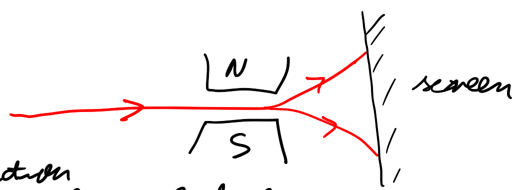
Chapter 9: Dirac Field

Motivation:

- Paul Dirac 1928:
  - unify principles of special relativity and quantum mechanics:
    Dirac equation = relativistic wave equation
  - describes massive spin 1/2 particles like electrons or quarks
  - prediction of anti-matter, first observation of an anti-particle was positron in cosmic radiation by Carl David Anderson
- Dirac equation for 4 complex fields = spinor
  - transforms differently under Lorentz transformations than a vector, e.g. a vector needs a rotation of 360° around a fixed axis to reproduce the original state but a spinor needs a rotation of 720°.
  - Non-relativistic limit: Pauli two-component wave function
  - Massless spin 1/2-particles: Weyl equation
- Outlook:
  - Group-theoretical derivation of Dirac equation: spinor representation of Lorentz group
  - Invariance under discrete symmetries like Charge conjugation, Parity transformation, Time inversion:
    > CPT theorem: exemplarily proven for Dirac theory: mirror universe with matter replaced by anti-matter would evolve under the same physical laws
    > Consequence: masses, life times, ... of matter and anti-matter are identical (see current experiment with anti-hydrogen at CERN)
  - canonical field quantization: description of many massive spin 1/2 particles and their anti-particles
  - Dirac propagator: free motion of a massive spin 1/2 (anti-)particles
    > important for perturbative QED

9.1 Pauli Matrices:

- Stern-Gerlach experiment 1922:
  atom beam $\begin{cases}\text{silver} & 5s^1 \\ \text{hydrogen} & 1s^1\end{cases}$
  valence electron



- Explanation:
  - spin angular momentum $s = 1/2$ of valence electron
  - leads to a magnetic moment
  - deflection in inhomogeneous magnetic field, splitting into 2 beams
- mathematical description due to Wolfgang Pauli by 3 complex 2×2 matrices:
  $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
- Anti-commutators: $[\sigma^k, \sigma^l]_+ = 2\delta_{kl} \cdot I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  Clifford algebra with $N = 3$
- N generators $\xi^1, \ldots, \xi^N$ form a Clifford algebra provided $[\xi^k, \xi^l]_+ = 2\delta_{kl}$
- Commutators: $[\sigma^k, \sigma^l]_- = 2i\,\varepsilon_{klm}\sigma^m$
  Lie algebra with $N = 3$
- N generators $\xi^1, \ldots, \xi^N$ form a Lie algebra provided $[\xi^k, \xi^l]_- = i\,\underbrace{C_{klm}}_{\text{structure constants}}\sigma^m$
- Important calculational rule:

$$\sigma^k \sigma^l + \sigma^l \sigma^k = 2\delta_{kl} I$$
$$\sigma^k \sigma^l - \sigma^l \sigma^k = 2i\,\varepsilon_{klm}\sigma^m$$

$$\sigma^k \sigma^l = \delta_{kl}\cdot I + i\,\varepsilon_{klm}\sigma^m \Leftarrow (*)$$

Product of Pauli matrices can be simplified

9.2 Spinor Representations of Lorentz Algebra:

- Outlook: Pauli matrices allow to construct two different spinor representations of Lorentz algebra
- generators of rotations: $L_k = \frac{1}{2}\sigma^k$

  $[L_k, L_l]_- = \frac{1}{4}[\sigma^k, \sigma^l]_- = \frac{1}{4} 2i\,\varepsilon_{klm}\sigma^m = i\,\varepsilon_{klm} L_m$ ✓ (see Chap. 6)
- generators of boosts: $M_k = \pm\frac{i}{2}\sigma^k$
  1) $[L_k, M_l]_- = \frac{\pm i}{4}[\sigma^k, \sigma^l]_- = \frac{\pm i}{4} 2i\,\varepsilon_{klm}\sigma^m = i\,\varepsilon_{klm} M_m$ ✓ "
  2) $[M_k, M_l]_- = -\frac{1}{4}[\sigma^k, \sigma^l]_- = -\frac{1}{4}2i\,\varepsilon_{klm}\sigma^m = -i\,\varepsilon_{klm} L_m$ ✓ "
- Two representation:

  $D^{(1/2,0)}$: $(L_k, M_k) = \left(\frac{1}{2}\sigma^k, -\frac{i}{2}\sigma^k\right)$

  $D^{(0,1/2)}$: $(L_k, M_k) = \left(\frac{1}{2}\sigma^k, +\frac{i}{2}\sigma^k\right)$
- general representation of Lorentz algebra:
  - $D^{(s_1,s_2)}$: $s_1, s_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$
  - $D^{(s_1,s_2)}$: contains particles with spin $[|s_1 - s_2|, s_1+s_2]$
  - $D^{(s,0)}, D^{(0,s)}$: describe particles with spin s
  - Trivial representation: $D^{(0,0)}$: $(L_k, M_k) = (1, 1)$
- Lie theorem: $\underbrace{D(\Lambda)}_{\text{representation of Lorentz group}} = e^{-i\vec{L}\cdot\vec{\varphi} - i\vec{M}\cdot\vec{\xi}}$ ← representation of Lorentz algebra

  $D^{(1/2,0)}(\Lambda) = \exp\left\{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} - \frac{1}{2}\vec{\sigma}\vec{\xi}\right\}$ rotations: $\vec{\xi} = \vec{0}$

  $D^{(0,1/2)}(\Lambda) = \exp\left\{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi} + \frac{1}{2}\vec{\sigma}\vec{\xi}\right\}$ boosts: $\vec{\varphi} = \vec{0}$

9.3 Spinor Representation of Rotations:

- $D^{(1/2,0)}(R(\vec{\varphi})) = D^{(0,1/2)}(R(\vec{\varphi})) = D(R(\vec{\varphi})) = e^{-\frac{i}{2}\vec{\sigma}\cdot\vec{\varphi}}$: same representations
- hermiticity: $(\sigma^k)^+ = \sigma^k \Rightarrow$ unitary representation $D(R(\vec{\varphi}))^+ = D(R(\vec{\varphi}))^{-1}$
- Taylor series of exponential function: consider odd and even terms separately

$$D(R(\vec{\varphi})) = \sum_{n=0} \frac{(-1)^n}{(2n)!}\frac{(\vec{\sigma}\cdot\vec{\varphi})^{2n}}{2^{2n}} - i\sum_{n=0}\frac{(-1)^n}{(2n+1)!}\frac{(\vec{\sigma}\cdot\vec{\varphi})^{2n+1}}{(2)^{2n+1}}$$

$$(\vec{\sigma}\vec{\varphi})^2 = \varphi_k\varphi_l\sigma^k\sigma^l \overset{(*)}{=} \varphi_k\varphi_l\left(\delta_{kl} I + \cancel{i\varepsilon_{klm}\sigma^m}\right) = |\vec{\varphi}|^2\cdot I$$ symmetry

$$D(R(\vec{\varphi})) = \underbrace{\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{|\vec{\varphi}|}{2}\right)^{2n}\right\}}_{=\cos\left(\frac{|\vec{\varphi}|}{2}\right)} I - i\underbrace{\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{|\vec{\varphi}|}{2}\right)^{2n+1}\right\}}_{=\sin\left(\frac{|\vec{\varphi}|}{2}\right)}\frac{(\vec{\sigma}\cdot\vec{\varphi})}{|\vec{\varphi}|}$$

Result: $D(R(\vec{\varphi})) = I\cos\left(\frac{|\vec{\varphi}|}{2}\right) - i\frac{\vec{\sigma}\cdot\vec{\varphi}}{|\vec{\varphi}|}\sin\left(\frac{|\vec{\varphi}|}{2}\right)$

- unitary
- rotation of 4π around fixed axis needed to recover identity, characteristic for spinor

9.4 Spinor Representations of Boosts:

- To evaluate: $D(B(\vec{\xi})) = e^{\mp\frac{1}{2}\vec{\sigma}\cdot\vec{\xi}}$ ($-$: $D^{(1/2,0)}$, $+$: $D^{(0,1/2)}$) $\xrightarrow{(\sigma^k)^+ = \sigma^k}$ $D(B(\vec{\xi}))^+ = D(B(\vec{\xi}))$ hermitian
- Taylor series:

$$D(B(\vec{\xi})) = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\frac{(\vec{\sigma}\cdot\vec{\xi})^{2n}}{2^{2n}} \mp \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{(\vec{\sigma}\cdot\vec{\xi})^{2n+1}}{2^{2n+1}}$$

$$\overset{(*)}{=} \underbrace{\left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{|\vec{\xi}|}{2}\right)^{2n}\right\}}_{=\cosh\left(\frac{|\vec{\xi}|}{2}\right)} I \mp \underbrace{\left\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{|\vec{\xi}|}{2}\right)^{2n+1}\right\}}_{=\sinh\left(\frac{|\vec{\xi}|}{2}\right)}\frac{\vec{\sigma}\cdot\vec{\xi}}{|\vec{\xi}|}$$

Result: $D(B(\vec{\xi})) = I\cosh\left(\frac{|\vec{\xi}|}{2}\right) \mp \frac{\vec{\sigma}\cdot\vec{\xi}}{|\vec{\xi}|}\sinh\left(\frac{|\vec{\xi}|}{2}\right) = e^{\mp\frac{1}{2}\vec{\sigma}\vec{\xi}}$ (**)

→ hermitian

- Aim: relate rapidity $\vec{\xi}$ to momentum $\vec{p}$

$$(p^\mu_R) = (Mc, \vec{0}) \longrightarrow p^\mu = \underbrace{B^\mu{}_\nu(\vec{\xi})}\, p^\nu_R$$

Chapter 6
$$= \left(\begin{array}{c|c} \cosh|\vec{\xi}| & \frac{\xi_j}{|\vec{\xi}|}\sinh|\vec{\xi}| \\ \hline \frac{\xi_i}{|\vec{\xi}|}\sinh|\vec{\xi}| & \delta_{ij} + \frac{\xi_i\xi_j}{|\vec{\xi}|^2}(\cosh|\vec{\xi}| - 1) \end{array}\right)$$

$$\Rightarrow (p^\mu) = (p^0, \vec{p}) = \left(mc \cosh|\vec{\zeta}|, mc\frac{\vec{\zeta}}{\zeta} \sinh|\vec{\zeta}|\right)$$

$$\Rightarrow \cosh|\vec{\zeta}| = \frac{p^0}{mc}, \quad \frac{\vec{\zeta}}{|\vec{\zeta}|} = \frac{\vec{p}}{mc}\frac{1}{\sinh|\vec{\zeta}|}$$

- half arguments:

$$\cosh\left(\frac{|\vec{\zeta}|}{2}\right) = \sqrt{\frac{\cosh|\vec{\zeta}|+1}{2}} = \sqrt{\frac{p^0+mc}{2mc}}, \quad \sinh\left(\frac{|\vec{\zeta}|}{2}\right) = \sqrt{\frac{\cosh|\vec{\zeta}|-1}{2}} = \sqrt{\frac{p^0-mc}{2mc}}$$

$$\sinh|\vec{\zeta}| = 2\sinh\left(\frac{|\vec{\zeta}|}{2}\right)\cosh\left(\frac{|\vec{\zeta}|}{2}\right) = \frac{\sqrt{(p^0+mc)(p^0-mc)}}{mc}$$

- Boost representation matrix in terms of ~~four~~-momentum vectors:

$$D(B(\vec{\zeta})) = I\sqrt{\frac{p^0+mc}{2mc}} + \vec{\sigma}\cdot\frac{\vec{p}}{\cancel{mc}}\frac{\cancel{mc}}{\sqrt{(p^0+mc)\cancel{(p^0-mc)}}}\sqrt{\frac{\cancel{p^0-mc}}{2mc}} = \frac{(p^0+mc)I + \vec{\sigma}\vec{p}}{\sqrt{2mc(p^0+mc)}}$$

- Four-vector of Pauli matrices:

$$(\sigma^\mu) = (\sigma^0, \sigma^k), \sigma^0 = I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \Rightarrow D^{(1/2,0)}(B(\vec{\zeta})) = e^{-\frac{1}{2}\vec{\sigma}\vec{\zeta}} = \frac{p\sigma + mc}{\sqrt{2mc(p^0+mc)}}$$

$$p\cdot\sigma = p_\mu\sigma^\mu = g_{\mu\nu}p^\mu\sigma^\nu = \sigma^0p^0 - \vec{p}\cdot\vec{\sigma}$$

- Spatially inverted four vector:

$$\tilde{x} = (\tilde{x}^0, \tilde{x}^k) = (x^0, -x^k), \quad \tilde{\sigma} = (\tilde{\sigma}^0, \tilde{\sigma}^k) = (\sigma^0, -\sigma^k)$$

$$D^{(0,1/2)}(B(\vec{\zeta})) = e^{+\frac{1}{2}\vec{\sigma}\vec{\zeta}} = \frac{p\tilde{\sigma}+mc}{\sqrt{2mc(p^0+mc)}}$$

$$p\tilde{\sigma} = \tilde{p}\sigma = g_{\mu\nu}p^\mu\tilde{\sigma}^\nu = \sigma^0p^0 + \vec{\sigma}\cdot\vec{p}$$

- Additional calculation for later purposes:

$$e^{\mp\frac{1}{2}\vec{\sigma}\vec{\zeta}} = \sqrt{e^{\mp\vec{\sigma}\vec{\zeta}}} = \sqrt{I\cosh|\vec{\zeta}| \mp \frac{\vec{\sigma}\vec{\zeta}}{|\vec{\zeta}|}\sinh|\vec{\zeta}|} = \sqrt{\frac{p^0}{mc} \mp \frac{\vec{\sigma}\cdot\vec{p}}{mc}}$$

$$\Rightarrow e^{-\frac{1}{2}\vec{\sigma}\vec{\zeta}} = \sqrt{\frac{p^0\sigma^0 - \vec{\sigma}\vec{p}}{mc}} = \sqrt{\frac{p\sigma}{mc}}, \quad e^{\frac{1}{2}\vec{\sigma}\vec{\zeta}} = \sqrt{\frac{p^0\sigma^0+\vec{\sigma}\vec{p}}{mc}} = \sqrt{\frac{p\tilde{\sigma}}{mc}}$$

9.5 Lorentz Invariant Combinations of Weyl Spinors:

- $D^{(1/2,0)}, D^{(0,1/2)}$: smallest non-trivial representations of Lorentz group
- Weyl spinors upon which representation matrices act:

$$\xi^\alpha(x) \longrightarrow \xi'_\alpha(x') = D^{(1/2,0)}(\Lambda)_\alpha{}^\beta \xi_\beta(x)$$

$$\xi^{\dot\alpha}(x) \longrightarrow \xi'_{\dot\alpha}(x') = D^{(0,1/2)}(\Lambda)^{\dot\alpha}{}_{\dot\beta}\,\xi^{\dot\beta}(x)$$

- Aim: Search for Lorentz invariant action on the basis of these Weyl spinors. Restrict to quadratic terms in Weyl spinors and their first partial derivatives
- First: no partial derivative $\hat{=}$ mass term

four combinations: $\xi^+\xi$, $\xi^+\chi$, $\chi^+\xi$, $\chi^+\chi$

Apply Lorentz transformation:

$\xi^+D^{(1/2,0)}(\Lambda)^+D^{(1/2,0)}(\Lambda)\xi$, $\chi^+D^{(0,1/2)}(\Lambda)^+D^{(0,1/2)}(\Lambda)\chi$

$\chi^+D^{(0,1/2)}(\Lambda)^+D^{(1/2,0)}(\Lambda)\xi$, $\xi^+D^{(1/2,0)}(\Lambda)^+D^{(0,1/2)}(\Lambda)\chi$

- Rotations: $\Lambda = R$
  - $D^{(1/2,0)}(R) = D^{(0,1/2)}(R) = D(R)$
  - $D(R)^+ = D(R)^{-1} \Rightarrow D(R)^+D(R) = 1$

  $\Rightarrow$ all four transformed combinations agree with original combinations
- Boosts: $\Lambda = B$
  - $D^{(1/2,0)}(B) = e^{-\frac{1}{2}\vec{\sigma}\vec{\zeta}}$, $D^{(0,1/2)}(B) = e^{\frac{1}{2}\vec{\sigma}\vec{\zeta}} \Rightarrow D^{(1/2,0)}(B)^{-1} = D^{(0,1/2)}(B)$
  - $D(B)^+ = D(B)$

  $\Rightarrow D^{(1/2,0)}(B)^+D^{(0,1/2)}(B) = D^{(0,1/2)}(B)^+D^{(1/2,0)}(B) = 1$

  $\Rightarrow$ From all transformed combinations only last two agree with original combination