7.8 Redefinition of Fourier operators:

$[\hat a^{(1)}_{\vec p}, \hat a^{(1)+}_{\vec p'}]_- = \delta(\vec p-\vec p')$ $\Rightarrow$ $\left\{ \begin{array}{l} \hat a^{(1)}_{\vec p}: \text{annihilation} \\ \hat a^{(1)+}_{\vec p}: \text{creation} \end{array} \right\}$ operator for particle sort $1=1$

$\underbrace{[\hat a^{(2)}_{\vec p}, \hat a^{(2)+}_{\vec p'}]_- = \bullet\,\delta(\vec p-\vec p')}$ $\Rightarrow$ $\left\{ \begin{array}{l} \hat a^{(2)}_{\vec p}: \text{creation} \\ \hat a^{(2)+}_{\vec p}: \text{annihilation} \end{array} \right\}$ operator for particle sort $2=2$

$[\hat a^{(2)+}_{\vec p}, \hat a^{(2)}_{\vec p}]_- = +\delta(\vec p-\vec p')$

$\Rightarrow$ This suggests redefinition

particle sort a: $\hat a_{\vec p} = \hat a^{(1)}_{\vec p}$, $\hat a^+_{\vec p} = \hat a^{(1)+}_{\vec p}$

particle sort b: $\hat b_{\vec p} = \hat a^{(2)+}_{\vec p}$, $\hat b^+_{\vec p} = \hat a^{(2)}_{\vec p}$

$[\hat a_{\vec p}, \hat a^+_{\vec p'}]_- = [\hat b_{\vec p}, \hat b^+_{\vec p'}]_- = i\hbar\,\delta(\vec p-\vec p')$; all other commutators vanish

$$\hat\Psi(\vec x,t) = \sum_{r=1}^{2} \int d^3p\, \hat a^{(r)}_{\vec p} u^{(r)}_{\vec p}(\vec x,t) = \int d^3p \{ \hat a_{\vec p} \underbrace{\overset{\sim e^{-\frac{i}{\hbar}E_{\vec p}t}}{u_{\vec p}(\vec x,t)}} + \hat b^+_{\vec p}\, \overset{*\ \sim\ e^{+\frac{i}{\hbar}E_{\vec p}t} = e^{-\frac{i}{\hbar}\underbrace{(E_{\vec p})t}_{=E_{\vec p}(-t)}}}{u_{\vec p}(\vec x,t)} \}$$

$$= u^{(1)}_{\vec p}(\vec x,t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p}}}\, e^{\frac{i}{\hbar}(\vec p\vec x - E_{\vec p}t)}$$

$$\hat\Psi^+(\vec x,t) = \int d^3p \{ \hat a^+_{\vec p} u^*_{\vec p}(\vec x,t) + \hat b_{\vec p} u_{\vec p}(\vec x,t) \}$$

Hamilton operator:

$$\hat H = \int d^3p\, E_{\vec p} (\hat a^+_{\vec p} \hat a_{\vec p} + \underbrace{\hat b_{\vec p} \hat b^+_{\vec p}}_{=\hat b^+_{\vec p}\hat b_{\vec p} + \delta(\vec 0)}) = \int d^3p\, E_{\vec p} (\hat a^+_{\vec p} \hat a_{\vec p} + \hat b^+_{\vec p} \hat b_{\vec p}) + \underbrace{\delta(\vec 0) \int d^3p\, E_{\vec p}}_{\text{twice infinite}}$$

$$\hat Q = \int d^3p\, (\hat a^+_{\vec p} \hat a_{\vec p} - \overbrace{\hat b_{\vec p} \hat b^+_{\vec p}}) = \int d^3p\, (\hat a^+_{\vec p} \hat a_{\vec p} - \hat b^+_{\vec p} \hat b_{\vec p}) - \delta(\vec 0) \int d^3p\ 1$$

vacuumstate $\hat a_{\vec p}|0\rangle = 0$, $\hat b_{\vec p}|0\rangle = 0$

$\Rightarrow$ $\langle 0|\hat H|0\rangle = \delta(\vec 0)\int d^3p\, E_{\vec p}$ $\langle 0|\hat Q|0\rangle = -\delta(\vec 0)\int d^3p\, 1$

$\Rightarrow$ normal ordering ; "get rid of infinite large vacuum contribution"

$$:\hat H: = \hat H - \langle 0|\hat H|0\rangle = \int d^3p\, E_{\vec p}(\hat a^+_{\vec p} \hat a_{\vec p} + \hat b^+_{\vec p} \hat b_{\vec p})$$

$$:\hat Q: = \hat Q - \langle 0|\hat Q|0\rangle = \int d^3p\, (\hat a^+_{\vec p} \hat a_{\vec p} - \hat b^+_{\vec p} \hat b_{\vec p})$$

Physical interpretation:

particle sort a: energy $E_{\vec p}$, charge $+1$ $\hat = \pi^+$ particle

" " b: energy $E_{\vec p}$, charge -1 $\hat = \pi^-$ antiparticle

7.9 Definition of Propagator:

- Propagators are needed to describe interacting quantum fields perturbatively. They represent the elementary building blocks of Feynman diagrams.
- Overview:

Schrödinger propagator sheet 2 + Appendix	⊛ ←	Klein Gordon propagator now ≙ chapter 7	⊛⊛ ←→	Dirac propagator chapter 9
non-relativistic quantum many-body theory		scalar quantum electrodynamics		quantum electro-dynamics

⊛ non-relativistic limit $c \to \infty$

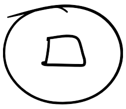
⊛⊛ Dirac propagator follows from Klein-Gordon propagator via partial derivatives

Reminder:

$$\underset{\uparrow}{\hat\Psi}(\vec x,t) = \int d^3p \{ \underbrace{\hat a_{\vec p}} u_{\vec p}(\vec x,t) + \underbrace{\hat b^+_{\vec p}} u^*_{\vec p}(\vec x,t) \}$$

annihilation of particle of sort a with charge +1

creation of antiparticle of sort b with charge −1



describes annihilation of charge +1 at space-point $(\vec x,t)$

Klein-Gordon propagator definition:

$$G(\vec{x},t;\vec{x}',t') = \langle 0|\hat{T}\left(\hat{\Psi}(\vec{x},t)\ \hat{\Psi}^+(\vec{x}',t')\right)|0\rangle$$

$\hat{T}$: time-ordering operator

$$\hat{T}(\hat{A}(t)\,\hat{B}(t')) = \Theta(t-t')\,\hat{A}(t)\,\hat{B}(t') + \Theta(t'-t)\,\hat{B}(t')\,\hat{A}(t)$$

Heaviside function: $\Theta(t) = \begin{cases} 1 & , t>0 \\ 0 & , t<0 \end{cases}$

time-ordering: operator at later time is left

Note: Time-ordering can become problematic at $t=t'$ when operators $\hat{A}(t)$ and $\hat{B}(t)$ do not commute. But in Klein-Gordon theory no problem due to $[\hat{\Psi}(\vec{x},t), \hat{\Psi}(\vec{x}',t)]_- = 0$

Motivation: Time ordering appears in perturbation theory naturally in the Dirac interaction picture

$$G(\vec{x},t;\vec{x}',t') = \Theta(t-t')\langle 0|\hat{\Psi}(\vec{x},t)\,\hat{\Psi}^+(\vec{x}',t')|0\rangle + \Theta(t'-t)\langle 0|\hat{\Psi}^+(\vec{x}',t')\,\hat{\Psi}(\vec{x},t)|0\rangle$$

Aim: equation of motion?

$$\frac{\partial G(\vec{x},t;\vec{x}',t')}{\partial t} = \cancel{\underbrace{\frac{\partial\Theta(t-t')}{\partial t}}_{=\delta(t-t')}\langle 0|\underbrace{\left[\hat{\Psi}(\vec{x},t), \underbrace{\hat{\Psi}^+(\vec{x}',t')}_{\hat{\Psi}^+(\vec{x}',t)}\right]_-}_{=0}|0\rangle}$$

$$+\Theta(t-t')\langle 0|\frac{\partial\hat{\Psi}(\vec{x},t)}{\partial t}\,\hat{\Psi}^+(\vec{x}',t')|0\rangle + \Theta(t'-t)\langle 0|\hat{\Psi}^+(\vec{x}',t')\,\frac{\partial\hat{\Psi}(\vec{x},t)}{\partial t}|0\rangle$$

$$\frac{\partial^2 G(\vec{x},t;\vec{x}',t')}{\partial t^2} = \delta(t-t')\langle 0|\underbrace{\left[\underbrace{\frac{\partial\hat{\Psi}(\vec{x},t)}{\partial t}}_{=\frac{2mc^2}{\hbar^2}\hat{\pi}^+(\vec{x},t)}, \underbrace{\hat{\Psi}^+(\vec{x}',t')}_{=\hat{\Psi}^+(\vec{x}',t)}\right]_-}_{\frac{2mc^2}{\hbar^2}(-i)\,\delta(\vec{x}-\vec{x}')}|0\rangle$$

$$+\Theta(t-t')\langle 0|\underbrace{\frac{\partial^2\hat{\Psi}(\vec{x},t)}{\partial t^2}}_{=c^2\Delta\hat{\Psi}(\vec{x},t)-\frac{m^2c^4}{\hbar^2}\hat{\Psi}(\vec{x},t)}\hat{\Psi}^+(\vec{x}',t)|0\rangle + \Theta(t'-t)\langle 0|\hat{\Psi}^+(\vec{x}',t')\underbrace{\frac{\partial^2\hat{\Psi}(\vec{x},t)}{\partial t^2}}_{\text{dito}}|0\rangle$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right)G(\vec{x},t;\vec{x}',t') = -\frac{2m}{\hbar}\,\delta(t-t')\,\delta(\vec{x}-\vec{x}') \quad (*)$$

Green function of Klein-Gordon equation = Klein-Gordon propagator

Note: George Green was by profession a miller who learned physics and mathematics autodidactically

$\Rightarrow$ Heisenberg equation for $\hat{\Psi}(\vec{x},t)$ coupled to other quantum fieldoperators can be solved perturbatively with the help of the Green function of Klein-Gordon equation

Non-relativistic limit:

$$G(\vec{x},t;\vec{x}',t') = g(\vec{x},t;\vec{x}',t)\cdot\exp\left\{-\frac{i}{\hbar}mc^2(t-t')\right\}$$

$\rightarrow$ insert this into (*)

$$\left(\underbrace{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\substack{=0 \\ c\to\infty}} - 2\frac{i}{\hbar}m\cancel{c^2}\frac{1}{\cancel{c^2}}\frac{\partial}{\partial t} - \cancel{\frac{m^2c^4}{\hbar^2}\frac{1}{c^2}} - \Delta + \cancel{\frac{m^2c^2}{\hbar^2}}\right) g(\vec{x},t;\vec{x}',t')$$

$$= -i\frac{2m}{\hbar}\delta(t-t')\,\delta(\vec{x}-\vec{x}') \quad \Big| \cdot \frac{\hbar^2}{2m}$$

$$\Rightarrow \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\underbrace{g(\vec{x},t;\vec{x}',t')}_{} = i\hbar\,\delta(t-t')\,\delta(\vec{x}-\vec{x}')$$

Schrödinger propagator = Green function of Schrödinger equation

7.10 Interpretation of Propagator:

charge operator: $\hat{Q} = \frac{i}{\hbar}\int d^3x\left\{\hat{\Phi}^+(\vec{x},t)\hat{\pi}^+(\vec{x},t) - \hat{\pi}(\vec{x},t)\hat{\Phi}(\vec{x},t)\right\}$

$$\Rightarrow \begin{cases} [\hat{Q},\hat{\Phi}(\vec{x},t)]_- = -\hat{\Phi}(\vec{x},t) & (1) \\ [\hat{Q},\hat{\Phi}^+(\vec{x},t)]_- = +\hat{\Phi}^+(\vec{x},t) & (2) \end{cases}$$

$$\hat{Q}|q\rangle = q|q\rangle \qquad (3)$$

$$\hat{Q}\{\hat{\Phi}(\vec{x},t)|q\rangle\} \overset{(1)}{=} \hat{\Phi}(\vec{x},t)\{\hat{Q}-1\}|q\rangle \overset{(3)}{=} (q-1)\,\hat{\Phi}(\vec{x},t)|q\rangle$$

$$\Rightarrow \hat{\Phi}(\vec{x},t)|q\rangle \sim |q-1\rangle$$

$\hat{\Phi}(\vec{x},t)$: decrease charge by one unit

analogously: (2) + (3) $\Rightarrow$ $\hat{\Phi}^+(\vec{x},t)$ increases charge by 1 unit

$$G(\vec{x},t;\vec{x}',t) = \Theta(t-t')\underbrace{\langle 0|\ \hat{\Phi}(\vec{x},t)\ \hat{\Phi}^+(\vec{x}',t')\ |0\rangle}_{\text{propagation of charge } +1 \text{ from } (\vec{x}',t') \text{ to } (\vec{x},t)}$$

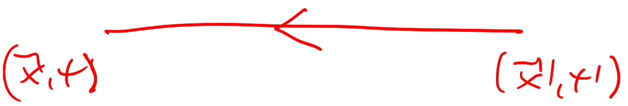
forward propagation

$$+\,\Theta(t'-t)\underbrace{\langle 0|\ \hat{\Phi}^+(\vec{x}',t')\ \hat{\Phi}(\vec{x},t)|0\rangle}_{\text{propagation of charge } -1 \text{ from } (\vec{x},t) \text{ to } (\vec{x}',t')}$$

$\textcircled{BD}$

backward propagation

$G(\vec{x},t;\vec{x}',t') \hat{=}$



charge propagation from $(\vec{x}',t')$ to $(\vec{x},t)$

7.11 Calculation of Propagator:

Insert Ⓑ into $\textcircled{BD}$:

$$G(\vec{x},t;\vec{x}',t') = \int d^3p\int d^3p'\,\Big\{\Theta(t-t')$$

$$\cdot\langle 0|\{\hat{a}_{\vec{p}}u_{\vec{p}}(\vec{x},t) + \cancel{\hat{b}^+_{\vec{p}}u^*_{\vec{p}}(\vec{x},t)}\}\{\hat{a}^+_{\vec{p}'}u^*_{\vec{p}'}(\vec{x}',t') + \cancel{\hat{b}_{\vec{p}'}u_{\vec{p}'}(\vec{x}',t')}\}|0\rangle$$

$$+\,\Theta(t'-t)$$

$$\cdot\langle 0|\{\cancel{\hat{a}^+_{\vec{p}'}u^*_{\vec{p}'}(\vec{x}',t')} + \hat{b}_{\vec{p}'}u_{\vec{p}'}(\vec{x}',t')\}\{\cancel{\hat{a}_{\vec{p}}u_{\vec{p}}(\vec{x},t)} + \hat{b}^+_{\vec{p}}u^*_{\vec{p}}(\vec{x},t)\}|0\rangle\Big\}$$

Note: $\hat{a}_{\vec{p}}|0\rangle = 0$, $\hat{b}_{\vec{p}}|0\rangle = 0$

$$\hat{a}_{\vec{p}}\hat{a}^+_{\vec{p}'} = \hat{a}^+_{\vec{p}'}\hat{a}_{\vec{p}} + \delta(\vec{p}-\vec{p}')\ ;\ \hat{b}_{\vec{p}'}\hat{b}^+_{\vec{p}} = \hat{b}^+_{\vec{p}}\hat{b}_{\vec{p}'} + \delta(\vec{p}-\vec{p}')$$

$$G(\vec x,t;\vec x',t') = \int d^3p\,\Big\{ \Theta(t-t')\, u_{\vec p}(\vec x,t)\, u^*_{\vec p}(\vec x',t') + \Theta(t'-t)\, u_{\vec p}(\vec x',t')\, u^*_{\vec p}(\vec x,t) \Big\}$$

$$u_{\vec p}(\vec x,t) = \sqrt{\frac{mc^2}{(2\pi\hbar)^3 E_{\vec p}}}\ \exp\Big\{\frac{i}{\hbar}\big[\vec p(\vec x-\vec x') - \underbrace{E_{\vec p}}_{}(t-t')\big]\Big\}\Big)$$

$$= \sqrt{\vec p^2c^2+m^2c^4}$$

$$= \frac{mc^2}{(2\pi\hbar)^3}\int d^3p\,\frac{1}{\sqrt{\vec p^2c^2+m^2c^4}}$$

$$\Big\{\Theta(t-t')\exp\Big\{\frac{i}{\hbar}\big[\vec p(\vec x-\vec x') - \sqrt{\vec p^2c^2+m^2c^4}\,(t-t')\big]\Big\}$$

$$+\Theta(t'-t)\underbrace{\exp\Big\{\frac{i}{\hbar}\big[\vec p(\vec x'-\vec x) - \sqrt{\vec p^2c^2+m^2c^4}\,(t'-t)\big]\Big\}}_{\vec p\to-\vec p\,:\ \vec p(\vec x-\vec x') - \sqrt{\vec p^2c^2+m^2c^4}\,(t'-t)}\Big\}\Big)$$

$$= \frac{mc^2}{(2\pi\hbar)^3}\int\frac{d^3p}{\sqrt{\vec p^2c^2+m^2c^4}}\,\exp\Big\{\frac{i}{\hbar}\big[\vec p(\vec x-\vec x') - \underbrace{\sqrt{\vec p^2c^2+m^2c^4}}_{=E_{\vec p}}\,|t-t'|\big]\Big\} \qquad \odot$$

- further calculations: spherical coordinates for momenta
- technical details: see later manuscript

$$G(\vec x,t;\vec x',t') = \frac{i\left(\frac{mc}{\hbar}\right)^2}{4\pi\sqrt{c^2(t-t') - (\vec x-\vec x')^2}}\,H_1^{(2)}\Big(\frac{mc}{\hbar}\sqrt{c^2(t-t')^2-(\vec x-\vec x')^2}\Big)$$

- Hankel function of second kind:

$$H_\nu^{(2)}(x) = \underbrace{J_\nu(x)}_{\text{Bessel function}} - i\underbrace{N_\nu(x)}_{\text{von Neumann function}} \qquad (8.405.2)$$

see formula collection book by Gradshteyn / Ryzhik

- Particle mass appears in terms of Compton wave length

$$\lambda_c = 2\pi\frac{\hbar}{mc}$$

Non-relativistic limit $c\to\infty \;\hat{=}\;$ large arguments in Hankel function

$$H_\nu^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}}\cdot e^{-i\left(x-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)},\ x\gg 1 \qquad (8.451.4)$$

Limit $c\to\infty$ for $t>t'$:

$$G(\vec x,t;\vec x',t') \xrightarrow{c\to\infty} \frac{i}{4\pi\cancel{c}(t-t')}\left(\frac{m\cancel{c}}{\hbar}\right)^2\sqrt{\frac{2}{\pi\frac{m\cancel{c}}{\hbar}\cancel{c}(t-t')}}\cdot e^{+i\frac{3\pi}{4}}$$

$$\cdot\exp\Big\{-i\frac{mc}{\hbar}c(t-t')\underbrace{\sqrt{1-\frac{(\vec x-\vec x')^2}{c^2(t-t')^2}}}_{=1-\frac{1}{2}\frac{(\vec x-\vec x')^2}{c^2(t-t')^2}}\Big\}$$

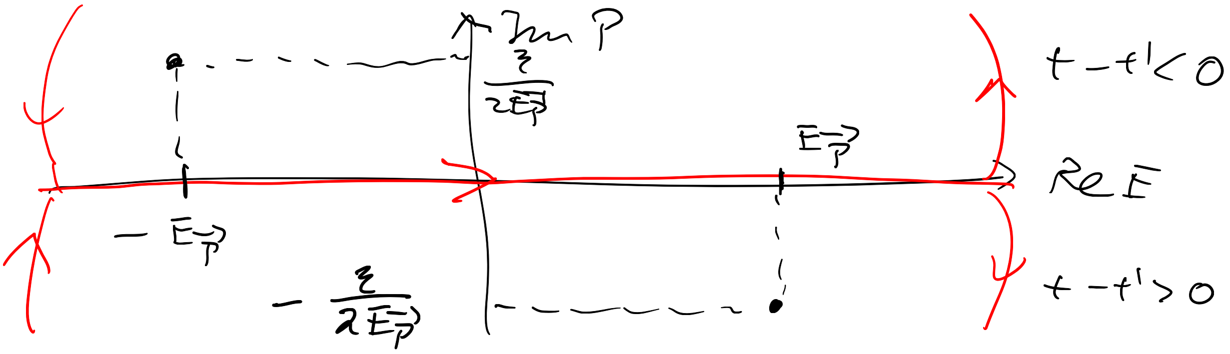
$$= \underbrace{\left(\frac{m}{2\pi i\hbar(t-t')}\right)^{\frac{3}{2}}\exp\Big\{\frac{i}{\hbar}\Big[\frac{m(\vec x-\vec x')^2}{2(t-t')}}_{\text{Schrödinger propagator}} - \underbrace{mc^2(t-t')\Big]\Big\}}_{\text{oscillation in time due to rest energy}}$$

7.12 Covariant Form of Propagator:

3D Fourier integral ⓪ ⇒ extend this 4D covariant Fourier integral

To this end: evaluate integral

$$I(t-t') = \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty}\frac{dE}{2\pi\hbar}\,\frac{e^{-\frac{i}{\hbar}E(t-t')}}{E^2-E_{\vec p}^2+\underbrace{i\varepsilon}_{\text{Feynman } i\varepsilon\text{-prescription}}}$$



convergence:

$$\left|e^{-\frac{i}{\hbar}E(t-t')}\right| = e^{-\frac{i}{\hbar}\,i\,\mathrm{Im}\,E(t-t')} = e^{\frac{1}{\hbar}\mathrm{Im}\,E(t-t')}$$

$$\mathrm{Im}\,E<0 \;\hat{=}\; t-t'>0\;;\quad \mathrm{Im}\,E>0 \;\hat{=}\; t-t'<0$$

$t>t'$:

$$I(t-t') = \frac{-2\pi i}{2\pi\hbar}\lim_{\varepsilon\downarrow 0}\underbrace{\mathrm{Res}_{E=\sqrt{E_{\vec p}^2-i\varepsilon}}\frac{e^{-\frac{i}{\hbar}E(t-t')}}{E^2-E_{\vec p}^2+i\varepsilon}}$$

$$=\lim_{E\to\sqrt{E_{\vec p}^2-i\varepsilon}}\frac{(E-\sqrt{E_{\vec p}^2-i\varepsilon})}{(E-\sqrt{E_{\vec p}^2-i\varepsilon})(E+\sqrt{E_{\vec p}^2-i\varepsilon})}\,e^{-\frac{i}{\hbar}E(t-t')}$$

$$=-\frac{i}{\hbar}\,\frac{1}{2E_{\vec p}}\cdot e^{-\frac{i}{\hbar}E_{\vec p}(t-t')}$$

$t<t'$:

$$I(t-t') = -\frac{i}{\hbar}\,\frac{1}{2E_{\vec p}}\,e^{+\frac{i}{\hbar}E_{\vec p}(t-t')}$$

combine both:

$$\lim_{\varepsilon\downarrow 0}\int_{-\infty}^{+\infty}\frac{dE}{2\pi\hbar}\,\frac{e^{-\frac{i}{\hbar}E(t-t')}}{E^2-E_{\vec p}^2+i\varepsilon} = \frac{-i}{2\hbar E_{\vec p}}\,e^{-\frac{i}{\hbar}E_{\vec p}|t-t'|}$$

$$G(\vec x,t;\vec x',t') = \frac{mc^2}{(2\pi\hbar)^3}\int d^3p\,e^{-\frac{i}{\hbar}\vec p(\vec x-\vec x')}\underbrace{\frac{e^{-\frac{i}{\hbar}E_{\vec p}|t-t'|}}{E_{\vec p}}}_{\text{insert here integral}}$$

$$= mc^2c\,2i\hbar\frac{1}{c^2}\lim_{\varepsilon\downarrow 0}\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{dE/c}{2\pi\hbar}\;\frac{e^{-\frac{i}{\hbar}\left[\frac{E}{c}(t-t')c-\vec p(\vec x-\vec x')\right]}}{\frac{1}{c^2}E^2-\frac{1}{c^2}E_{\vec p}^2+i\varepsilon}$$

$$G(x^\lambda;x'^\lambda) = 2i\hbar mc\lim_{\varepsilon\downarrow 0}\int\frac{d^4p}{(2\pi\hbar)^4}\;\frac{\exp\{-\frac{i}{\hbar}g_{\mu\nu}p^\mu(x^\nu-x'^\nu)\}}{g_{\mu\nu}p^\mu p^\nu-m^2c^2+i\varepsilon}$$

$$= \int \frac{d^4p}{(2\pi\hbar)^4} \underline{G(p^\lambda)} \, e^{-\frac{i}{\hbar} g_{\mu\nu} p^\mu (x^\nu - x'^\nu)}$$

$$\lim_{\varepsilon \downarrow 0} \frac{2i\hbar m c}{g_{\mu\nu} p^\mu p^\nu - m^2 c^2 + i\varepsilon}$$

singularity at Fourier transform: $E = E_{\vec{p}} = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$

non-relativistic limit $c \to \infty$

$$g(\vec{p}, E) = \lim_{c \to \infty} \frac{1}{c} G(\vec{p}, E + mc^2)$$

$$= \lim_{\varepsilon \downarrow 0} \lim_{c \to \infty} \frac{2i\hbar m}{\underline{\left(\frac{E}{c} + mc\right)^2} - \vec{p}^2 - \cancel{m^2 c^2} + i\varepsilon}$$

$$\underline{\left(\frac{E}{c}\right)^2} + 2mc\frac{E}{c} + \cancel{m^2 c^2}$$

$\to 0$ for $c \to \infty$

$$= \lim_{\varepsilon \downarrow 0} \frac{i\hbar}{E - \frac{\vec{p}^2}{2m} + i\varepsilon}$$

$$g(\vec{x}, t; \vec{x}', t') = \int \frac{d^3p}{(2\pi\hbar)^3} \int_{-\infty}^{+\infty} \frac{dE}{2\pi\hbar} \, g(\vec{p}, E) \, e^{\frac{i}{\hbar}\left[\vec{p}(\vec{x} - \vec{x}') - E(t - t')\right]}$$