\vec{k} \to -\vec{k}$:

$\phi \to \phi + \pi$: $\quad \sin\phi \to -\sin\phi, \quad \cos\phi \to -\cos\phi$

$\Theta \to \Theta - \pi$: $\quad \sin\Theta \to \sin\Theta, \quad \cos\Theta \to -\cos\Theta$

$$\Rightarrow \vec{\varepsilon}(-\vec{k},\lambda) = \ldots = \vec{\varepsilon}(\vec{k},-\lambda) = \vec{\varepsilon}(\vec{k},\lambda)^*$$

Result: first formula on problem sheet 7

$$\vec{A}(\vec{x},t) = \underbrace{\sum_{\lambda=\pm1}}_{} \underbrace{\int d^3k}_{} \underbrace{N_{\vec{k}}}_{\text{determine such that}} \left\{ \vec{\varepsilon}(\vec{k},\lambda)\,e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\,\hat{a}_{\vec{k},\lambda} + \vec{\varepsilon}(\vec{k},\lambda)^*\,e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\,\hat{a}^+_{\vec{k},\lambda} \right\}$$

$\hat{a}_{\vec{k},\lambda}$: annihilation operator of a photon of wave vector k and helicity λ

$\hat{a}^+_{\vec{k},\lambda}$: creation operator

two polarization degrees of freedom due to Coulomb gauge: $\operatorname{div}\vec{A} = 0$

properties of polarization vectors: $\vec{\varepsilon}(\vec{k},\lambda)$

transversality: $\quad \vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) = 0$

orthonormality: $\quad \vec{\varepsilon}(\vec{k},\lambda)\cdot\vec{\varepsilon}(\vec{k},\lambda')^* = \delta_{\lambda\lambda'}$

behaviour under inversion: $\quad \vec{\varepsilon}(-\vec{k},\lambda) = \vec{\varepsilon}(\vec{k},-\lambda) = \vec{\varepsilon}(\vec{k},\lambda)^*$

8.12 Fourier operators:

$$\hat{\vec\pi}(\vec x,t) = \varepsilon_0 \frac{\partial \hat{\vec A}(\vec x,t)}{\partial t} \qquad (\hat= \ \vec p = m\dot{\vec x})$$

$$= \sum_{\lambda=\pm1} \varepsilon_0 \int d^3k\, N_{\vec k}\left\{-i\omega_{\vec k}\,\vec\varepsilon(\vec k,\lambda)\, e^{i(\vec k\vec x-\omega_{\vec k}t)}\hat a_{\vec k,\lambda} + i\omega_{\vec k}\,\vec\varepsilon(\vec k,\lambda)^* e^{-i(\vec k\vec x-\omega_{\vec k}t)}\hat a^+_{\vec k,\lambda}\right\}$$

Fourier operators $\hat a_{\vec k,\lambda}, \hat a^+_{\vec k,\lambda}$ are expressed in terms of $\vec A(\vec x,t)$, $\hat{\vec\pi}(\vec x,t)$:

$$\hat a_{\vec k,\lambda} = \frac{1}{2(2\pi)^3 N_{\vec k}} \int d^3x\, \vec\varepsilon(\vec k,\lambda)^* e^{-i(\vec k\vec x-\omega_{\vec k}t)}\left\{\hat{\vec A}(\vec x,t) + i\frac{\hat{\vec\pi}(\vec x,t)}{\varepsilon_0\omega_{\vec k}}\right\}$$

$$\hat a^+_{\vec k,\lambda} = \dots$$

Commutation relations:

$$[\hat A_k(\vec x,t), \hat A_l(\vec x',t)]_- = 0$$

$$[\hat\pi_k(\vec x,t), \hat\pi_l(\vec x',t)]_- = 0$$

$$[\hat A_k(\vec x,t),\hat\pi_l(\vec x',t)]_- = i\hbar\,\underbrace{\delta^T_{kl}(\vec x-\vec x')}$$

$$= \delta_{kl}\delta(\vec x-\vec x') + \frac{1}{4\pi}\partial_k\partial_l\frac{1}{|\vec x-\vec x'|}$$

$\Longleftrightarrow$

$$[\hat a_{\vec k,\lambda},\hat a_{\vec k',\lambda'}]_- = 0$$

$$[\hat a^+_{\vec k,\lambda},\hat a^+_{\vec k',\lambda'}]_- = 0$$

$$[\hat a_{\vec k,\lambda}, \hat a^+_{\vec k',\lambda'}]_- = \delta_{\lambda\lambda'}\,\delta(\vec k-\vec k')\cdot \underbrace{\frac{\hbar}{2(2\pi)^3\varepsilon_0\,\omega_{\vec k}N_{\vec k}}}_{\overset{!}{=}1}$$

normalization constant: $N_{\vec k} = \sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\,\omega_{\vec k}}}$, $\omega_{\vec k}=c|\vec k|$

$\hat a_{\vec k,\lambda}/\hat a^+_{\vec k,\lambda}$: annihilation/creation operator of a photon with wave vector $\vec k$ and polarisation λ

Question: properties of a single photon? energy? momentum? spin angular momentum?

8.13 Energy:

$$H = \int d^3x\left\{\frac{\varepsilon_0}{2}\vec E(\vec x,t)^2 + \frac{1}{2\mu_0}\vec B(\vec x,t)^2\right\}$$

$$= \frac12\int d^3x\left\{\frac{1}{\varepsilon_0}\pi_k(\vec x,t)\pi_k(\vec x,t) + \frac{1}{\mu_0}\partial_k A_l(\vec x,t)\partial_k A_l(\vec x,t)\right\}$$

$$\hat H = \frac12\int d^3x\left\{\frac{1}{\varepsilon_0}\hat\pi_k(\vec x,t)\hat\pi_k(\vec x,t) + \frac{1}{\mu_0}\partial_k\hat A_l(\vec x,t)\partial_k\hat A_l(\vec x,t)\right\}$$

Inserting plane wave decompositions:

$$\hat H = \frac12\sum_{\lambda=\pm1}\int d^3k\,\hbar\omega_{\vec k}\left(\hat a^+_{\vec k,\lambda}\hat a_{\vec k,\lambda} + \hat a_{\vec k,\lambda}\hat a^+_{\vec k,\lambda}\right)$$

vacuum state: $\hat a_{\vec k,\lambda}|0\rangle = 0$ $\quad \langle 0|\hat a^+_{\vec k,\lambda} = 0$

$$\langle 0|\hat H|0\rangle = \frac12\, 2\int d^3k\,\omega_{\vec k}\hbar = $$ divergent due to linear dispersion $\omega_{\vec k} = c|\vec k|$

$$:\hat H: = \hat H - \langle 0|\hat H|0\rangle = \sum_{\lambda=\pm1}\int d^3k\,\hbar\omega_{\vec k}\,\underbrace{\hat a^+_{\vec k,\lambda}\hat a_{\vec k,\lambda}}_{\text{occupation number operator}}$$

Renormalized Hamiltonian: normal ordered

8.14 Momentum:

$\vec P = \int d^3x\,\frac{1}{c^2}\vec S(\vec x,t)$, Poynting vector $\vec S(\vec x,t) = \frac{1}{\mu_0}\vec E(\vec x,t)\times\vec B(\vec x,t)$

$$= \int d^3x\,[\vec\nabla\times\vec A(\vec x,t)]\times\vec\pi(\vec x,t)$$

$$\hat{\vec P} = \int d^3x\,[\vec\nabla\times\hat{\vec A}(\vec x,t)]\times\hat{\vec\pi}(\vec x,t)$$

Inserting plane wave decompositions

$$\hat{\vec P} = \sum_{\lambda=\pm1}\int d^3k\,\frac{\hbar\vec k}{2}\left(\hat a^+_{\vec k,\lambda}\hat a_{\vec k,\lambda} + \hat a_{\vec k,\lambda}\hat a^+_{\vec k,\lambda}\right)$$

$$\langle 0|\hat{\vec P}|0\rangle = \frac12\, 2\int d^3k\,\hbar\underbrace{\vec k}_{\text{odd}} = \vec 0$$

$$\hat{\vec P} = :\hat{\vec P}: = \hat{\vec P} - \langle 0|\hat{\vec P}|0\rangle = \sum_{\lambda=\pm1}\int d^3k\,\hbar\vec k\,\hat a^+_{\vec k,\lambda}\hat a_{\vec k,\lambda}$$

8.15 Spin Angular Momentum:

Result from Noether Theorem:

$$\vec S = \int d^3x\,\varepsilon_0\vec E(\vec x,t)\times\vec A(\vec x,t) = \int d^3x\,\vec A(\vec x,t)\times\vec\pi(\vec x,t) \quad (\hat= \ \vec x\times\vec p)$$

$\hat{\vec{S}} = \int d^3x\, \hat{\vec{A}}(\vec{x},t)\times\hat{\vec{\pi}}(\vec{x},t)$

Inserting plane wave decomposition and

$$\vec\varepsilon(\vec k,\lambda)\times\vec\varepsilon(\vec k,\lambda')^* = -i\lambda\frac{\vec k}{k}\,\delta_{\lambda\lambda'}$$

$$\hat{\vec S} = \sum_{\lambda=\pm1}\int d^3k\,\lambda\frac{\hbar}{2}\frac{\vec k}{k}\left(\hat a^\dagger_{\vec k\lambda}\hat a_{\vec k\lambda}+\hat a_{\vec k\lambda}\hat a^\dagger_{\vec k\lambda}\right)$$

$$\langle 0|\hat{\vec S}|0\rangle = \hbar\cdot\frac{1}{2}\underbrace{\left(\sum_{\lambda=\pm1}\lambda\right)}_{odd}\underbrace{\left(\int d^3k\,\frac{\vec k}{k}\right)}_{odd} = 0\cdot 0 = 0$$

$$\hat{\vec S} = :\hat{\vec S}: = \hat{\vec S} - \langle 0|\hat{\vec S}|0\rangle = \sum_{\lambda=\pm1}\int d^3k\,\hbar\lambda\frac{\vec k}{|\vec k|}\,\hat a^\dagger_{\vec k\lambda}\hat a_{\vec k\lambda}$$

Result: A photon has the energy $\hbar\omega_{\vec k}$, the momentum $\hbar\vec k$, and the spin angular momentum $\lambda\hbar\frac{\vec k}{k}$, i.e. $\lambda\hbar$ helicity.

8.16 Definition of Maxwell Propagator:

$$D^{\mu\nu}(\vec x,t;\vec x',t') = \langle 0|\hat T\left(\hat A^\mu(\vec x,t)\,\hat A^\nu(\vec x',t')\right)|0\rangle$$

$$=\Theta(t-t')\cdot\hat A^\mu(\vec x,t)\,\hat A^\nu(\vec x',t') + \Theta(t'-t)\,\hat A^\nu(\vec x',t')\,\hat A^\mu(\vec x,t)$$

radiation gauge: $\hat A^0(\vec x,t) = 0$

$\Rightarrow D^{\mu\nu}(\vec x,t;\vec x',t') = 0$ if either $\mu=0$ or $\nu=0$

only spatial components non-vanishing

Equation of motion?

$$\frac{\partial D^{jk}(\vec x,t;\vec x',t')}{\partial t} = \delta(t-t')\langle 0|\underbrace{\left[\hat A^j(\vec x,t),\hat A^k(\vec x',t)\right]_-}_{=0\ \text{due to independence}}|0\rangle$$

$$+\Theta(t-t')\langle 0|\frac{\partial\hat A^j(\vec x,t)}{\partial t}\hat A^k(\vec x',t')|0\rangle + \Theta(t'-t)\langle 0|\hat A^k(\vec x',t')\frac{\partial\hat A^j(\vec x,t)}{\partial t}|0\rangle$$

$$\frac{\partial^2 D^{jk}(\vec x,t;\vec x',t')}{\partial t^2} = \delta(t-t')\langle 0|\underbrace{\Big[\underbrace{\frac{\partial\hat A^j(\vec x,t)}{\partial t}}_{=\frac{1}{\varepsilon_0}\hat\pi^j(\vec x,t)},\ \hat A^k(\vec x',t)\Big]_-}_{=-\frac{1}{\varepsilon_0}i\hbar\,\delta^T_{jk}(\vec x-\vec x')}|0\rangle$$

$$\underbrace{+\Theta(t-t')\langle 0|\underbrace{\frac{\partial^2\hat A^j(\vec x,t)}{\partial t^2}}_{=c^2\Delta\hat A^j(\vec x,t)}\hat A^k(\vec x',t')|0\rangle + \Theta(t'-t)\langle 0|\hat A_k(\vec x',t')\underbrace{\frac{\partial^2\hat A_j(\vec x,t)}{\partial t^2}}_{=c^2\Delta\hat A^j(\vec x,t)}|0\rangle}_{=c^2\Delta D^{jk}(\vec x,t;\vec x',t')}$$

$\rightarrow$ transversal Maxwell propagator

$$\Rightarrow\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)D^{jk}(\vec x,t;\vec x',t') = -i\hbar\mu_0\,\delta(t-t')\cdot\delta^T_{jk}(\vec x-\vec x')$$

Maxwell propagator = Green function of wave equation

8.17 Calculation of Maxwell Propagator:

Insert plane wave decomposition:

$$D^{jk}(\vec x,t;\vec x',t') = \Theta(t-t')\langle 0|\Big\{\sum_{\lambda=\pm1}\int d^3k\sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec k}}}\Big[\varepsilon_j(\vec k,\lambda)e^{i(\vec k\vec x-\omega_{\vec k}t)}\hat a_{\vec k\lambda}$$

$$+\cancel{\varepsilon_j^*(\vec k,\lambda)e^{-i(\vec k\vec x-\omega_{\vec k}t)}\hat a^\dagger_{\vec k\lambda}}\Big]\Big\}\cdot\Big\{\sum_{\lambda'=\pm1}\int d^3k'\sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec k'}}}\Big[\cancel{\varepsilon_k(\vec k',\lambda')e^{i(\vec k'\vec x'-\omega_{\vec k'}t')}\hat a_{\vec k'\lambda'}}$$

$$+\varepsilon_k^*(\vec k',\lambda')e^{-i(\vec k'\vec x'-\omega_{\vec k'}t')}\hat a^\dagger_{\vec k'\lambda'}\Big]\Big\}|0\rangle + t\leftrightarrow t',\ \vec x\leftrightarrow\vec x',\ j\leftrightarrow k$$

Note: $\langle 0|\hat a_{\vec k\lambda}\hat a^\dagger_{\vec k'\lambda'}|0\rangle = \delta_{\lambda\lambda'}\,\delta(\vec k-\vec k') + \cancel{\langle 0|\hat a^\dagger_{\vec k'\lambda'}\hat a_{\vec k\lambda}|0\rangle}$

$$D^{jk}(\vec x,t;\vec x',t') = \sum_{\lambda=\pm1}\int d^3k\,\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_{\vec k}}\Big\{\Theta(t-t')\,\varepsilon_j(\vec k,\lambda)\,\varepsilon_k^*(\vec k,\lambda)\,e^{i[\vec k(\vec x-\vec x')-\omega_{\vec k}(t-t')]}$$

$$+\Theta(t'-t)\,\varepsilon_k^*(\vec k,\lambda)\,\varepsilon_j(\vec k,\lambda)\,e^{-i[\vec k(\vec x-\vec x')-\omega_{\vec k}(t-t')]}\Big\}$$

second term: $\lambda\to-\lambda$, $\vec\varepsilon(\vec k,-\lambda) = \vec\varepsilon(\vec k,\lambda)^*$

$$D^{jk}(\vec x,t;\vec x',t') = \int d^3k\,\frac{\hbar P^{jk}(\vec k)}{2(2\pi)^3\varepsilon_0\omega_{\vec k}}\Big\{\Theta(t-t')\,e^{i[\vec k(\vec x-\vec x')-\omega_{\vec k}(t-t')]} + \Theta(t'-t)\,e^{+i[\vec k(\vec x-\vec x')+\omega_{\vec k}(t-t')]}\Big\}$$

polarization sum: $P^{jk}(\vec{k}) = \sum_{\lambda=\pm 1} \varepsilon_j(\vec{k},\lambda)\, \varepsilon_k^*(\vec{k},\lambda)$

$$P^{jk}(-\vec{k}) = \sum_{\lambda=\pm1} \varepsilon_j(-\vec{k},\lambda)\varepsilon_k^*(-\vec{k},\lambda) = \sum_{\lambda=\pm1} \varepsilon_j(\vec{k},-\lambda)\,\varepsilon_k^*(\vec{k},-\lambda)$$

$$\underline{\lambda'=-\lambda} \sum_{\lambda'=\pm1} \varepsilon_j(\vec{k},\lambda')\,\varepsilon_k^*(\vec{k},\lambda') = P^{jk}(\vec{k}) \qquad = e^{-i\omega_k(t-t')}$$

second term: $\vec{k} \to -\vec{k}$

$$D^{jk}(\vec{x},t;\vec{x}',t') = \int d^3k \,\frac{\hbar\, P^{jk}(\vec{k})}{2(2\pi)^3 \varepsilon_0\, \omega_k}\, e^{i\vec{k}(\vec{x}-\vec{x}')}\left\{ \Theta(t-t')\, e^{-i\omega_k(t-t')} + \Theta(t'-t)\, e^{i\omega_k(t-t')} \right\}$$

explicit evaluation polarization sum:

$$\vec{k} = k\begin{pmatrix} \sin\Theta\cos\phi \\ \sin\Theta \sin\phi \\ \cos\Theta\end{pmatrix}, \quad \vec{\varepsilon}(\vec{k},\lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\Theta\cos\phi - i\lambda \sin\phi \\ \cos\Theta\sin\phi + \lambda i \cos\phi \\ -\sin\Theta\end{pmatrix}$$

$$\left(P^{jk}(\vec{k})\right) = \frac{1}{2}\left(\cos\Theta\cos\phi - i\sin\phi,\ \cos\Theta\sin\phi + i\cos\phi,\ -\sin\Theta\right)\begin{pmatrix} \cos\Theta\cos\phi + i\sin\phi \\ \cos\Theta\sin\phi - i\cos\phi \\ -\sin\Theta\end{pmatrix} + \text{c.c.}$$

$$= \cdots = \begin{pmatrix} 1-\frac{k_x^2}{k^2} & -\frac{k_x k_y}{k^2} & -\frac{k_x k_z}{k^2} \\ -\frac{k_x k_y}{k^2} & 1-\frac{k_y^2}{k^2} & -\frac{k_y k_z}{k^2} \\ -\frac{k_x k_z}{k^2} & -\frac{k_y k_z}{k^2} & 1-\frac{k_z^2}{k^2}\end{pmatrix} = \left(\delta_{jk} - \frac{k_j k_k}{k^2}\right)$$

transversality condition: $k_j P^{jk}(\vec{k}) \equiv 0 \implies \partial_j D^{jk}(\vec{x},t;\vec{x}',t') \equiv 0$

8.18 Four-dimensional Fourier representation:

integral identity (see Klein-Gordon chapter):

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{\omega^2-\omega_k^2+i\varepsilon} = \frac{-i}{2\omega_k}\left\{\Theta(t-t')\, e^{-i\omega_k(t-t')} + (t\to t')\right\}$$

← Feynman prescription

$$D^{jk}(\vec{x},t;\vec{x}',t') = \lim_{\varepsilon\downarrow 0}\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\,\frac{i\hbar}{\varepsilon_0}\, P^{jk}(\vec{k})\,\frac{e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega(t-t')]}}{\omega^2-\omega_k^2+i\varepsilon}$$

solves equation of motion:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)D^{jk}(\vec{x},t;\vec{x}',t') = \lim_{\varepsilon\downarrow 0}\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\,\frac{i\hbar}{\varepsilon_0}\, P_{jk}(\vec{k})\,\frac{-\frac{\omega^2}{c^2}+k^2}{\omega^2-\omega_k^2+i\varepsilon}\, e^{i[\vec{k}(\vec{x}-\vec{x}')-\omega(t-t')]}$$

$\xrightarrow{\varepsilon\to 0} -\frac{1}{c^2}$

$$= -\frac{i\hbar}{\varepsilon_0}\cdot\varepsilon_0\mu_0\,\delta(t-t')\underbrace{\int\frac{d^3k}{(2\pi)^3}\left(\delta_{jk}-\frac{k_j k_k}{k^2}\right)e^{i\vec{k}(\vec{x}-\vec{x}')}}_{=\delta^T_{jk}(\vec{x}-\vec{x}')} = -i\hbar\mu_0\,\delta(t-t')\,\delta^T_{jk}(\vec{x}-\vec{x}')$$

Remark: Transversal Maxwell propagator is NOT Lorentz invariant due to chosen Coulomb gauge

Aim: decompose into Lorentz invariant and Lorentz not-invariant contribution

contravariant four-wave vector: $(k^\lambda) = \begin{pmatrix}\omega/c \\ \vec{k}\end{pmatrix}$, $d^4k = \frac{1}{c}\, d^3k\, d\omega$

$$\omega^2-\omega_k^2 = \omega^2-c^2|\vec{k}|^2 = c^2\left\{\left(\frac{\omega}{c}\right)^2-\vec{k}^2\right\} = c^2 k^\lambda k_\lambda$$

$$\vec{k}(\vec{x}-\vec{x}')-\omega(t-t') = -k_\lambda(x^\lambda-x'^\lambda)\quad;\quad x^0=ct,\ k^0=\frac{\omega}{c}$$

$$D^{\mu\nu}(x^\lambda;x'^\lambda) = \lim_{\varepsilon\downarrow 0}\frac{i\hbar}{c\,\varepsilon_0}\int\frac{d^4k}{(2\pi)^4}\,\frac{P^{\mu\nu}(k^\lambda)}{k^\lambda k_\lambda+i\varepsilon}\, e^{-ik_\lambda(x^\lambda-x'^\lambda)}$$

$$\left(P^{\mu\nu}(k^\lambda)\right) = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & P^{jk}(\vec{k}) & \\ 0 & & & \end{array}\right) = \underbrace{\left(-g^{\mu\nu}\right)}_{\text{covariant}} + \underbrace{\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & -\frac{k_j k_k}{k^2} & \\ 0 & & & \end{array}\right)}_{\text{non-covariant}}$$

projects onto 2D subspace perpendicular to $\begin{pmatrix}0\\ \vec{k}\end{pmatrix}$

Aim: investigate non-covariant part in more detail

time-like vector: $(\zeta^\lambda) = \begin{pmatrix}1\\ \vec{0}\end{pmatrix}$

space-like vector: $(\bar{k}^\lambda) = \begin{pmatrix}0\\ \vec{k}/|\vec{k}|\end{pmatrix}$

$\zeta^\lambda \bar{k}_\lambda = 0$

1) $k^\lambda\zeta_\lambda = k^0 = \frac{\omega}{c}$, $k^2 = \left(\frac{\omega}{c}\right)^2 - \vec{k}^2 \implies \sqrt{(k\zeta)^2-k^2} = \sqrt{|\vec{k}|^2} = |\vec{k}|$

2) $\left(k\lambda - (k\xi)\xi\lambda\right) = \begin{pmatrix} \frac{\omega}{c} \\ \vec{k} \end{pmatrix} - \frac{\omega}{c}\begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{k} \end{pmatrix}$

$$\Rightarrow \bar{k}\lambda \overset{(1|2)}{=} \frac{k\lambda - (k\xi)\xi\lambda}{\sqrt{(k\xi)^2 - k^2}}$$

decomposition of polarization sum:

$$P^{\mu\nu} = -g^{\mu\nu} + \underbrace{\xi^\mu\xi^\nu - \bar{k}^\mu\bar{k}^\nu}$$

$$= \xi^\mu\xi^\nu - \frac{[k^\mu - (k\xi)\xi^\mu]\cdot[k^\nu - (k\xi)\xi^\nu]}{(k\xi)^2 - k^2}$$

$$\underbrace{=}_{1} - \underbrace{k^2\frac{\xi^\mu\xi^\nu}{(k\xi)^2 - k^2}}_{2} - \underbrace{\frac{k^\mu k^\nu - (k\xi)(k^\mu\xi^\nu - \xi^\mu k^\nu)}{(k\xi)^2 - k^2}}_{3}$$

$$D^{\mu\nu}(x,x') = D^{\mu\nu}_F(x,x') - D^{\mu\nu}_C(x,x') - D^{\mu\nu}_R(x,x')$$

Feynman 1 Coulomb 2 3 residual

$$= \lim_{\varepsilon\downarrow 0}\frac{\hbar}{c\varepsilon_0}\int\frac{d^4k}{(2\pi)^4}\frac{-ig^{\mu\nu}}{k^2+i\varepsilon}e^{-ik(x-x')}$$

covariant, $\hat{=}$ Gupta-Bleuler quantization

$$= \lim_{\varepsilon\downarrow 0}\frac{i\hbar}{c\varepsilon_0}\int\frac{d^4k}{(2\pi)^4}\frac{k^2\xi^\mu\xi^\nu}{(k\xi)^2 - k^2}\frac{e^{-ik(x-x')}}{k^2+i\varepsilon}$$

$$= \frac{i\hbar\mu_0}{4\pi}\underbrace{\delta^{\mu 0}\delta^{\nu 0}}_{\text{couples to charge density}}\frac{\delta(t-t')}{|\vec{x}-\vec{x}'|}$$ ← instantaneous ← coulomb potential

$\Rightarrow$ This term is cancelling instantaneous coulomb interactions in perturbation theory

$$= \lim_{\varepsilon\downarrow 0}\frac{i\hbar}{c\varepsilon_0}\lim_{\varepsilon\downarrow 0}\int\frac{d^4k}{(2\pi)^4}\underbrace{\frac{k^\mu k^\nu - (k\xi)(k^\mu\xi^\nu - \xi^\mu k^\nu)}{(k\xi)^2 - k^2}}_{\sim \text{ either } k^\mu \text{ or } k^\nu}\cdot\frac{e^{ik(x-x')}}{k^2+i\varepsilon}$$

perturbative calculations

$$\int d^4x\int d^4x'\, j_\mu(x) D^{\mu\nu}_R(x,x')\, j_\nu(x') = \int\frac{d^4k}{(2\pi)}\underbrace{j_\mu(-k)\, D^{\mu\nu}_R(k)\, j_\nu(k)}_{j_\mu D^{\mu\nu}_R \text{ or } D^{\mu\nu}_R j_\nu} \equiv 0$$

continuity equation $\partial_\mu j^\mu = 0 \Rightarrow k_\mu j^\mu(k) = 0$

Moral: Non-covariant contributions of transversal maxwell propagator vanish at final result