charge as conserved quantity: $Q = \int d^3x\, j^0(\vec x,t) = k\int d^3x\, \underbrace{\overline{\Psi}(x)}_{\Psi^\dagger(x)\gamma^0}\gamma^0\Psi(x) = k\int d^3x\,\Psi^\dagger(x)\Psi(x)$

- Dirac adjoint Dirac spinor:

$$\overline{\Psi}'_C(x) = \Psi'^\dagger_C(x)\gamma^0 = (C\gamma^0\Psi^*(x))^\dagger\gamma^0 = \Psi^T(x)\underbrace{(\gamma^0)^\dagger}_{\gamma^0}\underbrace{C^\dagger}_{=-C}\overset{=1}{\gamma^0}$$

$$= -\Psi^T(x)\underbrace{\gamma^0 C\gamma^0}\ = \Psi^T(x)C$$

$$\begin{pmatrix}0&I\\I&0\end{pmatrix}\begin{pmatrix}c&0\\0&-c\end{pmatrix}\begin{pmatrix}0&I\\I&0\end{pmatrix} = \begin{pmatrix}-c&0\\0&c\end{pmatrix} = -C$$

- Parity transformation acting upon current density:

$$j'^\mu_C(x) = k\,\overline{\Psi}'_C(x)\gamma^\mu\Psi'_C(x) = k\,\underbrace{\Psi^T(x)C\gamma^\mu C\gamma^0\Psi^*(x)}$$

fixed μ $\hat=$ scalar $= [\Psi^T(x)C\gamma^\mu C\gamma^0\Psi^*(x)]^T$

$$= \Psi^\dagger(x)\underbrace{(\gamma^0)^T}_{=\gamma^0}\underbrace{(C\gamma^\mu C)^T}_{\gamma^\mu}\Psi(x)$$

$$= k\,\underbrace{\Psi^\dagger(x)\gamma^0}_{\overline{\Psi}(x)}\gamma^\mu\Psi(x) = j^\mu(x)$$

- Interpretation of this discrete transformation as a charge conjugation becomes only clear when implementing the second quantization. Only then the sign of four-current density changes.

9.11 Time Inversion:

- definition: $x = (x^0, x^k) \longrightarrow x'_T = Tx = (-x^0, x^k) =: -\tilde{x}$ $\hat=$ movie runs backwards in time
- Involutoric property: $T^2 = 1 \iff T^{-1} = T$
- representation matrix: $T = \begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
- Explicit calculation:

$$T^{-1}L_kT = L_k \iff [T, L_k]_- = 0$$

$$T^{-1}M_kT = -M_k \iff [T, M_k]_+ = 0$$

e.g. $T^{-1}L_1T = -i\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&+1\\0&0&-1&0\end{pmatrix}\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = -i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix} = L_1$ ✓

- Time inversion is intriguing, look first at Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\Psi(\vec x,t) = 0$$

$$\Psi'_T(\vec x,t) = \Psi^*(\vec x,-t) \Rightarrow \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta\right)\Psi'_T(\vec x,t) = 0$$

- By analogy: time inversion for Dirac spinor

$$\Psi(x) \longrightarrow \Psi'_T(x) = \underbrace{D(T)}\Psi^*(-\tilde{x})$$

representation matrix for time inversion in space of Dirac spinors

- We demand that with $\Psi(x)$ also $\Psi'_T(x)$ satisfies Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \xrightarrow{T} (i\gamma^\mu\partial_\mu - m)\Psi'_T(x) = 0$$

$$\Rightarrow (i\gamma^\mu\partial_\mu - m)D(T)\Psi^*(-\tilde x) = \{i\gamma^\mu D(T)\partial_\mu\Psi^*(-\tilde x) - m D(T)\Psi^*(-\tilde x) = 0 \quad | D(T)^{-1}\cdot$$

$$\Rightarrow \{i D(T)^{-1}\gamma^\mu D(T)\partial_\mu - m\underbrace{D(T)^{-1}D(T)}_{=1}\}\Psi^*(-\tilde x) = 0 \quad |\ \text{complex conjugation}$$

$$\Rightarrow -i\{D(T)^{-1}\gamma^\mu D(T)\}^*\partial_\mu\Psi(-\tilde x) - m\Psi(-\tilde x) = 0 \qquad (1)$$

- Comparison with time inverted Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0 \xrightarrow{x\to-\tilde x} -i\underbrace{\gamma^\mu\tilde\partial_\mu}_{=\tilde\gamma^\mu\partial_\mu}\Psi(-\tilde x) - m\Psi(-\tilde x) = 0 \qquad (2)$$

$(1)+(2)$: $\{D(T)^{-1}\gamma^\mu D(T)\}^* = \tilde\gamma^\mu \iff D(T)^{-1}\gamma^\mu D(T) = (\tilde\gamma^\mu)^* = (\gamma^\mu)^T$

- Complex conjugate Dirac matrices:

$$(\sigma^0)^* = \sigma^0,\ (\sigma^1)^* = \sigma^1,\ (\sigma^2)^* = -\sigma^2,\ (\sigma^3)^* = \sigma^3$$

$$\gamma^\mu = \begin{pmatrix}0&\sigma^\mu\\\tilde\sigma^\mu&0\end{pmatrix} \Rightarrow (\gamma^0)^* = \gamma^0,\ (\gamma^1)^* = \gamma^1,\ (\gamma^2)^* = -\gamma^2,\ (\gamma^3)^* = \gamma^3$$

$(\sigma^\mu)^\dagger = \sigma^\mu$

- Transposed Pauli matrices:

$$(\sigma^0)^T = \sigma^0,\ (\sigma^1)^T = \sigma^1,\ (\sigma^2)^T = -\sigma^2,\ (\sigma^3)^T = \sigma^3$$

- Evaluation for $(\tilde\gamma^\mu)^T$:

$$(\tilde\gamma^0)^T = (\gamma^0)^T = \begin{pmatrix}0&(\sigma^0)^T\\(\sigma^0)^T&0\end{pmatrix} = \begin{pmatrix}0&\sigma^0\\\sigma^0&0\end{pmatrix} = \gamma^0$$

Comparison: $(\gamma^\mu)^* = (\tilde\gamma^\mu)^T$

$(\tilde\gamma^1)^T = -(\gamma^1)^T = -\begin{pmatrix} 0 & -(\sigma^1)^T \\ (\sigma^1)^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} = \gamma^1$

$(\tilde\gamma^2)^T = -(\gamma^2)^T = -\begin{pmatrix} 0 & -(\sigma^2)^T \\ (\sigma^2)^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} = -\gamma^2$ $\Leftrightarrow (\gamma^\mu)^+ = \tilde\gamma^\mu$

$(\tilde\gamma^3)^T = -(\gamma^3)^T = -\begin{pmatrix} 0 & -(\sigma^3)^T \\ (\sigma^3)^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \gamma^3$

- Intermediate result: $D(T)^{-1}\tilde\gamma^\mu D(T) = (\gamma^\mu)^T$ $\Big\}$ $D(T)^{-1}\gamma^\mu D(T) = -C^{-1}\gamma^\mu C \,\Big|\, C\cdot \ \cdot C^{-1}$

Observation: $C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$

$\underbrace{(C\,D(T)^{-1})}_{=\{D(T)C^{-1}\}^{-1}}\gamma^\mu(D(T)\,C^{-1}) = -\gamma^\mu$ has to be solved for $D(T)\,C^{-1}$

- Solution: $\boxed{D(T)\,C^{-1} = -i\gamma^5}$ parity

time inversion charge conjugation

subtle connection between all 3 discrete transformation

Note: $\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \Rightarrow (\gamma^5)^2 = 1,\ (\gamma^5)^{-1} = \gamma^5 \Rightarrow \{D(T)C^{-1}\}^{-1} = i\gamma^5$

- Explicit verification:

$i\gamma^5\gamma^\mu(-i)\gamma^5 = \gamma^5\gamma^\mu\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = -\begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix} = -\gamma^\mu$ ✓

- Representation matrix $D(T)$:

$D(T) = -i\gamma^5 C = -i\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} = i\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

Note: $C^+ = C^{-1} = C^T = -C = C^*$, $\quad c = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow D(T) = D(T)^{-1} = D(T)^+ = -D(T)^* = -D(T)^T$

- Although T is involutoric and $D(T)$ is involutoric; still the time inversion of a Dirac spinor is NOT involutoric:

$\psi_T''(x) = D(T)\,\psi_T'^*(-\tilde x) = D(T)\{D(T)\,\psi^*(-\tilde x)\}^*_{x\to-\tilde x} = \underbrace{D(T)D(T)^*}\psi(x)$

$= -1$ and NOT $+1$

This is a consequence of spin 1/2!

- Check: $D(T)$ and $D(L_R)$

$D(T)^{-1}D(L_R)D(T) = D^*(L_R) \neq D(L_R)$

- Check: $D(T)$ and $D(M_R)$ $\Big\}$ expectations not fulfilled

$D(T)^{-1}D(M_R)D(T) = -D^*(M_R) \neq D(M_R)$

- Time inversion represents in second quantized an anti-linear operator:

$\hat{T}(\alpha_1\hat{O}_1 + \alpha_2\hat{O}_2) = \alpha_1^*\hat{T}(\hat{O}_1) + \alpha_2^*\hat{T}(\hat{O}_2)$

complex numbers

operators in second quantization

9.12 Dirac Representation:

so far:

Weyl (chiral) representation

adequate for Lorentz transformation

block diagonal $D(\Lambda) = \begin{pmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{pmatrix}$

i.e. both Weyl spinors ξ and η are dealt with on equal footing

$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ diagonal

$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ non-diagonal

now

Dirac (standard) representation

adequate for non-relativistic limit

i.e. upper Weyl spinor ξ_D survives but lower Weyl spinor η_D negligible

γ^5_D non-diagonal

γ^0_D diagonal

- Transformation from Weyl to Dirac representation:

$\psi_D(x) = S_D\,\psi(x)$, $\quad S_D = \frac{1}{\sqrt2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$, $\quad S_D^{-1} = \frac{1}{\sqrt2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = S_D^T$, S_D: orthonormal, i.e. unitary

- Dirac adjoint Dirac spinor in Dirac representation:

$\overline{\psi}_D(x) = \psi_D^+(x)\gamma^0 = \psi^+(x)\,S_D^+\gamma^0 = \overline{\psi}(x)\underbrace{\gamma^0 S_D^+\gamma^0}$

$= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \frac{1}{\sqrt2}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = S_D^{-1}$

– Dirac matrices in Dirac representation

$$\gamma^0_D = S_D \gamma^0 S_D^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} I & -I \\ +I & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \text{ diagonal}$$

$$\gamma^k_D = S_D \gamma^k S_D^{-1} = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$$

– chirality matrix:

$$\gamma^5_D = S_D \gamma^5 S_D^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{non-diagonal}$$

– generators for rotations:

$$D(L_k)_D = S_D D(L_k) S_D^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} \sigma^k/2 & 0 \\ 0 & \sigma^k/2 \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

– generators for boosts:

$$D(M_k)_D = S_D D(M_k) S_D^{-1} = \frac{i}{2}\begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

5.13 Non-Relativistic Limit:

– Dirac representation: historic one used by Paul Dirac

– Dirac equation in Dirac representation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0 \quad \Rightarrow \quad (i\gamma^\mu_D \partial_\mu - m)\,\psi_D(x) = 0$$

$$\longrightarrow\; i S_D \gamma^\mu \partial_\mu S_D^{-1} \psi_D(x) - m S_D S_D^{-1} \psi_D(x) = 0$$

– separation into temporal and spatial degrees of freedom:

$$i\gamma^0_D \frac{1}{c}\frac{\partial}{\partial t}\psi_D(\vec{x},t) + i\gamma^k_D \partial_k \psi_D(\vec{x},t) - m\,\psi_D(\vec{x},t) = 0 \quad \Big|\; \hbar c\,\gamma^0_D \,\cdot$$

$$i\hbar \underbrace{(\gamma^0_D)^2}\frac{\partial}{\partial t}\psi_D(\vec{x},t) + \underline{i\hbar c\,\gamma^0_D \gamma^k_D \partial_k \psi_D(\vec{x},t) - \hbar c m\,\psi_D(\vec{x},t)} = 0$$

$$= S_D \underbrace{(\gamma^0)^2}_{=1} S_D^{-1} = S_D S_D^{-1} = 1$$

– rewrite Dirac equation formally as a Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}\psi_D(\vec{x},t) = \hat{H}\,\psi_D(\vec{x},t) \quad \longleftarrow \text{ Dirac Hamilton operator}$$

$$\hat{H} = -i\hbar c\,\vec{\alpha}\cdot\vec{\nabla} + \hbar c m\,\beta$$

– newly introduced matrices:

$$\beta = \gamma^0_D = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\alpha^k = \gamma^0_D \gamma^k_D = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

– Anticommutator relations:

$$[\beta,\beta]_+ = 2\,I, \quad I = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad 4\times 4 \text{ unity-matrix}$$

$$[\alpha^k,\beta]_+ = 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 4\times 4 \text{ zero-matrix}$$

$$[\alpha^k,\alpha^l]_+ = 2\,\delta_{kl}\,I$$

α^k, β form a Clifford algebra with $N = 4$

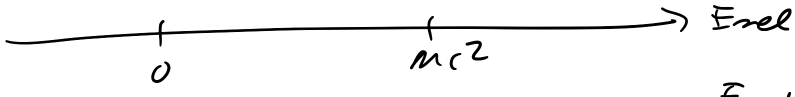
– decomposition: $\psi_D(\vec{x},t) = \begin{pmatrix} \zeta_D(\vec{x},t) \\ \xi_D(\vec{x},t) \end{pmatrix}$

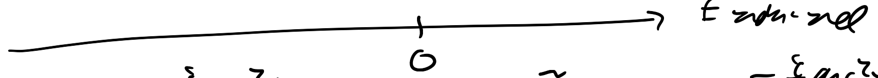
Schrödinger equation:

$$i\hbar \frac{\partial \zeta_D(\vec{x},t)}{\partial t} = -i\hbar c\,\vec{\sigma}\cdot\vec{\nabla}\,\xi_D(\vec{x},t) + \hbar c m\,\zeta_D(\vec{x},t)$$

$$i\hbar \frac{\partial \xi_D(\vec{x},t)}{\partial t} = -i\hbar c\,\vec{\sigma}\cdot\vec{\nabla}\,\zeta_D(\vec{x},t) - \hbar c m\,\xi_D(\vec{x},t)$$

– Energy scales: $E_{rel} = E_{non\text{-}rel} + Mc^2$





$$\zeta_D(\vec{x},t) = \tilde{\zeta}_D(\vec{x},t)\,e^{-\frac{i}{\hbar}Mc^2 t}, \quad \xi_D(\vec{x},t) = \tilde{\xi}_D(\vec{x},t)\,e^{-\frac{i}{\hbar}Mc^2 t}$$

$$i\hbar \frac{\partial \tilde{\zeta}_D(\vec{x},t)}{\partial t} = -i\hbar c\,\vec{\sigma}\vec{\nabla}\,\tilde{\xi}_D(\vec{x},t) + \overbrace{(\hbar c m - Mc^2)}^{=0}\,\tilde{\zeta}_D(\vec{x},t)$$

$$i\hbar \frac{\partial \tilde{\xi}_D(\vec{x},t)}{\partial t} = -i\hbar c\,\vec{\sigma}\vec{\nabla}\,\tilde{\zeta}_D(\vec{x},t) + \underbrace{(-\hbar c m - Mc^2)}_{=-2Mc^2}\,\tilde{\xi}_D(\vec{x},t)$$

$m = \frac{Mc}{\hbar}$

each wavefunction should fulfill a Klein-Gordon equation

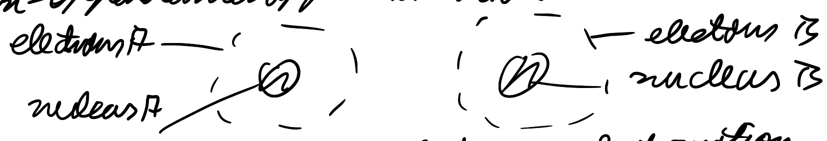
– Non-relativistic limit: $c \to \infty$ adiabatic elimination for $\tilde{\xi}_D$

$$\left| i\hbar \frac{\partial \tilde{\xi}_D(\vec{x},t)}{\partial t} \right| \ll \left| 2Mc^2 \tilde{\xi}_D(\vec{x},t) \right| \Rightarrow \underbrace{\tilde{\xi}_D(\vec{x},t)}_{\text{fast}} = -\frac{i\hbar}{2Mc^2}\, \vec{\sigma}\cdot\vec{\nabla}\, \underbrace{\tilde{\zeta}_D(\vec{x},t)}_{\text{slow}}$$

$\tilde{\xi}_D$ is no longer a dynamical degree of freedom

– Other applications of adiabatic elimination:

- Born-Oppenheimer approximation for molecules:



nuclei: slow motion, electrons: fast motion

→ motion of ~~electronics~~ adiabatically eliminated

- laser physics:

electric field
slow: u

"enslaving"

matter degrees of freedom:
population inversion
polarisation
fast: s

physics:
s adiabatically eliminated

$s = s(u)$
mathematics:
center manifold

Hermann Haken: synergetics, this is a self-organization theory

→ order parameter equations (slow varying)

– Resulting evolution equation for $\tilde{\zeta}_D$:

$$i\hbar \frac{\partial \tilde{\zeta}_D(\vec{x},t)}{\partial t} = \underbrace{-i\hbar c \frac{-i\hbar c}{2Mc^2}}_{= -\frac{\hbar^2}{2M}} \underbrace{(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{\nabla})}_{= \frac{1}{2}\underbrace{[\sigma^k, \sigma^l]_+}_{= \delta_{kl}} \partial_k \partial_l = \Delta} \tilde{\zeta}_D(\vec{x},t) = -\frac{\hbar^2}{2M} \Delta \tilde{\zeta}_D(\vec{x},t)$$

– Exercises:

- Extension to minimal coupling of Dirac field to Maxwell field
- Non-relativistic limit, systematically ⇒ Foldy-Wouthuysen
- Pauli equation with Landé factor $g_S = 2$ for a point-like massive spin 1/2 particle
- Proton and neutron, which are massive spin 1/2 particles, have Landé factor of 2.79 and -1.91

⇒ nucleons are not point-like but are composite particle. Indeed, proton and neutron consist each of 3 quarks, which are point-like, massive spin 1/2 particles