$$\Rightarrow (p^\mu) = (p^0, \vec p) = \left(mc\cosh|\vec\zeta|, \ mc\frac{\vec\zeta}{\zeta}\ \sinh|\vec\zeta| \right)$$

$$\Rightarrow \cosh|\vec\zeta| = \frac{p^0}{mc}, \qquad \frac{\vec\zeta}{|\vec\zeta|} = \frac{\vec p}{mc}\ \frac{1}{\sinh|\vec\zeta|}$$

- half arguments:

$$\cosh\left(\frac{|\vec\zeta|}{2}\right) = \sqrt{\frac{\cosh|\vec\zeta|+1}{2}} = \sqrt{\frac{p^0+mc}{2mc}}, \qquad \sinh\left(\frac{|\vec\zeta|}{2}\right) = \sqrt{\frac{\cosh|\vec\zeta|-1}{2}} = \sqrt{\frac{p^0-mc}{2mc}}$$

$$\sinh|\vec\zeta| = 2\sinh\left(\frac{|\vec\zeta|}{2}\right)\cosh\left(\frac{|\vec\zeta|}{2}\right) = \frac{\sqrt{(p^0+mc)(p^0-mc)}}{mc}$$

- Boost representation matrix in terms of four-momentum vectors:

$$D(B(\vec\zeta)) = I\sqrt{\frac{p^0+mc}{2mc}} \mp \vec\sigma\cdot\frac{\vec p}{\cancel{mc}}\ \frac{\cancel{mc}}{\sqrt{(p^0+mc)(\cancel{p^0-mc})}}\sqrt{\frac{\cancel{p^0-mc}}{2mc}} = \frac{(p^0+mc)I \mp \vec\sigma\vec p}{\sqrt{2mc(p^0+mc)}}$$

- Four-vector of Pauli matrices:

$$(\sigma^\mu) = (\sigma^0, \sigma^k), \sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow D^{(1/2,0)}(B(\vec\zeta)) = e^{-\frac{1}{2}\vec\sigma\vec\zeta} = \frac{p\sigma + mc}{\sqrt{2mc(p^0+mc)}}$$

$$p\cdot\sigma = p_\mu\sigma^\mu = g_{\mu\nu}p^\mu\sigma^\nu = \sigma^0p^0 - \vec p\cdot\vec\sigma$$

- Spatially inverted four vector:

$$\tilde x = (\tilde x^0, \tilde x^k) = (x^0, -x^k), \qquad \tilde\sigma = (\tilde\sigma^0, \tilde\sigma^k) = (\sigma^0, -\sigma^k)$$

$$D^{(0,1/2)}(B(\vec\zeta)) = e^{+\frac{1}{2}\vec\sigma\vec p} = \frac{p\tilde\sigma + mc}{\sqrt{2mc(p^0+mc)}}$$

$$p\tilde\sigma = \tilde p\sigma = g_{\mu\nu}p^\mu\tilde\sigma^\nu = \sigma^0p^0 + \vec\sigma\cdot\vec p$$

- Additional calculation for later purposes:

$$e^{\mp\frac{1}{2}\vec\sigma\vec\zeta} = \sqrt{e^{\mp\vec\sigma\vec\zeta}} = \sqrt{I\cosh|\vec\zeta| \mp \frac{\vec\sigma\vec\zeta}{|\vec\zeta|}\sinh|\vec\zeta|} = \sqrt{\frac{p^0}{mc} \mp \frac{\vec\sigma\cdot\vec p}{mc}}$$

$$\Rightarrow e^{-\frac{1}{2}\vec\sigma\vec\zeta} = \sqrt{\frac{p^0\sigma^0 - \vec\sigma\vec p}{mc}} = \sqrt{\frac{p\sigma}{mc}}, \qquad e^{\frac{1}{2}\vec\sigma\vec\zeta} = \sqrt{\frac{p^0\sigma^0 + \vec\sigma\vec p}{mc}} = \sqrt{\frac{p\tilde\sigma}{mc}}$$

9.5 Lorentz Invariant Combinations of Weyl Spinors:

- $D^{(1/2,0)}, D^{(0,1/2)}$: smallest non-trivial representations of Lorentz group
- Weyl spinors upon which representation matrices act:

$$\zeta^\alpha(x) \longrightarrow \zeta'_\alpha(x') = D^{(1/2,0)}(\Lambda)_\alpha{}^\beta\zeta_\beta(x)$$

$$\xi^{\dot\alpha}(x) \longrightarrow \xi'_\alpha(x') = D^{(0,1/2)}(\Lambda)^{\dot\alpha}{}_\beta\,\xi^{\dot\beta}(x)$$

- Aim: Search for Lorentz invariant action on the basis of these Weyl spinors. Restrict to quadratic terms in Weyl spinors and their first partial derivatives
- First: no partial derivative ≙ <u>mass term</u>

four combinations: $\zeta^+\zeta, \ \xi^+\xi, \ \underline{\xi^+\zeta, \ \zeta^+\xi}$

Apply Lorentz transformation:

$$\zeta^+D^{(1/2,0)}(\Lambda)^+D^{(1/2,0)}(\Lambda)\zeta, \quad \xi^+D^{(0,1/2)}(\Lambda)^+D^{(0,1/2)}(\Lambda)\xi$$

$$\underline{\xi^+D^{(0,1/2)}(\Lambda)^+D^{(1/2,0)}(\Lambda)\zeta}, \quad \underline{\zeta^+D^{(1/2,0)}(\Lambda)^+D^{(0,1/2)}(\Lambda)\xi}$$

- Rotations: $\Lambda = R$
  - $D^{(1/2,0)}(R) = D^{(0,1/2)}(R) = D(R)$
  - $D(R)^+ = D(R)^{-1} \Rightarrow D(R)^+D(R) = 1$

  ⇒ all four transformed combinations agree with original combinations
- Boosts: $\Lambda = B$
  - $D^{(1/2,0)}(B) = e^{-\frac{1}{2}\vec\sigma\vec\zeta}$, $D^{(0,1/2)}(B) = e^{\frac{1}{2}\vec\sigma\vec\zeta} \Rightarrow D^{(1/2,0)}(B)^{-1} = D^{(0,1/2)}(B)$
  - $D(B)^+ = D(B)$

  $\Rightarrow D^{(1/2,0)}(B)^+D^{(0,1/2)}(B) = D^{(0,1/2)}(B)^+D^{(1/2,0)}(B) = 1$

  ⇒ From all transformed combinations only last two agree with original combination
- Result: Lorentz-invariant action without partial derivatives is only possible by combining both Weyl spinors ≙ mass term
- Second step: motion of particle in space-time needs partial derivatives
- Restrict ourselves at first to spatial derivatives:

four combinations: $\zeta^+\sigma^k\partial_k\zeta, \ \xi^+\sigma^k\partial_k\xi, \ \xi^+\sigma^k\partial_k\zeta, \ \zeta^+\sigma^k\partial_k\xi$

- Apply Lorentz transformation Λ:

$$\zeta^+D^{(1/2,0)}(\Lambda)^+\sigma^kD^{(1/2,0)}(\Lambda)\partial'_k\zeta, \quad \xi^+D^{(0,1/2)}(\Lambda)^+\sigma^kD^{(0,1/2)}(\Lambda)\partial'_k\xi$$

$$\xi^+D^{(0,1/2)}(\Lambda)^+\sigma^kD^{(1/2,0)}(\Lambda)\partial'_k\zeta, \quad \zeta^+D^{(1/2,0)}(\Lambda)^+\sigma^kD^{(0,1/2)}(\Lambda)\partial'_k\xi$$

- Rotations: $\Lambda = R$, $D^{(1/2,0)}(R) = D^{(0,1/2)}(R) = D(R)$

$$\Rightarrow D(R)^+\sigma^kD(R) = ?$$

- Insertion of expression for $D(R)$:

$$=\left\{\cos\left(\frac{|\vec\varphi|}{2}\right)+i\frac{\vec\sigma\vec\varphi}{|\vec\varphi|}\sin\left(\frac{|\vec\varphi|}{2}\right)\right\}\sigma^k\left\{\cos\left(\frac{|\vec\varphi|}{2}\right)-i\frac{\vec\sigma\vec\varphi}{|\vec\varphi|}\sin\left(\frac{|\vec\varphi|}{2}\right)\right\}$$

$$=\underbrace{\cos^2\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}(1+\cos|\vec\varphi|)}\sigma^k+i\underbrace{\sin\left(\frac{|\vec\varphi|}{2}\right)\cos\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}\sin|\vec\varphi|}\frac{\varphi^l}{|\vec\varphi|}\underbrace{[\sigma^l,\sigma^k]_-}_{=2i\varepsilon_{lkm}\sigma^m}+\underbrace{\sin^2\left(\frac{|\vec\varphi|}{2}\right)}_{=\frac{1}{2}(1-\cos|\vec\varphi|)}\frac{\varphi^l\varphi^m}{|\vec\varphi|^2}\underbrace{\sigma^l\sigma^k\sigma^m}_{=?}$$

- Side calculation:

$$(\sigma^l\sigma^k)\sigma^m\overset{(*)}{=}(\delta_{lk}+i\varepsilon_{lkn}\sigma^n)\sigma^m=\delta_{lk}\sigma^m+i\varepsilon_{lkn}\underbrace{\sigma^n\sigma^m}_{=\delta_{nm}+i\varepsilon_{nmp}\sigma^p}$$

$$=\delta_{lk}\sigma^m+i\varepsilon_{lkm}-\underbrace{\varepsilon_{lkn}\varepsilon_{mpn}}_{=\delta_{lm}\delta_{kp}-\delta_{lp}\delta_{km}}\sigma^p$$

$$=i\varepsilon_{lkm}+\delta_{lk}\sigma^m-\delta_{lm}\sigma^k+\delta_{km}\sigma^l$$

- Insertion of side calculation:

$$D(R)^+\sigma^kD(R)=\frac{1}{2}(1+\cos|\vec\varphi|)\sigma^k-\overset{=-\varepsilon_{klm}}{\varepsilon_{lkm}}\sin|\vec\varphi|\frac{\varphi^l}{|\vec\varphi|}\sigma^m$$

$$+\frac{1}{2}(1-\cos|\vec\varphi|)\underbrace{\frac{\varphi^l\varphi^m}{|\vec\varphi|^2}\left[i\cancel{\varepsilon_{lkm}}+\delta_{lk}\sigma^m-\delta_{lm}\sigma^k+\delta_{km}\sigma^l\right]}_{=\frac{1}{|\vec\varphi|^2}\left\{2\varphi^k(\vec\varphi\cdot\vec\sigma)-|\vec\varphi|^2\sigma^k+\cancel{(\vec\varphi\cdot\vec\sigma)\varphi^k}\right\}}$$

$$=\underbrace{\cos|\vec\varphi|\,\sigma^k+\varepsilon_{klm}\frac{\varphi^l}{|\vec\varphi|}\sin|\vec\varphi|\,\sigma^m+(1-\cos|\vec\varphi|)\frac{\varphi^k}{|\vec\varphi|}\frac{(\vec\sigma\cdot\vec\varphi)}{|\vec\varphi|}}_{R_{kl}\sigma^l}$$

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representation of rotation matrix in 3D space

- Result:

$$D(R)^+\sigma^kD(R)\partial'_k=R_{kl}\sigma^l\cdot R_{km}\partial_m=\underbrace{R_{kl}R_{km}}_{=\delta_{lm}}\cdot\sigma^l\partial_m=\sigma^k\partial_k$$

=> All transformed ψ combinations agree with original combinations

- How to generalise these combinations relativistically?

$$\sigma^k\ \begin{matrix}\nearrow & (\sigma^\mu)=(\sigma^0,\sigma^k)\\ \searrow & (\tilde\sigma^\mu)=(\sigma^0,-\sigma^k)\end{matrix}$$

=> We have two relativistic extensions

=> eight combinations of two Weyl spinors with spatio-temporal derivatives:

$\zeta^+\sigma^\mu\partial_\mu\zeta$, $\xi^+\sigma^\mu\partial_\mu\xi$, $\cancel{\xi^+\sigma^\mu\partial_\mu\zeta}$, $\cancel{\zeta^+\sigma^\mu\partial_\mu\xi}$

$\zeta^+\tilde\sigma^\mu\partial_\mu\zeta$, $\xi^+\tilde\sigma^\mu\partial_\mu\xi$, $\cancel{\xi^+\tilde\sigma^\mu\partial_\mu\zeta}$, $\cancel{\zeta^+\tilde\sigma^\mu\partial_\mu\xi}$

- additional term $\sigma^0\partial_0=\tilde\sigma^0\partial_0$ is invariant with respect to rotations:

$$D(R)^+\underbrace{\sigma^0}_{\text{unity matrix}}D(R)\partial'_0=\underbrace{D(R)^+D(R)}_{=1\ (\text{unitary})}\underbrace{\partial'_0}_{=\partial_0}=\partial_0=\sigma^0\partial_0$$

- But: What about invariance under boosts?

$$D(B)^+\sigma^\mu D(B)=?\qquad D(B)^+\tilde\sigma^\mu D(B)=?\quad\text{with}\quad D(B)=D^{(1/2,0)}(B)\ \text{or}\ D^{(0,1/2)}(B)$$

- Special case: $\mu=0$ and left/right representation are different

$$\underbrace{D^{(1/2,0)}(B)^+}_{=D^{(1/2,0)}(B)}\underbrace{\sigma^0}_{\text{unity}}D^{(0,1/2)}(B)=\underbrace{D^{(1/2,0)}(B)}_{=D^{(0,1/2)}(B)^{-1}}\cdot D^{(0,1/2)}(B)=1=\sigma^0$$

$$D^{(0,1/2)+}(B)\sigma^0D^{(1/2,0)}(B)=1=\sigma^0$$

=> this does not correspond to a transformation property of a ~~boost~~ to a vector

=> Result: 3./4./7./8. term of the eight combinations have to be excluded!

=> have to be identical

=> left/right Weyl spinor

$$\underbrace{D(B)^+}_{=D(B)}\underbrace{\sigma^0}_{\text{unity}}D(B)=D(B)^2=\left(e^{\mp\frac{1}{2}\vec\sigma\vec\zeta}\right)^2=e^{\mp\vec\sigma\vec\zeta}\overset{(**)}{=}\cosh|\vec\zeta|\mp\frac{\vec\sigma\cdot\vec\zeta}{|\vec\zeta|}\sinh|\vec\zeta|\quad\text{(I)}$$

$$D(B)^+\sigma^kD(B)\overset{(**)}{=}\left\{\cosh\left(\frac{|\vec\zeta|}{2}\right)\mp\frac{\vec\sigma\vec\zeta}{|\vec\zeta|}\sinh\left(\frac{|\vec\zeta|}{2}\right)\right\}\sigma^k\left\{\cosh\left(\frac{|\vec\zeta|}{2}\right)\mp\frac{\vec\sigma\vec\zeta}{|\vec\zeta|}\sinh\left(\frac{|\vec\zeta|}{2}\right)\right\}$$

= . . . quite analogous to rotations . . . $=\sigma^k\mp\frac{\zeta^k}{|\vec\zeta|}\sinh|\vec\zeta|+\frac{\zeta^k}{|\vec\zeta|}\frac{\vec\zeta\vec\sigma}{|\vec\zeta|}(\cosh|\vec\zeta|-1)$

$$D(B)^+(\mp\sigma^k)D(B)=\mp\sigma^k+\frac{\zeta^k}{|\vec\zeta|}\sinh|\vec\zeta|+(\cosh|\vec\zeta|-1)\frac{\zeta^k}{|\vec\zeta|}\frac{\vec\zeta\cdot(\mp\vec\sigma)}{|\vec\zeta|}\quad\text{(II)}$$

- summary of (I) and (II):

$$D^{(1/2,0)}(B)^{+}\,\tilde{\sigma}^{\mu}\,D^{(\frac{1}{2},0)}(B) = B^{\mu}{}_{\nu}\,\tilde{\sigma}^{\nu}$$
$$D^{(0,1/2)}(B)^{+}\,\sigma^{\mu}\,D^{(0,1/2)}(B) = B^{\mu}{}_{\nu}\,\sigma^{\nu} \quad \text{with } (B^{\mu}{}_{\nu}) = \begin{pmatrix} \cosh|\zeta| & \frac{\zeta^{l}}{|\zeta|}\sinh|\zeta| \\ \frac{\zeta^{k}}{|\zeta|}\sinh|\zeta| & \delta_{kl}+(\cosh|\zeta|-1)\frac{\zeta^{k}\zeta^{l}}{|\zeta|^{2}} \end{pmatrix}$$

chapter 6 + problem sheet 3

- Transformation of partial derivatives: $\partial_{\mu} \xrightarrow{B} \partial'_{\mu} = B_{\mu}{}^{\nu}\,\partial_{\nu}$
- Result:

1) $$\zeta^{+}\tilde{\sigma}^{\mu}\partial_{\mu}\zeta \xrightarrow{B} \zeta^{+}\underbrace{D^{(\frac{1}{2},0)}(B)^{+}\,\tilde{\sigma}^{\mu}\,D^{(\frac{1}{2},0)}(B)}_{=B^{\mu}{}_{\nu}\,\tilde{\sigma}^{\nu}}\,\underbrace{\partial'_{\mu}}_{B_{\mu}{}^{\lambda}\partial_{\lambda}}\zeta = \zeta^{+}B^{\mu}{}_{\nu}\,B_{\mu}{}^{\lambda}\,\tilde{\sigma}^{\nu}\,\partial_{\lambda}\zeta$$

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Extra calculation: some additional material for chapter 6

definition of Lorentz transformation:

$$\Lambda^{T}\,g\,\Lambda = g \quad \Leftrightarrow \quad \Lambda^{\sigma}{}_{\mu}\,g_{\sigma\rho}\,\Lambda^{\rho}{}_{\nu} = g_{\mu\nu} \quad \Big|\cdot g^{\nu\lambda}$$

1) $$g_{\mu\nu}\,g^{\nu\lambda} = g_{\mu}{}^{\lambda} \underset{\text{chap. 6}}{=} \delta^{\lambda}{}_{\mu} = \delta_{\mu}{}^{\lambda}$$

2) $$\Lambda^{\sigma}{}_{\mu}\,g_{\sigma\rho}\,\Lambda^{\rho}{}_{\nu}\,g^{\nu\lambda} = \Lambda^{\sigma}{}_{\mu}\,g_{\sigma\rho}\,\Lambda^{\rho\lambda} = \Lambda^{\sigma}{}_{\mu}\,\underbrace{g_{\sigma}{}^{\rho}}_{=\delta_{\sigma}{}^{\rho}}\,\Lambda_{\rho}{}^{\lambda} = \Lambda^{\rho}{}_{\mu}\,\Lambda_{\rho}{}^{\lambda} \overset{\text{chap. 6}}{=} (\Lambda^{T})_{\mu}{}^{\rho}\,\Lambda_{\rho}{}^{\lambda}$$

$$\boxed{\Lambda^{T} = \Lambda^{-1}}$$

1. Remark: rotations
$$e^{-i\vec{\tau}\vec{\varphi}} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right) \Rightarrow R^{T} = R^{-1} \quad \text{orthonormality}$$

2. Remark: boosts $B^{\rho}{}_{\mu}\,B_{\rho}{}^{\lambda} = \delta_{\mu}{}^{\lambda}$

- -

Return: $$\zeta^{+}\tilde{\sigma}^{\mu}\partial_{\mu}\zeta \Rightarrow \zeta^{+}\underbrace{B^{\mu}{}_{\nu}\,B_{\mu}{}^{\lambda}}_{=\delta_{\nu}{}^{\lambda}}\,\tilde{\sigma}^{\nu}\,\partial_{\lambda}\zeta = \zeta^{+}\tilde{\sigma}^{\nu}\partial_{\nu}\zeta$$ ✓ invariance

2) $\xi^{+}\sigma^{\mu}\partial_{\mu}\xi = \ldots$ similar calculation $\ldots = \xi^{+}\sigma^{\mu}\partial_{\mu}\xi$ ✓

⇒ 2./5. term are invariant under boost

⇒ 1./6. " " not " " "

9.6 Dirac Action:

- Ansatz for Lorentz-invariant action for describing massive spin 1/2 particles

$$S = S[\zeta(\cdot), \zeta^{+}(\cdot); \xi(\cdot), \xi^{+}(\cdot)]$$
$$= \frac{1}{c}\int \underbrace{d^{4}x}_{=c\,dt\,d^{3}x}\, \mathcal{L}\big(\zeta(x), \partial_{\mu}\zeta(x), \zeta^{+}(x), \partial_{\mu}\zeta^{+}(x); \xi(x), \partial_{\mu}\xi(x), \xi^{+}(x), \partial_{\mu}\xi^{+}(x)\big)$$

Section 9.5 ⇒ $$\mathcal{L} = \alpha\, i\,\zeta^{+}\tilde{\sigma}^{\mu}\partial_{\mu}\zeta + i\beta\,\xi^{+}\sigma^{\mu}\partial_{\mu}\xi + \gamma\,\zeta^{+}\xi + \delta\,\xi^{+}\zeta$$

$\alpha, \beta, \gamma, \delta$: constants not yet defined

- Later on: additional demand on invariance with respect to parity transformation leads to the fact that both Weyl spinors ζ and ξ have to appear on equal footing

$$\Rightarrow \mathcal{L} = \alpha\big(i\zeta^{+}\tilde{\sigma}^{\mu}\partial_{\mu}\zeta + i\xi^{+}\sigma^{\mu}\partial_{\mu}\xi - m\,\zeta^{+}\zeta - m\,\xi^{+}\xi\big)$$

α, m: constants not yet determined, define physical dimension
⇒ fixed by comparison with non-relativistic limit later on

- Equations of motion:

$$\frac{\delta S}{\delta\zeta^{+}(x)} = \frac{\partial\mathcal{L}}{\partial\zeta^{+}(x)} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\zeta^{+}(x))} = i\tilde{\sigma}^{\mu}\partial_{\mu}\zeta - m\xi = 0 \quad (1)$$
$$\frac{\delta S}{\delta\xi^{+}(x)} = \frac{\partial\mathcal{L}}{\partial\xi^{+}(x)} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\xi^{+}(x))} = i\sigma^{\mu}\partial_{\mu}\xi - m\zeta = 0 \quad (2)$$

- Calculational rules:

$$\sigma^{\mu}\tilde{\sigma}^{\nu} + \sigma^{\nu}\tilde{\sigma}^{\mu} = 2g^{\mu\nu}I, \quad \tilde{\sigma}^{\mu}\sigma^{\nu} + \tilde{\sigma}^{\nu}\sigma^{\mu} = 2g^{\mu\nu}I \quad (\cancel{*}\cancel{*}\cancel{*})$$

specialize μ,ν to spatial/~~temporal~~ components

- multiply (1) with $i\sigma^{\nu}\partial_{\nu}$ from the left:

$$-\underbrace{\sigma^{\nu}\tilde{\sigma}^{\mu}}\partial_{\mu}\partial_{\nu}\zeta - \underbrace{m\,i\sigma^{\nu}\partial_{\nu}\xi}_{m\zeta} = 0 \Rightarrow g^{\mu\nu}\partial_{\mu}\partial_{\nu}\zeta + m^{2}\zeta = 0$$
$$= \frac{1}{2}(\sigma^{\nu}\tilde{\sigma}^{\mu} + \sigma^{\mu}\tilde{\sigma}^{\nu})$$
$$= \frac{1}{2}g^{\mu\nu}I$$

analogous: $g^{\mu\nu}\partial_{\mu}\partial_{\nu}\xi + m^{2}\xi = 0$

- Result: ζ and ξ fulfill a Klein-Gordon equation ⇒ $m = \frac{Mc}{\hbar}$

- massive spin 1/2 particles involve both Weyl spinors ζ and ξ, so they can be combined to a Dirac spinor
$$\Psi(x) = \begin{pmatrix} \zeta(x) \\ \xi(x) \end{pmatrix}$$
- Rewrite Dirac Lagrange density:
$$\mathcal{L} = \alpha \left\{ \underbrace{(\zeta^+(x), \xi^+(x))}_{=\Psi^+(x)} \begin{pmatrix} \tilde{\sigma}^\mu & \mathbf{0} \\ \mathbf{0} & \sigma^\mu \end{pmatrix} i\partial_\mu \begin{pmatrix} \zeta(x) \\ \xi(x) \end{pmatrix} - \overbrace{(\zeta^+(x)\ \xi^+(x))}^{\Psi^+(x)} \begin{pmatrix} \mathbf{0} & mI \\ mI & \mathbf{0} \end{pmatrix} \begin{pmatrix} \zeta(x) \\ \xi(x) \end{pmatrix} \right\}$$
2×2 zero matrix: $\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, 2×2 unity matrix: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Dirac adjoint Dirac spinor:
$$\overline{\Psi}(x) \overset{!}{=} (\xi^+(x), \zeta^+(x)) = \Psi^+(x) \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$$
$$\Psi^+(x) = (\zeta^+(x), \xi^+(x)) = \overline{\Psi}(x) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$\mathcal{L} = \alpha \Big\{ \overline{\Psi}(x) \underbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}}_{= \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} = \gamma^\mu \text{ Dirac matrices}} i\partial_\mu \Psi(x) - \overline{\Psi}(x) \underbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & mI \\ mI & 0 \end{pmatrix}}_{m \cdot \underbrace{\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}}_{4\times 4 \text{ unity matrix}} \equiv m} \Psi(x) \Big\}$$
$$\mathcal{L} = \alpha \overline{\Psi} \big(i \underbrace{\gamma^\mu \partial_\mu}_{= \not{\partial} \text{ (Feynman dagger) as a shortcut notation}} - m \big) \Psi \qquad \not{a} = \gamma^\mu a_\mu$$
- Clifford algebra:
$$[\gamma^\mu, \gamma^\nu]_+ = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \tilde{\sigma}^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \tilde{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}$$
$$= \begin{pmatrix} [\sigma^\mu, \tilde{\sigma}^\nu]_+ & 0 \\ 0 & [\tilde{\sigma}^\mu, \sigma^\nu] \end{pmatrix} \overset{(***)}{=\!=} 2g^{\mu\nu} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \equiv 2g^{\mu\nu}$$
- Result: $\mathcal{A} = \mathcal{A}[\Psi(\cdot); \overline{\Psi}(\cdot)]$
$$= \frac{1}{c} \int d^4x\, \mathcal{L}\big(\Psi(x), \partial_\mu \Psi(x); \overline{\Psi}(x), \partial_\mu \overline{\Psi}(x)\big)$$
$$\frac{\delta \mathcal{A}}{\delta \overline{\Psi}(x)} = \frac{\partial \mathcal{L}}{\partial \overline{\Psi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\Psi}(x))} = \alpha \big(i \underbrace{\gamma^\mu \partial_\mu}_{= \not{\partial}} - m \big) \Psi(x) = 0$$
Dirac equation: $(i\not{\partial} - m)\Psi(x) = 0$