$$\int d^3x \frac{-\varepsilon_0}{2}(\vec\nabla \varphi(\vec x,t))^2 = \frac{\varepsilon_0}{2}\int d^3x\, \varphi(\vec x,t)\Delta\varphi(\vec x,t) = \frac{\varepsilon_0}{2}\int d^3x \int d^3x' \varphi(\vec x,t)\frac{q\,\psi^+(\vec x',t)\psi(\vec x',t)}{4\pi\varepsilon_0}$$

- $\Delta \frac{1}{|\vec x-\vec x'|} = -4\pi\delta(\vec x-\vec x')$

$$= -\frac{q}{2}\int d^3x\, \varphi(\vec x,t)\psi^+(\vec x,t)\psi(\vec x,t)$$

– Hamilton function of QED: $H = H^{(0)} + H^{(int)}$

$$H^{(0)} = \int d^3x \Big\{\psi^+(\vec x,t)(-i\hbar c\vec\alpha\vec\nabla + mc^2\beta)\psi(\vec x,t) + \frac{\varepsilon_0}{2}\frac{\partial A_k(\vec x,t)}{\partial t}\frac{\partial A_k(\vec x,t)}{\partial t} + \frac{1}{2\mu_0}\partial_k A_l(\vec x,t)\partial_k A_l(\vec x,t)\Big\}$$

$$H^{(int)} = -qc\int d^3x\, \overline{\psi}(\vec x,t)\vec\gamma\psi(\vec x,t)\vec A(\vec x,t)$$ → stems from free Dirac Hamiltonian by minimal coupling $\vec\nabla \to \vec\nabla - \frac{iq}{\hbar}\vec A$

$$+ \frac{q^2}{8\pi\varepsilon_0}\int d^3x\int d^3x' \frac{\overline{\psi}(\vec x,t)\gamma^0\psi(\vec x,t)\overline{\psi}(\vec x',t)\gamma^0\psi(\vec x',t)}{|\vec x-\vec x'|}$$

instantaneous Coulomb self-interaction; does not contradict special relativity; is cancelled in a concrete calculation for cross section by a corresponding term in Maxwell propagator

10.4 Dirac Picture:

Introduction:

– Free QFT: "trivial"
  - Hamiltonian is quadratic in field operators
  - plane wave expansions: field operators contain annihilation/creation of particles with well-defined property
– Interacting QFT: "non-trivial"
  - nonlinearities lead to interesting physical processes
  - dynamics of field operators change their character
  - example: $t=0 \hat=$ annihilation operator ⟶ $t>0 \hat=$ superposition of annihilation and creation operators
  - not exactly solvable
– weak interaction:
  - perturbation theory
  - Schrödinger picture: state vectors time-dependent, operators time-independent — NO
  - Heisenberg picture: operators " , state vectors " — NO
  - Dirac picture: time dependence appropriately distributed between state vectors and operators — YES

10.4.1 Derivation:

– Starting point (also in QED): $\hat H_S = \hat H_S^{(0)} + \hat H_S^{(int)}$ (Schrödinger picture; free part, interacting part)

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat H_S|\psi_S(t)\rangle \Rightarrow |\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat H_S t}|\psi_S(0)\rangle$$

– Dirac picture notion: redo the temporal evolution of $\hat H_S^{(0)}$

$$|\psi_D(t)\rangle = e^{+\frac{i}{\hbar}\hat H_0^{(0)}t}|\psi_S(t)\rangle \Leftrightarrow |\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat H_S^{(0)}t}|\psi_D(t)\rangle$$

– Operators in Dirac picture:

$$\langle\psi_D(t)|\hat O_D(t)|\psi_D(t)\rangle = \langle\psi_S(t)|\hat O_S|\psi_S(t)\rangle$$
$$= \langle\psi_D(t)|e^{\frac{i}{\hbar}\hat H_S^{(0)}t} \quad = e^{-\frac{i}{\hbar}\hat H_S^{(0)}t}|\psi_D(t)\rangle$$
$$\Rightarrow \hat O_D(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)}t}\hat O_S e^{-\frac{i}{\hbar}\hat H_S^{(0)}t} \quad (*)$$

– Example: $\hat O_S = \hat H_S^{(0)}$

$$\hat H_D^{(0)}(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)}t}\hat H_S^{(0)}e^{-\frac{i}{\hbar}\hat H_S^{(0)}t} = \hat H_S^{(0)}$$

– Equation of motion: state vector

$$i\hbar\frac{\partial}{\partial t}|\psi_D(t)\rangle = e^{\frac{i}{\hbar}\hat H_S^{(0)}t}\Big\{i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle - \hat H_S^{(0)}|\psi_S(t)\rangle\Big\} = e^{\frac{i}{\hbar}\hat H_S^{(0)}t}\hat H_S^{(int)}|\psi_S(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat H_S|\psi_S(t)\rangle = (\hat H_S^{(0)} + \hat H_S^{(int)})|\psi_S(t)\rangle, \quad |\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat H_S^{(0)}t}|\psi_D(t)\rangle$$

Schrödinger equation for state vector in Dirac picture = Tomonaga-Schwinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi_D(t)\rangle = \hat H_D^{(int)}(t)|\psi_D(t)\rangle, \quad \hat H_D^{(int)}(t) = e^{\frac{i}{\hbar}\hat H_S^{(0)}t}\hat H_S^{(int)}e^{-\frac{i}{\hbar}\hat H_S^{(0)}t}$$

– Operator: equation of motion → Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\hat{O}_D(t) \overset{(*)}{=} e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\left(-\hat{H}_S^{(0)}\hat{O}_S+\hat{O}_S\hat{H}_S^{(0)}\right)e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}$$

$$=\underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\hat{O}_S e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}}_{\overset{(*)}{=}\hat{O}_D(t)}\hat{H}_S^{(0)}-\hat{H}_S^{(0)}\underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\hat{O}_S e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}}_{\overset{(*)}{=}\hat{O}_D(t)}=\left[\hat{O}_D(t),\hat{H}_S^{(0)}\right]_-$$

- Moral: dynamics of state vector (operator) in Dirac picture is determined by interacting (free) Hamilton operator
- Consequence: field operators in Dirac picture turn out to retain their respective properties of the free theory to annihilate/create a particle

10.4.2 Example:

- Non-relativistic bosons
- Schrödinger picture: $\hat{\Psi}_S(\vec{x}), \hat{\Psi}_S^+(\vec{x})$, $[\hat{\Psi}_S(\vec{x}),\hat{\Psi}_S^+(\vec{x}')]_-=\delta(\vec{x}-\vec{x}')$
  $[\hat{\Psi}_S(\vec{x}),\hat{\Psi}_S(\vec{x}')]_-=[\hat{\Psi}_S^+(\vec{x}),\hat{\Psi}_S^+(\vec{x}')]_-=0$,
- Basis $u_{\vec{p}}(\vec{x})$:
  $\int d^3x\, u^*_{\vec{p}}(\vec{x})u_{\vec{p}'}(\vec{x})=\delta(\vec{p}-\vec{p}')$, $\int d^3p\, u^*_{\vec{p}}(\vec{x})u_{\vec{p}}(\vec{x}')=\delta(\vec{x}-\vec{x}')$
  $\hat{\Psi}_S(\vec{x})=\int d^3p\, u_{\vec{p}}(\vec{x})\hat{a}_S(\vec{p}) \iff \hat{a}_S(\vec{p})=\int d^3x\,\hat{\Psi}_S(\vec{x})u^*_{\vec{p}}(\vec{x})$
  $\hat{\Psi}^+_S(\vec{x})=\int d^3p\, u^*_{\vec{p}}(\vec{x})\hat{a}^+_S(\vec{p}) \iff \hat{a}^+_S(\vec{p})=\int d^3x\,\hat{\Psi}^+_S(\vec{x})u_{\vec{p}}(\vec{x})$
  $[\hat{a}_S(\vec{p}),\hat{a}_S(\vec{p}')]_-=[\hat{a}^+_S(\vec{p}),\hat{a}^+_S(\vec{p}')]_-=0$, $[\hat{a}_S(\vec{p}),\hat{a}^+_S(\vec{p}')]_-=\delta(\vec{p}-\vec{p}')$
- Free diagonal Hamiltonian in Schrödinger picture: $\hat{H}_S^{(0)}=\int d^3p\,E_{\vec{p}}\,\hat{a}^+_S(\vec{p})\hat{a}_S(\vec{p})$
- Heisenberg equation in Dirac picture

$$i\hbar\frac{\partial}{\partial t}\hat{a}_D(\vec{p},t)=[\hat{a}_D(\vec{p},t),\hat{H}_S^{(0)}]_-=e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}[\hat{a}_S(\vec{p}),\hat{H}_S^{(0)}]_-\,e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}$$

$$=\int d^3p' E_{\vec{p}'}e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\underbrace{[\hat{a}_S(\vec{p}),\hat{a}^+_S(\vec{p}')\hat{a}_S(\vec{p}')]_-}_{=\delta(\vec{p}-\vec{p}')\hat{a}_S(\vec{p}')}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}=E_{\vec{p}}\underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\hat{a}_S(\vec{p})e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}}_{=\hat{a}_D(\vec{p},t)}$$

$$\Rightarrow \hat{a}_D(\vec{p},t)=\underbrace{e^{-\frac{i}{\hbar}E_{\vec{p}}t}}_{\text{pick up a phase}}\underbrace{\hat{a}_D(\vec{p},t=0)}_{=\hat{a}_S(\vec{p})},\quad \text{analogous: } \hat{a}^+_D(\vec{p},t)=e^{\frac{i}{\hbar}E_{\vec{p}}t}\hat{a}^+_S(\vec{p})$$

Note: annihilation/creation character remains!

- Consequence:
  $[\hat{a}_D(\vec{p},t),\hat{a}_D(\vec{p}',t)]_-=[\hat{a}^+_D(\vec{p},t),\hat{a}^+_D(\vec{p}',t)]_-=0$, $[\hat{a}_D(\vec{p},t),\hat{a}^+_D(\vec{p}',t)]_-=\delta(\vec{p}-\vec{p}')$
- Field operators in Dirac picture
  $\hat{\Psi}_D(\vec{x},t)=e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t}\hat{\Psi}_S(\vec{x})e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t}=\int d^3p\, u_{\vec{p}}(\vec{x})\hat{a}_D(\vec{p},t)$ $\hat{=}$ Heisenberg picture
  $\hat{\Psi}^+_D(\vec{x},t)=$ " $\hat{\Psi}^+_S(\vec{x})$ " $=\int d^3p\, u^*_{\vec{p}}(\vec{x})\hat{a}^+_D(\vec{p},t)$
  $[\hat{\Psi}_D(\vec{x},t),\hat{\Psi}_D(\vec{x}',t)]_-=[\hat{\Psi}^+_D(\vec{x},t),\hat{\Psi}^+_D(\vec{x}',t)]_-=0$, $[\hat{\Psi}_D(\vec{x},t),\hat{\Psi}^+_D(\vec{x}',t)]_-=\delta(\vec{x}-\vec{x}')$
  $\hat{\Psi}_D(\vec{x},t)$ ($\hat{\Psi}^+_D(\vec{x},t)$) annihilate (create) a boson at $\vec{x}$ and t

10.5 Canonical Field Quantization:

- Work from now on in Dirac picture: omit index D, it is superfluous
- Equal-time (anti-)commutator relations
- Spinor QED:
- Dirac field:
  $[\hat{\Psi}_\alpha(\vec{x},t),\hat{\Psi}_\beta(\vec{x}',t)]_+=0=[\hat{\pi}_\alpha(\vec{x},t),\hat{\pi}_\beta(\vec{x}',t)]_+$, $[\hat{\Psi}_\alpha(\vec{x},t),\hat{\pi}_\beta(\vec{x}',t)]_+=i\hbar\,\delta_{\alpha\beta}\,\delta(\vec{x}-\vec{x}')$
- Maxwell field:
  $[\hat{A}_k(\vec{x},t),\hat{A}_l(\vec{x}',t)]_-=0=[\hat{\pi}_k(\vec{x},t),\hat{\pi}_l(\vec{x}',t)]_-$, $[\hat{A}_l(\vec{x},t),\hat{\pi}_l(\vec{x}',t)]_-=i\hbar\,\delta^{tr}_{kl}(\vec{x}-\vec{x}')$
  (transversal delta function due to Coulomb gauge)
- Independence of Dirac/Maxwell operators:
  $[\hat{\Psi}_\alpha(\vec{x},t),\hat{A}_k(\vec{x}',t)]_-=[\hat{\Psi}_\alpha(\vec{x},t),\hat{\pi}_k(\vec{x}',t)]_-=0$
  $[\hat{\Psi}^+_\alpha(\vec{x},t),\hat{A}_k(\vec{x}',t)]_-=[\hat{\Psi}^+_\alpha(\vec{x},t),\hat{\pi}_k(\vec{x}',t)]_-=0$
- Remark:
  $\hat{\pi}(\vec{x},t)=i\hbar\,\hat{\bar{\Psi}}(\vec{x},t)\gamma^0=i\hbar\,\hat{\Psi}^+(\vec{x},t)$
  $\hat{\vec{\pi}}(\vec{x},t)=\varepsilon_0\left\{\frac{\partial\hat{\vec{A}}(\vec{x},t)}{\partial t}+\vec{\nabla}\hat{\varphi}(\vec{x},t)\right\}$, $\hat{\varphi}(\vec{x},t)=\int d^3x'\,\frac{q\,\hat{\Psi}^+(\vec{x}',t)\hat{\Psi}(\vec{x}',t)}{|\vec{x}-\vec{x}'|\,4\pi\varepsilon_0}$
  $\hat{\pi}_\alpha(\vec{x},t),\ \hat{\pi}_k(\vec{x},t) \longleftrightarrow \hat{\Psi}^+_\alpha(\vec{x},t),\ \frac{\partial\hat{A}_k(\vec{x},t)}{\partial t}$ (leads to non-local commutation relation)

– Interacting Hamilton operator in gauge QED with added normal ordering

$$\hat{H}^{(int)}(t) = -qc \int d^3x \, :\hat{\bar{\Psi}}(\vec{x},t)\, \vec{\gamma}\, \hat{\Psi}(\vec{x},t): \hat{\vec{A}}(\vec{x},t)$$
$$+ \frac{q^2}{8\pi\varepsilon_0} \int d^3x \int d^3x' \, \frac{:\hat{\bar{\Psi}}(\vec{x},t)\gamma^0\hat{\Psi}(\vec{x},t)\,\hat{\bar{\Psi}}(\vec{x}',t)\gamma^0\hat{\Psi}(\vec{x}',t):}{|\vec{x}-\vec{x}'|}$$

Note: We are in Dirac picture!

10.6 Time Evolution Operator:

– Formal solution of Tomonaga-Schwinger equation

$$|\Psi_D(t_2)\rangle = \underbrace{\hat{U}(t_2,t_1)}_{\text{time evolution operator in Dirac picture}} |\Psi_D(t_1)\rangle$$

– Formal solution of Schrödinger equation in Schrödinger picture

$$|\Psi_S(t_2)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}|\Psi_S(t_1)\rangle$$

– Combination:

$$|\Psi_D(t_2)\rangle = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}|\Psi_S(t_2)\rangle = \underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\cdot e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}\cdot e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}}_{=\hat{U}(t_2,t_1)}|\Psi_D(t_1)\rangle$$

where $|\Psi_S(t_1)\rangle = e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}|\Psi_D(t_1)\rangle$

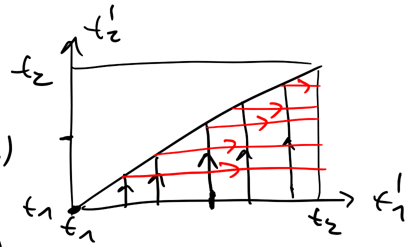
Note: $[\hat{H}_S^{(0)}, \hat{H}_S]_- \neq 0 \Rightarrow$ operator ordering is important

– List properties of $\hat{U}(t_2,t_1)$:

- initial condition: $\hat{U}(t_1,t_1) = 1$
- group property:
$$\hat{U}(t_3,t_2)\hat{U}(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_3}\underbrace{e^{-\frac{i}{\hbar}\hat{H}_S(t_3-t_2)}\underbrace{e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}}_{=1}e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}}_{=e^{-\frac{i}{\hbar}\hat{H}_S(t_3-t_1)}}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}$$
$$= \hat{U}(t_3,t_1)$$
- inverse: $t_3 = t_1$
$$\hat{U}(t_1,t_2)\hat{U}(t_2,t_1) = \hat{U}(t_1,t_1) = 1 \Rightarrow \hat{U}^{-1}(t_2,t_1) = \hat{U}(t_1,t_2)$$
- unitary: $\hat{U}^{+}(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}e^{+\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2} = \hat{U}(t_1,t_2) = \hat{U}^{-1}(t_2,t_1)$
- differential equation:
$$i\hbar\frac{\partial}{\partial t_2}\hat{U}(t_2,t_1) = e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\underbrace{(-\hat{H}_S^{(0)}+\hat{H}_S)}_{=\hat{H}_S^{(int)}}e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}$$
$$= \underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}\hat{H}_S^{(int)}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}}_{=\hat{H}_D^{(int)}(t_2)}\underbrace{e^{\frac{i}{\hbar}\hat{H}_S^{(0)}t_2}e^{-\frac{i}{\hbar}\hat{H}_S(t_2-t_1)}e^{-\frac{i}{\hbar}\hat{H}_S^{(0)}t_1}}_{=\hat{U}(t_2,t_1)}$$
- Initial value problem:
$$\left.\begin{aligned}&\hat{U}(t_1,t_1) = 1\\ &i\hbar\frac{\partial}{\partial t_2}\hat{U}(t_2,t_1) = \hat{H}_D^{(int)}(t_2)\hat{U}(t_2,t_1)\end{aligned}\right\} \Leftrightarrow \hat{U}(t_2,t_1) = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1')\hat{U}(t_1',t_1)$$
integral equation
- Iteratively reinsert:
$$\hat{U}(t_2,t_1) = 1 - \frac{i}{\hbar}\int_{t_1}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1') + \left(\frac{-i}{\hbar}\right)^2\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\,\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2')$$
$$+\dots+\left(\frac{-i}{\hbar}\right)^n\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\cdots\int_{t_1}^{t_{n-1}'}dt_n'\,\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2')\cdots\hat{H}_D^{(int)}(t_n')+\dots$$

time ordering: $t_1' > t_2' > t_3' > \dots > t_n'$

Free Dyson: all integrals can be performed in $[t_1,t_2]$



Example: $n=2$

$$\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\,\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2') = \int_{t_1}^{t_2}dt_2'\int_{t_2'}^{t_2}dt_1'\,\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2')$$
$$=\frac{1}{2}\left(\text{———} + \text{———}\right)$$
$$=\frac{1}{2}\left\{\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_1'}dt_2'\,\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2') + \int_{t_1}^{t_2}dt_1'\int_{t_1'}^{t_2}dt_2'\,\hat{H}_D^{(int)}(t_2')\hat{H}_D^{(int)}(t_1')\right\}$$
$$=\frac{1}{2}\int_{t_1}^{t_2}dt_1'\int_{t_1}^{t_2}dt_2'\left\{\Theta(t_1'-t_2')\hat{H}_D^{(int)}(t_1')\hat{H}_D^{(int)}(t_2') + \Theta(t_2'-t_1')\hat{H}_D^{(int)}(t_2')\hat{H}_D^{(int)}(t_1')\right\}$$

$$= \hat{T}\left(\hat{H}_D^{(int)}(t_1), \hat{H}_D^{(int)}(t_2) \right)$$

generalizable to any order n via complete induction

Von Neumann series:

$$\hat{U}(t_1,t_2) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \int_{t_1}^{t_2} dt_1' \dots \int_{t_1}^{t_2} dt_n' \, \hat{T}\left(\hat{H}_D^{(int)}(t_1) \dots \hat{H}_D^{(int)}(t_n) \right)$$

formally $\hat{T} \exp\left\{ -\frac{i}{\hbar} \int_{t_1}^{t_2} dt \, \hat{H}_D^{(int)}(t) \right\}$

with this we can perturbatively calculate cross sections

10.7 Scattering Operator:

- generic scenario of a scattering problem: $\lim_{t\to-\infty} |\psi(t)\rangle = |\psi_i\rangle$

  evolution: $|\psi(t)\rangle = \hat{U}(t,-\infty)|\psi_i\rangle$,

- What is the probability amplitude for a transition to the state $|\psi_f\rangle$?

  scattering matrix element $S_{fi} = \lim_{t\to\infty} \langle \psi_f | \psi(t) \rangle = \lim_{t\to\infty} \langle \psi_f | \hat{U}(t,-\infty) | \psi_i \rangle$

  $\Rightarrow S_{fi} = \langle \psi_f | \underbrace{\hat{S}}_{\text{scattering operator}} | \psi_i \rangle$, $\hat{S} = \hat{U}(+\infty,-\infty)$

- QED:

$$\hat{S} = \hat{T} \exp\left\{ \frac{iq}{\hbar} \int d^4x \, :\hat{\bar{\psi}}(x)\vec{\gamma}\hat{\psi}(x): \hat{\vec{A}}(x) - \frac{iq^2}{8\pi\hbar\varepsilon_0} \int dt \int d^3x \int d^3x' \frac{:\hat{\bar{\psi}}(\vec{x},t)\gamma^0\hat{\psi}(\vec{x},t)\,\hat{\bar{\psi}}(\vec{x}',t)\gamma^0\hat{\psi}(\vec{x}',t):}{|\vec{x}-\vec{x}'|} \right\}$$

Expand in powers of q:

$$\hat{S} = 1 + \frac{iq}{\hbar}\int d^4x \, :\hat{\bar{\psi}}(x)\vec{\gamma}\hat{\psi}(x): \hat{\vec{A}}(x)$$
$$+ \frac{1}{2}\left(\frac{iq}{\hbar}\right)^2 \int d^4x \int d^4x' \, \hat{T}\left\{ \left(:\hat{\bar{\psi}}(x)\vec{\gamma}\hat{\psi}(x): \hat{\vec{A}}(x) \right)\left(:\hat{\bar{\psi}}(x')\vec{\gamma}\hat{\psi}(x'): \hat{\vec{A}}(x') \right)\right.$$
$$- \frac{iq^2}{8\pi\hbar\varepsilon_0} \int dt \int d^3x \int d^3x' \frac{:\hat{\bar{\psi}}(\vec{x},t)\gamma^0\hat{\psi}(\vec{x},t)\,\hat{\bar{\psi}}(\vec{x}',t)\gamma^0\hat{\psi}(\vec{x}',t):}{|\vec{x}-\vec{x}'|} + \dots$$