$D^{(1/2,0)}(B)^+ \tilde\sigma^\mu D^{(\frac12,0)}(B) = B^\mu{}_\nu \tilde\sigma^\nu$

$D^{(0,1/2)}(B)^+ \sigma^\mu D^{(0,1/2)}(B) = B^\mu{}_\nu \sigma^\nu$ with $(B^\mu{}_\nu) = \begin{pmatrix} \cosh|\vec\zeta| & \frac{\vec\zeta^T}{|\vec\zeta|}\sinh|\vec\zeta| \\ \frac{\vec\zeta}{|\vec\zeta|}\sinh|\vec\zeta| & \mathbb{1} + (\cosh|\vec\zeta|-1)\frac{\vec\zeta\vec\zeta^T}{|\vec\zeta|^2}\end{pmatrix}$

chapter 6 + problem sheet 3

- Transformation of spatial derivatives: $\partial_\mu \xrightarrow{B} \partial'_\mu = B_\mu{}^\nu \partial_\nu$
- Result:

1) $\xi^+ \tilde\sigma^\mu \partial_\mu \xi \xrightarrow{B} \xi^+ \underbrace{D^{(\frac12,0)}(B)^+\tilde\sigma^\mu D^{(\frac12,0)}(B)}_{=B^\mu{}_\nu\tilde\sigma^\nu} \underbrace{\partial'_\mu}_{B_\mu{}^\varkappa\partial_\varkappa}\xi = \xi^+ B^\mu{}_\nu B_\mu{}^\varkappa \tilde\sigma^\nu \partial_\varkappa \xi$

Side calculation: some additional material for chapter 6

definition of Lorentz transformation:

$\Lambda^T g \Lambda = g \Leftrightarrow \Lambda^\sigma{}_\mu g_{\sigma\rho}\Lambda^\rho{}_\nu = g_{\mu\nu} \quad | \cdot g^{\nu\varkappa}$

1) $g_{\mu\nu} g^{\nu\varkappa} = g_\mu{}^\varkappa \underset{\text{chap. 6}}{=} \delta^\varkappa{}_\mu = \delta_\mu{}^\varkappa$

2) $\Lambda^\sigma{}_\mu g_{\sigma\rho}\Lambda^\rho{}_\nu g^{\nu\varkappa} = \Lambda^\sigma{}_\mu g_{\sigma\rho}\Lambda^{\rho\varkappa} = \Lambda^\sigma{}_\mu \underbrace{g_\sigma{}^\rho}_{=\delta_\sigma{}^\rho}\Lambda_\rho{}^\varkappa = \Lambda^\rho{}_\mu\Lambda_\rho{}^\varkappa \overset{\text{chap. 6}}{=} (\Lambda^T)_\mu{}^\rho \Lambda_\rho{}^\varkappa$

$\Lambda^T = \Lambda^{-1}$

1. Remark: rotations $e^{-i\vec\tau\vec\varphi} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \Rightarrow R^T = R^{-1}$ orthonormality

2. Remark: boosts $B^\rho{}_\mu B_\rho{}^\varkappa = \delta_\mu{}^\varkappa$

Return: $\xi^+\tilde\sigma^\mu\partial_\mu\xi \Rightarrow \xi^+ \underbrace{B^\mu{}_\nu B_\mu{}^\varkappa}_{=\delta_\nu{}^\varkappa}\tilde\sigma^\nu\partial_\varkappa\xi = \xi^+\tilde\sigma^\nu\partial_\nu\xi$ ✓ invariance

2) $\eta^+\sigma^\mu\partial_\mu\eta = \ldots$ similar calculation $\ldots = \eta^+\sigma^\mu\partial_\mu\eta$ ✓

⇒ 2./5. term are invariant under boost

⇒ 1./6. " " not " " "

9.6 Dirac Action:

Lorentz-invariant action for describing massive spin 1/2 particles

- Ansatz for Lorentz-invariant action

$$A = A[\xi(\cdot),\xi^+(\cdot);\eta(\cdot),\eta^+(\cdot)] = \frac1c\int \underbrace{d^4x}_{=c\,dt\,d^3x}\, \mathcal L(\xi(x),\partial_\mu\xi(x),\xi^+(x),\partial_\mu\xi^+(x);\eta(x),\partial_\mu\eta(x),\eta^+(x),\partial_\mu\eta^+(x))$$

Section 9.5 ⇒ $\mathcal L = A\,i\,\xi^+\tilde\sigma^\mu\partial_\mu\xi + iB\eta^+\sigma^\mu\partial_\mu\eta + C\xi^+\eta + D\eta^+\xi$

A, B, C, D: constants not yet defined

- Later on: additional demand on invariance with respect to parity transformation leads to the fact that both key spinors ξ and η have to appear on equal footing

⇒ $\mathcal L = \alpha(i\xi^+\tilde\sigma^\mu\partial_\mu\xi + i\eta^+\sigma^\mu\partial_\mu\eta - m\,\xi^+\eta - m\,\eta^+\xi)$

α, m: constants not yet determined, define physical dimension

⇒ fixed by comparison with non-relativistic limit later on

- Equations of motion:

$$\frac{\delta A}{\delta\xi^+(x)} = \frac{\partial\mathcal L}{\partial\xi^+(x)} - \partial_\mu\frac{\partial\mathcal L}{\partial(\partial_\mu\xi^+(x))} = \alpha(i\tilde\sigma^\mu\partial_\mu\xi - m\eta) = 0 \quad (1)$$

$$\frac{\delta A}{\delta\eta^+(x)} = \frac{\partial\mathcal L}{\partial\eta^+(x)} - \partial_\mu\frac{\partial\mathcal L}{\partial(\partial_\mu\eta^+(x))} = \alpha(i\sigma^\mu\partial_\mu\eta - m\xi) = 0 \quad (2)$$

- Calculational rules:

$\sigma^\mu\tilde\sigma^\nu + \sigma^\nu\tilde\sigma^\mu = 2g^{\mu\nu}I$, $\quad\tilde\sigma^\mu\sigma^\nu+\tilde\sigma^\nu\sigma^\mu = 2g^{\mu\nu}I$ $\quad(\ast\ast\ast)$

specialize μ,ν to spatial/temporal components

- multiply (1) with $i\sigma^\nu\partial_\nu$ from the left:

$-\sigma^\nu\tilde\sigma^\mu\partial_\mu\partial_\nu\xi - m\underbrace{i\sigma^\nu\partial_\nu\eta}_{m\xi} = 0 \Rightarrow g^{\mu\nu}\partial_\mu\partial_\nu\xi + m^2\xi = 0$

$= \frac12(\sigma^\nu\tilde\sigma^\mu+\sigma^\mu\tilde\sigma^\nu)$

$= \frac12 g^{\mu\nu} I$

analogous: $g^{\mu\nu}\partial_\mu\partial_\nu\eta + m^2\eta = 0$

- Result: ξ and η fulfill a Klein-Gordon equation ⇒ $m = \frac{Mc}{\hbar}$

– Massive spin 1/2 particles involve both Weyl-spinors ζ and ξ, so they can be combined to a Dirac spinor

$$\psi(x) = \begin{pmatrix}\zeta(x)\\ \xi(x)\end{pmatrix}$$

– Rewrite Dirac Lagrange density:

$$\mathcal{L} = \hbar\left\{ \underbrace{(\zeta^+(x), \xi^+(x))}_{=\psi^+(x)} \begin{pmatrix}\tilde\sigma^\mu & O\\ O & \sigma^\mu\end{pmatrix} i\partial_\mu \begin{pmatrix}\zeta(x)\\ \xi(x)\end{pmatrix} - \overbrace{(\zeta^+(x)\ \xi^+(x))}^{\psi^+(x)} \begin{pmatrix}O & mI\\ mI & O\end{pmatrix}\begin{pmatrix}\zeta(x)\\ \xi(x)\end{pmatrix}\right\}$$

2×2 zero matrix: $O = \begin{pmatrix}0&0\\0&0\end{pmatrix}$, 2×2 unity matrix: $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

– Dirac adjoint Dirac spinor:

$$\bar\psi(x) \hat{=} (\xi^+(x), \zeta^+(x)) = \psi^+(x)\begin{pmatrix}O & I\\ I & O\end{pmatrix}$$

$$\psi^+(x) = (\zeta^+(x), \xi^+(x)) = \bar\psi(x)\begin{pmatrix}O & I\\ I & O\end{pmatrix}$$

$$\mathcal{L} = \hbar\left\{\bar\psi(x)\underbrace{\begin{pmatrix}0&I\\I&0\end{pmatrix}\begin{pmatrix}\tilde\sigma^\mu&0\\0&\sigma^\mu\end{pmatrix}}_{=\begin{pmatrix}0&\sigma^\mu\\\tilde\sigma^\mu&0\end{pmatrix}=\gamma^\mu \text{ Dirac matrices}} i\partial_\mu\psi(x) - \bar\psi(x)\underbrace{\begin{pmatrix}0&I\\I&0\end{pmatrix}\begin{pmatrix}0&mI\\mI&0\end{pmatrix}}_{m\cdot\underbrace{\begin{pmatrix}I&0\\0&I\end{pmatrix}}_{4\times4 \text{ unity matrix}}\equiv m}\psi(x)\right\}$$

$$\mathcal{L} = \hbar\bar\psi\,(i\underbrace{\gamma^\mu\partial_\mu}_{=\not\partial} - m)\,\psi \qquad \not a = \gamma^\mu a_\mu$$

(Feynman dagger) as a shortcut notation

– Clifford algebra:

$$[\gamma^\mu,\gamma^\nu]_+ = \gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = \begin{pmatrix}0&\sigma^\mu\\\tilde\sigma^\mu&0\end{pmatrix}\begin{pmatrix}0&\sigma^\nu\\\tilde\sigma^\nu&0\end{pmatrix} + \begin{pmatrix}0&\sigma^\nu\\\tilde\sigma^\nu&0\end{pmatrix}\begin{pmatrix}0&\sigma^\mu\\\tilde\sigma^\mu&0\end{pmatrix}$$

$$= \begin{pmatrix}[\sigma^\mu,\tilde\sigma^\nu]_+ & 0\\ 0 & [\tilde\sigma^\mu,\sigma^\nu]_+\end{pmatrix} \overset{(***)}{=} 2g^{\mu\nu}\begin{pmatrix}I&0\\0&I\end{pmatrix} \equiv 2g^{\mu\nu}$$

– Result:

$$\mathcal{A} = \mathcal{A}[\psi(\cdot);\bar\psi(\cdot)] = \frac{1}{c}\int d^4x\,\mathcal{L}(\psi(x),\partial_\mu\psi(x);\bar\psi(x),\partial_\mu\bar\psi(x))$$

$$\frac{\delta\mathcal{A}}{\delta\bar\psi(x)} = \frac{\partial\mathcal{L}}{\partial\bar\psi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi(x))} = \hbar\,(i\underbrace{\gamma^\mu\partial_\mu}_{=\not\partial} - m)\,\psi(x) = 0$$

Dirac equation: $(i\not\partial - m)\psi(x) = 0$

9.12 Spinor Representation of Lorentz Group:

– By construction Dirac action is invariant under Lorentz transformations

– Aim: Prove this again by studying the representation of the Lorentz group in the space of Dirac spinors

$$\psi(x) = \begin{pmatrix}\zeta(x)\\\xi(x)\end{pmatrix} \xrightarrow{\Lambda} \psi'(x') = \begin{pmatrix}\zeta'(x')\\\xi'(x')\end{pmatrix} = \underbrace{D(\Lambda)}_{\text{representation matrix for Dirac spinor}}\psi(x)$$

$$\Rightarrow D(\Lambda) = \begin{pmatrix}D^{(1/2,0)}(\Lambda) & O\\ O & D^{(0,1/2)}(\Lambda)\end{pmatrix}$$

– Lorentz transformation of Dirac adjoint Dirac spinor:

$$\gamma^\mu = \begin{pmatrix}0&\sigma^\mu\\\tilde\sigma^\mu&0\end{pmatrix},\quad \gamma^0 = \begin{pmatrix}0&I\\I&0\end{pmatrix},\quad \tilde\sigma^0 = \sigma^0 = I$$

$$\bar\psi(x) = \psi^+(x)\gamma^0 \Leftrightarrow \psi^+(x) = \bar\psi(x)\gamma^0 \quad \text{as } (\gamma^0)^2 = 1$$

$$\psi'^+(x') = \psi^+(x)D^+(\Lambda)$$

$$\bar\psi'(x') = \psi'^+(x')\gamma^0 = \psi^+(x)D(\Lambda)^+\gamma^0 = \bar\psi(x)\underbrace{\gamma^0D^+(\Lambda)\gamma^0}_{=\bar D(\Lambda)} = \bar\psi(x)\bar D(\Lambda)$$

$$\bar D(\Lambda) = \begin{pmatrix}0&I\\I&0\end{pmatrix}\underbrace{\begin{pmatrix}D^{(1/2,0)}(\Lambda)^+ & O\\ O & D^{(0,1/2)}(\Lambda)^+\end{pmatrix}\begin{pmatrix}0&I\\I&0\end{pmatrix}}_{=\begin{pmatrix}0 & D^{(1/2,0)}(\Lambda)^+\\ D^{(0,1/2)}(\Lambda)^+ & 0\end{pmatrix}} = \begin{pmatrix}D^{(0,1/2)}(\Lambda)^+ & 0\\ 0 & D^{(1/2,0)}(\Lambda)^+\end{pmatrix}$$

$$\Rightarrow \bar D(\Lambda) = D(\Lambda)^{-1}$$

$$\left.\begin{aligned}D^{(1/2,0)}(\Lambda) &= \exp\{-\tfrac{i}{2}\vec\sigma\vec\varphi - \tfrac12\vec\sigma\vec\zeta\}\\ D^{(0,1/2)}(\Lambda) &= \exp\{-\tfrac{i}{2}\vec\sigma\vec\varphi + \tfrac12\vec\sigma\vec\zeta\}\end{aligned}\right\} \xrightarrow{(\sigma^k)^+ = \sigma^k} \begin{aligned}D^{(1/2,0)}(\Lambda)^{-1} &= D^{(0,1/2)}(\Lambda)^+\\ D^{(0,1/2)}(\Lambda)^{-1} &= D^{(1/2,0)}(\Lambda)^+\end{aligned}$$

– Section 9.5: for $\Lambda = R$ or $\Lambda = B$

$$D^{(1/2,0)}(\Lambda)^\dagger\, \tilde\sigma^\mu\, D^{(1/2,0)}(\Lambda) = \Lambda^\mu{}_\nu\, \tilde\sigma^\nu \qquad (1)$$
$$D^{(0,1/2)}(\Lambda)^\dagger\, \sigma^\mu\, D^{(0,1/2)}(\Lambda) = \Lambda^\mu{}_\nu\, \sigma^\nu \qquad (2)$$

– Aim: show that these relations are valid for any Lorentz transformation $\Lambda = BR$

$$D^{(1/2,0)}(\Lambda) = D^{(1/2,0)}(B)\, D^{(1/2,0)}(R), \quad D^{(0,1/2)}(\Lambda) = D^{(0,1/2)}(B)\, D^{(0,1/2)}(R)$$

– Now we get:

$$D^{(1/2,0)}(\Lambda)^\dagger\, \tilde\sigma^\mu\, D^{(1/2,0)}(\Lambda) = D^{(1/2,0)}(R)^\dagger \underbrace{D^{(1/2,0)}(B)^\dagger\, \tilde\sigma^\mu\, D^{(1/2,0)}(B)}_{\text{Sect. 9.5}} D^{(1/2,0)}(R)$$

$$= B^\mu{}_\nu \underbrace{D^{(1/2,0)}(R)^\dagger\, \tilde\sigma^\nu\, D^{(1/2,0)}(R)}_{=\ \text{Sect. 9.5}\ R^\nu{}_\kappa \tilde\sigma^\kappa} \qquad = B^\mu{}_\nu\, \tilde\sigma^\nu = \underbrace{(B^\mu{}_\nu R^\nu{}_\kappa)}_{=\Lambda^\mu{}_\kappa} \tilde\sigma^\kappa \quad ✓$$

Analogous for $D^{(0,1/2)}(\Lambda)$

– Combine relations (1), (2):

$$\bar D(\Lambda)\, \gamma^\mu\, D(\Lambda) = \begin{pmatrix} D^{(0,1/2)}(\Lambda)^\dagger & 0 \\ 0 & D^{(1/2,0)}(\Lambda)^\dagger \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \overbrace{D^{(0,1/2)}(\Lambda)^\dagger \sigma^\mu D^{(0,1/2)}(\Lambda)}^{(2)} \\ \underbrace{D^{(1/2,0)}(\Lambda)^\dagger \tilde\sigma^\mu D^{(1/2,0)}(\Lambda)}_{(1)} & 0 \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} 0 & \sigma^\nu \\ \tilde\sigma^\nu & 0 \end{pmatrix} = \Lambda^\mu{}_\nu\, \gamma^\nu$$

$\Rightarrow$ Lorentz transformation for Dirac matrices: like a four-vector

– Proof of invariance of Dirac action with respect to Lorentz transformations:

$$A = \frac{\hbar}{c} \int d^4x\, \bar\Psi(x) \{ i\gamma^\mu \partial_\mu - m \} \Psi(x)$$
$$A' = \frac{\hbar}{c} \int d^4x'\, \bar\Psi'(x') \{ i\gamma^\mu \partial'_\mu - m \} \Psi'(x')$$

$\underbrace{\det\Lambda}_{=1} \cdot d^4x = d^4x$ for special Lorentz transformation

$$= \frac{\hbar}{c} \int d^4x\, \bar\Psi(x) \{ i \underbrace{\underbrace{\bar D(\Lambda)\gamma^\mu D(\Lambda)}_{\Lambda^\mu{}_\nu \gamma^\nu}\ \underbrace{\partial'_\mu}_{\Lambda_\mu{}^\kappa \partial_\kappa}}_{\underbrace{\Lambda^\mu{}_\nu \Lambda_\mu{}^\kappa}_{=\delta_\nu{}^\kappa} \gamma^\nu \partial_\kappa\ =\ \gamma^\nu \partial_\nu} - m \underbrace{\underbrace{\bar D(\Lambda)}_{D^{-1}(\Lambda)} D(\Lambda)}_{=1} \} \Psi(x) = A \quad ✓$$

– Representation of Lorentz transformations in space of Dirac spinors

$$D(\Lambda) = \begin{pmatrix} e^{-\frac{i}{2}\vec\sigma\vec\varphi - \frac{1}{2}\vec\sigma\vec\zeta} & 0 \\ 0 & e^{-\frac{i}{2}\vec\sigma\vec\varphi + \frac{1}{2}\vec\sigma\vec\zeta} \end{pmatrix}$$

Compare this with covariant formulation of Lie theorem:

$$D(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}} = \exp\Big\{ -\frac{i}{2} 2 \big(\underbrace{\omega_{0i}}_{\zeta_i} S^{0i} + \cancel{\underbrace{\omega_{i0}}_{=-\omega_{0i}} \underbrace{S^{i0}}_{=-S^{0i}}} \big) - \frac{i}{2} \underbrace{\omega_{ij}}_{=\varepsilon_{ijk}\varphi_k} S^{ij} \Big\}$$

$$D(M_k) = S^{0k} = \begin{pmatrix} -\frac{i}{2}\sigma^k & 0 \\ 0 & +\frac{i}{2}\sigma^k \end{pmatrix}$$

$$D(L_k) = \underbrace{S^k}_{\text{spin vector of spin 1/2 particle according to chapter 6}} = \frac{1}{2}\varepsilon_{kij} S^{ij} = \begin{pmatrix} \frac{1}{2}\sigma^k & 0 \\ 0 & \frac{1}{2}\sigma^k \end{pmatrix}, \quad S^{ij} = \varepsilon_{ijk} S^k = \varepsilon_{ijk} \begin{pmatrix} \sigma^k/2 & 0 \\ 0 & \sigma^k/2 \end{pmatrix}$$

– Covariant summary: $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]_-$

$$S^{0k} = \frac{i}{4} [\gamma^0, \gamma^k]_- = \ldots\ldots, \quad S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j]_- = \ldots\ldots$$

– See exercises:

$$[S^{\mu\nu}, \gamma^\lambda]_- = i(g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu)$$
$$[S^{\mu\nu}, S^{\kappa\lambda}]_- = i(g^{\mu\lambda}S^{\nu\kappa} + g^{\nu\kappa}S^{\mu\lambda} - g^{\mu\kappa}S^{\nu\lambda} - g^{\nu\lambda}S^{\mu\kappa})$$

9.8 Parity transformation:

– four vector x is transformed to spatially inverted $\tilde x$:

$$x = (x^0, x^k) \longrightarrow x'_P = Px = \tilde x = (x^0, -x^k)$$

– Involutoric property:

$$P^2 = 1 \iff P^{-1} = P$$

- Representation matrix:
$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
- Explicit calculations:
$$P^{-1} L_k P = L_k \quad \Leftrightarrow \quad [P, L_k]_- = 0$$
$$\text{e.g. } P^{-1} L_1 P = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = L_1$$
$$P^{-1} M_k P = -M_k \quad \Leftrightarrow \quad [P, M_k]_+ = 0$$
$$\text{e.g. } P^{-1} M_1 P = \ldots\ldots = -M_1$$
- Parity transformation for Dirac spinor: $\psi(x) \xrightarrow{P} \psi'_P(x) = D(P)\,\psi(\tilde{x})$

 $D(P)$: representation matrix of parity transformations in space of Dirac spinors
- Demands on $D(P)$: same properties as P

 1) $D(P)^2 = 1$, 2) $D(P)^{-1} D(L_k) D(P) = D(L_k)$, 3) $D(P)^{-1} D(M_k) D(P) = -D(M_k)$
- determine $D(P)$ from requirement that Dirac equation is invariant under parity transformation:
$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \xrightarrow{P} (i \underbrace{\gamma^\mu \tilde{\partial}_\mu}_{=\tilde{\gamma}^\mu \partial_\mu} - m)\underbrace{\psi(\tilde{x})}_{=D(P)^{-1}\psi'_P(x)} = 0 \qquad | D(P)\cdot$$
$$\Rightarrow \{ i \underbrace{D(P)\tilde{\gamma}^\mu D(P)^{-1}}_{\overset{!}{=}\gamma^\mu} \partial_\mu - m \underbrace{D(P)D(P)^{-1}}_{=1} \} \psi'_P(x) = 0$$
(condition for parity invariance)
- Claim: $D(P) = \gamma^0$
- Involution property: $D(P)^2 = (\gamma^0)^2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ ✓
- Condition:
$$\mu = 0: \quad \gamma^0 \tilde{\gamma}^0 \gamma^0 = \gamma^0\gamma^0\gamma^0 = (\gamma^0)^3 = \gamma^0 \ ✓$$
$$\mu = k: \quad \gamma^0 \tilde{\gamma}^k \gamma^0 = -\gamma^0 \gamma^k \gamma^0 \ \underbrace{[\gamma^0, \gamma^k]_+ = 2g^{0k} = 0}_{\text{Clifford algebra}} \ \gamma^k \gamma^0 \gamma^0 = \gamma^k \ ✓$$
- Additional properties:
$$D(P)^{-1} D(L_k) D(P) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sigma^k & 0 \\ 0 & \frac{1}{2}\sigma^k \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sigma^k & 0 \\ 0 & \frac{1}{2}\sigma^k \end{pmatrix} = D(L_k) \ ✓$$
$$D(P)^{-1} D(M_k) D(P) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} -\frac{i}{2}\sigma^k & 0 \\ 0 & \frac{i}{2}\sigma^k \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2}\sigma^k & 0 \\ 0 & -\frac{i}{2}\sigma^k \end{pmatrix} = -D(M_k) \ ✓$$
- Effect of $D(P)$: exchange of Weyl spinors
$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \longrightarrow \psi'_P(x) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \xi(\tilde{x}) \\ \eta(\tilde{x}) \end{pmatrix} = \begin{pmatrix} \eta(\tilde{x}) \\ \xi(\tilde{x}) \end{pmatrix}$$
- Theory where both $\psi(x)$ and $\psi'_P(x)$ represent physically realised states: both ξ and η must appear. This leads according to Section 9.5 to mass terms.
- Parity transformation invariant theory implies that spinors ξ and η must appear on equal footing.

9.9 Neutrinos:

- Neutrino: spin 1/2 particle interacting only via weak force and gravity
- Postulated in 1930 by Wolfgang Pauli as an additional particle involved in the β-decay of the neutron into a proton and an electron in order to explain conservation of energy and momentum
- Name:
 > electrically neutral, like neutron
 > small or zero rest mass: suffix -ino
- In accordance with many experiments neutrinos were considered to be massless spin 1/2 particles and described either by Weyl spinor ξ or η. Two possible Lagrangians
$$\mathcal{L} = A\, i\, \xi^+ \tilde{\sigma}^\mu \partial_\mu \xi \quad \text{or} \quad \mathcal{L} = A\, i\, \eta^+ \sigma^\mu \partial_\mu \eta$$
Note: like in Maxwell theory also in Weyl theory no Planck constant but still valid first quantized theory for spin 1/2 particles without rest mass
- Lagrangian invariant under Lorentz transformations like in Sect. 9.5 but not invariant under parity transformations, see Sect. 9.8.
- description with Dirac spinors: project out upper or lower Weyl spinor
- To this end: introduce new matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
$$= i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \sigma^1\sigma^2\sigma^3 \end{pmatrix}$$
$$\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \Rightarrow \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$
- Projection matrices:
$$P_L = \frac{1}{2}(1 - \gamma^5) = \frac{1}{2}\left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \right\} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

$P_R = \frac{1}{2}(1+\gamma^5) = \frac{1}{2}\left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$

– Projection: $P_L \psi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $P_R \psi = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}$

– Observation: $\gamma^5 \frac{1}{2}(1 \mp \gamma^5) = \mp \frac{1}{2}(1 \mp \gamma^5)$

Neutrino state can be classified according to the eigenvalues of γ^5-matrix:

γ^5: chirality operator, $\frac{1}{2}(1 \mp \gamma^5)\psi$: -1 (left) chirality, $+1$ (right) chirality

– different notation:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{24}\varepsilon_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda$$

– Lorentz invariance of γ^5:

$$\underbrace{\bar{D}(\Lambda)}_{=D^{-1}(\Lambda)}\gamma^5 D(\Lambda) = \frac{i}{24}\varepsilon_{\mu\nu\kappa\lambda}\underbrace{\{\bar{D}(\Lambda)\gamma^\mu D(\Lambda)\}}_{=\Lambda^\mu{}_{\mu'}\gamma^{\mu'}}\underbrace{\{\bar{D}(\Lambda)\gamma^\nu D(\Lambda)\}}_{=\Lambda^\nu{}_{\nu'}\gamma^{\nu'}}\underbrace{\{\bar{D}(\Lambda)\gamma^\kappa D(\Lambda)\}}_{=\Lambda^\kappa{}_{\kappa'}\gamma^{\kappa'}}\underbrace{\{\bar{D}(\Lambda)\gamma^\lambda D(\Lambda)\}}_{=\Lambda^\lambda{}_{\lambda'}\gamma^{\lambda'}}$$

$$= \frac{i}{24}\underbrace{\varepsilon_{\mu\nu\kappa\lambda}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\kappa{}_{\kappa'}\Lambda^\lambda{}_{\lambda'}}_{=\varepsilon_{\mu'\nu'\kappa'\lambda'}}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\kappa'}\gamma^{\lambda'} = \gamma^5 \checkmark$$

Determinant expansion for determinant of Λ:

$$\underbrace{(\det\Lambda)}_{=1}\varepsilon_{\mu'\nu'\kappa'\lambda'} = \varepsilon_{\mu\nu\kappa\lambda}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\kappa{}_{\kappa'}\Lambda^\lambda{}_{\lambda'}$$

$=1$ special Lorentz transformation

– Neutrino Lagrangian: $\mathcal{L} = \hbar i\,\bar{\psi}(x)\gamma^\mu\partial_\mu \frac{1}{2}(1\mp\gamma^5)\psi(x)$

$(-)$: $\hbar i(\xi^+, \zeta^+)\underbrace{\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}}_{=\begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}}\partial_\mu\begin{pmatrix} \xi \\ 0 \end{pmatrix} = \hbar i\,\xi^+\bar{\sigma}^\mu\partial_\mu\xi$

$(+)$: $\hbar i\,\zeta^+\sigma^\mu\partial_\mu\zeta$

– Both Lagrangians are Lorentz invariant

– What about parity transformation?

$$D(P)^{-1}\gamma^5 D(P) = \frac{i}{24}\varepsilon_{\mu\nu\kappa\lambda}\underbrace{\{D(P)^{-1}\gamma^\mu D(P)\}}_{=\tilde{\gamma}^\mu}\underbrace{\{D(P)^{-1}\gamma^\nu D(P)\}}_{=\tilde{\gamma}^\nu}\underbrace{\{D(P)^{-1}\gamma^\kappa D(P)\}}_{=\tilde{\gamma}^\kappa}\underbrace{\{D(P)^{-1}\gamma^\lambda D(P)\}}_{=\tilde{\gamma}^\lambda}$$

$$= -\frac{i}{24}\varepsilon_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = -\gamma^5$$

$$\hbar i\,\bar{\psi}(x)\gamma^\mu\partial_\mu\frac{1}{2}(1\mp\gamma^5)\psi(x) \xleftrightarrow{P} \hbar i\,\bar{\psi}(x)\gamma^\mu\partial_\mu\frac{1}{2}(1\pm\gamma^5)\psi(x)$$

– History:

- Lagrangians first proposed in 1929 by mathematician Hermann Weyl to describe massless spin 1/2 particles
- But at that time only parity invariant interactions like electromagnetic or strong one were known, but this Weyl action is not parity invariant. Therefore it was considered to be NOT physical.
- 1956: Chien-Shiung Wu, β-decay experiment of $^{60}_{27}Co$ => weak interaction is not invariant under parity transformations
- since then neutrinos were assumed to be described by these Weyl Lagrangians for decades
- 1987: Kamiokande experiment managed to resolve the flavours of neutrinos (electron, muon, tauon) from the sun, detection of neutrino oscillations between different flavours
- conclusion: neutrinos do have a finite rest mass but their precise value is not yet known
- due to charge neutrality two possible descriptions:
  > Paul Dirac: neutrino and antimatter neutrino are different particles
  > Ettore Majorana: neutrino " " are the same particle masquerading as two
  => experimental decision between both possible theoretical descriptions is still lacking

– Weyl equation:

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}(x)} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}(x))} = \hbar i\,\gamma^\mu\partial_\mu\frac{1}{2}(1\mp\gamma^5)\psi(x) = 0 \quad \text{Weyl equation}$$

– Fixed momentum: $\psi(x) = \psi e^{-ipx}$:

$$\Rightarrow \vec{\gamma}\cdot\vec{p}\,\frac{1}{2}(1\mp\gamma^5)\psi = \gamma^0 p_0\frac{1}{2}(1\mp\gamma^5)\psi \quad \Big|\ \gamma^5\gamma^0\cdot$$

1) $\gamma^5\gamma^0\gamma^k = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

2) $\gamma^5\gamma^0\gamma^0 = \gamma^5 \quad (\gamma^{0\,2} = 1)$

$$\Rightarrow \underbrace{\underbrace{\frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}_{=S^k} p^k}_{\vec{S}\cdot\vec{p}}\frac{1}{2}(1\mp\gamma^5)\psi = \gamma^5\underbrace{p_0}_{=\mathrm{sgn}(p_0)|\vec{p}|}\frac{1}{2}(1\mp\gamma^5)\psi$$

$$\Rightarrow \underbrace{\frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}}_{\text{helicity operator}}\frac{1}{2}(1\mp\gamma^5)\psi = \underbrace{\gamma^5}_{\text{chirality operator}}\mathrm{sgn}(p_0)\frac{1}{2}(1\mp\gamma^5)\psi$$

9.10 Charge Conjugation:

- Dirac theory is also invariant with respect to another discrete symmetry transformation where components of $\psi(x)$ are replaced by components of $\psi^*(x)$. Ansatz

$$\psi'_c(x) = \underset{\sim}{C}\,\overline{\psi}^T(x) = C\gamma^0\psi^*(x)$$

representation matrix to be determined

$$\overline{\psi}(x) = \psi^+(x)\gamma^0, \quad \overline{\psi}^T(x) = \underbrace{(\gamma^0)^T}_{=\gamma^0}\underbrace{(\psi^{*T})^T(x)}_{=\psi^*(x)}$$

- C is defined by:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \;\rightarrow\; (i\gamma^\mu\partial_\mu - m)\psi'_c(x) = 0$$

- Conclusion: $(i\gamma^\mu\partial_\mu - m)\,C\,\overline{\psi}^T(x) = 0 \quad | \, C^{-1}\cdot$

$$(i\,C^{-1}\gamma^\mu C\,\partial_\mu - m\underbrace{C^{-1}C}_{=1})\,\overline{\psi}^T(x) = 0 \quad | \, T$$

$$\Rightarrow \; i\,\partial_\mu\overline{\psi}(x)\,(C^{-1}\gamma^\mu C)^T - m\,\overline{\psi}(x) = 0 \qquad (1)$$

- Comparison with equation for Dirac adjoint Dirac spinor.

$$i\gamma^\mu\partial_\mu\psi(x) - m\psi(x) = 0 \quad | \, +$$

$$-i\,\partial_\mu\psi^+(x)(\gamma^\mu)^+ - m\psi^+(x) = 0$$

$$\overline{\psi}(x) = \psi^+(x)\gamma^0 \;\Leftrightarrow\; \psi^+(x) = \overline{\psi}(x)\gamma^0$$

$$-i\,\partial_\mu\overline{\psi}(x)\underbrace{\gamma^0(\gamma^\mu)^+\gamma^0} - m\,\overline{\psi}(x)\underbrace{\gamma^0\gamma^0}_{=1} = 0 \quad | \cdot\gamma^0$$

$$= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}^+\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix} = \gamma^\mu$$

$$\Rightarrow \; i\,\partial_\mu\overline{\psi}(x)\gamma^\mu + m\,\overline{\psi}(x) = 0 \qquad (2)$$

- Comparison (1) + (2): $(C^{-1}\gamma^\mu C)^T = -\gamma^\mu \;\Leftrightarrow\; \underline{C^{-1}\gamma^\mu C = -(\gamma^\mu)^T}$
- Solution ansatz: $C = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \; C^{-1} = \begin{pmatrix} c^{-1} & 0 \\ 0 & -c^{-1} \end{pmatrix}$

$$\begin{pmatrix} c^{-1} & 0 \\ 0 & -c^{-1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} = \begin{pmatrix} 0 & -c^{-1}\sigma^\mu c \\ -c^{-1}\hat{\sigma}^\mu c & 0 \end{pmatrix} = -\begin{pmatrix} 0 & \sigma^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}^T = -\begin{pmatrix} 0 & (\hat{\sigma}^\mu)^T \\ (\sigma^\mu)^T & 0 \end{pmatrix}$$

1) $c^{-1}\hat{\sigma}^\mu c = +(\sigma^\mu)^T$, 2) $c^{-1}\sigma^\mu c = +(\hat{\sigma}^\mu)^T$

$\mu = 0$: $c^{-1}\sigma^0 c = (\sigma^0)^T$ — for both 1) and 2)

$\mu = k$: $c^{-1}\sigma^k c = -(\sigma^k)^T$

- Pauli matrices:

$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; (\sigma^0)^T = \sigma^0; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; (\sigma^1)^T = \sigma^1$

$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; (\sigma^2)^T = -\sigma^2; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; (\sigma^3)^T = \sigma^3$

- Ansatz: $c = -i\sigma^2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \underline{c^+ = c^{-1} = c^T = -c = -c^*}$
- Check for equations:

$c^{-1}\sigma^0 c = i\sigma^2\sigma^0(-i)\sigma^2 = \sigma^2\sigma^0\sigma^2 = (\sigma^2)^2 = \sigma^0 = (\sigma^0)^T$ ✓

$c^{-1}\sigma^1 c = \ldots = \underset{\sim}{\sigma^2\sigma^1}\sigma^2 = -\sigma^1(\sigma^2)^2 = -\sigma^1 = -(\sigma^1)^T$ ✓

$\qquad = -\sigma^1\sigma^2$

$c^{-1}\sigma^2 c = \ldots = \sigma^2\sigma^2\sigma^2 = (\sigma^2)^3 = \sigma^2 = -(\sigma^2)^T$ ✓

$c^{-1}\sigma^3 c = \ldots = \sigma^2\sigma^3\sigma^2 = -\sigma^3(\sigma^2)^2 = -\sigma^3 = -(\sigma^3)^T$ ✓

- Conclusion: $C^+ = C^{-1} = C^T = -C = -C^*$

representation of C by Dirac matrices:

$$i\gamma^0\gamma^2 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} = C$$

- Involutoric property:

$$\psi''_c(x) = C\gamma^0\psi'^*_c(x) = C\gamma^0\underbrace{C^*}_{=C}\underbrace{(\gamma^0)^*}_{=\gamma^0}\psi(x) = \underbrace{\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}}\underbrace{\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}}\psi(x) = \psi(x)$$

$$\begin{pmatrix} -c^2 & 0 \\ 0 & -c^2 \end{pmatrix} = 1$$

- Continuity equation:

$$i\gamma^\mu\partial_\mu\psi(x) - m\psi(x) = 0 \quad | \, \overline{\psi}(x)\cdot$$

$$i\,\partial_\mu\overline{\psi}(x)\gamma^\mu + m\overline{\psi}(x) = 0 \quad | \cdot\psi(x)$$

$$\Rightarrow \quad i\,\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \cancel{m\overline{\psi}(x)\psi(x)} = 0$$

$$\underline{i\,\partial_\mu\overline{\psi}(x)\gamma^\mu\psi(x) + \cancel{m\overline{\psi}(x)\psi(x)} = 0}$$

$$\partial_\mu\{\overline{\psi}(x)\gamma^\mu\psi(x)\} = 0 \;\Rightarrow\; \partial_\mu j^\mu(x) = 0, \quad j^\mu(x) = K\,\overline{\psi}(x)\gamma^\mu\psi(x)$$

charge as conserved quantity: $Q = \int d^3x\, j^0(\vec{x},t) = K \int d^3x\, \underbrace{\overline{\psi}(x)}_{\psi^+(x)\gamma^0}\, \gamma^0 \psi(x) = K \int d^3x\, \psi^+(x)\psi(x)$

- Dirac adjoint Dirac spinor:

$$\overline{\psi}_c'(x) = \psi_c'^{+}(x)\,\gamma^0 = \left(C\gamma^0\psi^*(x)\right)^+\gamma^0 = \psi^T(x)\underbrace{(\gamma^0)^+}_{\gamma^0}\underbrace{C^+\gamma^0}_{=-C}$$

$$= - \psi^T(x)\underbrace{\gamma^0 C\gamma^0} = \psi^T(x)\, C$$

$$\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & -c\end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & c\end{pmatrix} = -C$$

- Parity transformation acting upon current density:

$$j_c'^{\mu}(x) = K\,\overline{\psi}_c'(x)\gamma^\mu\psi_c'(x) = K\,\overbrace{\psi^T(x)\,C\,\gamma^\mu\, C\gamma^0\psi^*(x)}$$

fixed μ $\hat{=}$ scalar

$$= \left[\psi^T(x)\, C\,\gamma^\mu C\,\gamma^0\psi^*(x)\right]^T$$

$$= \psi^+(x)\underbrace{(\gamma^0)^T}_{=\gamma^0}\underbrace{(C\gamma^\mu C)^T}_{\gamma^\mu}\psi(x)$$

$$= K\,\underbrace{\psi^+(x)\gamma^0}_{\overline{\psi}(x)}\gamma^\mu\psi(x) = j^\mu(x)$$

- Interpretation of this discrete transformation as a charge conjugation becomes only clear when implementing the second quantization. Only then the sign of four-current density changes.