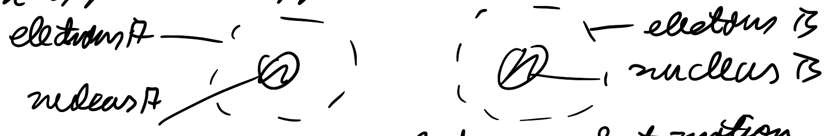
- Non-relativistic limit: $c \to \infty$ adiabatic elimination for $\tilde\xi_D$

$$\left| i\hbar \frac{\partial \tilde\xi_D(\vec x,t)}{\partial t} \right| \ll \left| 2Mc^2 \tilde\xi_D(\vec x,t) \right| \Rightarrow \underset{\text{fast}}{\tilde\xi_D(\vec x,t)} = -\frac{i\hbar}{2Mc^2}\vec\sigma\cdot\vec D\, \underset{\text{slow}}{\tilde\zeta_D(\vec x,t)}$$

$\tilde\xi_D$ is no longer a dynamical degree of freedom

- Other applications of adiabatic elimination:
  - Born-Oppenheimer approximation for molecules:



nuclei: slow motion, electrons: fast motion

→ motion of ~~electronics~~ adiabatically eliminated

  - laser physics:

electric field
slow: u

"enslaving"

matter degrees of freedom:
population inversion
polarisation
fast: s

physics:
s adiabatically eliminated

$s = s(u)$
mathematics:
center manifold

Hermann Haken: synergetics, this is a self-organisation theory

→ order parameter equations (slow varying)

- Resulting evolution equation for $\tilde\zeta_D$:

$$i\hbar\frac{\partial \tilde\zeta_D(\vec x,t)}{\partial t} = \underbrace{-i\hbar c \frac{-i\hbar c}{2Mc^2}}_{= -\frac{\hbar^2}{2M}} \underbrace{(\vec\sigma\cdot\vec D)(\vec\sigma\cdot\vec D)}_{=\frac{1}{2}\underbrace{[\sigma^k,\sigma^l]_+}_{=\delta_{kl}}\partial_k\partial_l = \Delta} \tilde\zeta_D(\vec x,t) \quad = -\frac{\hbar^2}{2m}\Delta\tilde\zeta_D(\vec x,t)$$

- Exercises:
  - Extension to minimal coupling of Dirac field to Maxwell field
  - Non-relativistic limit, systematically ⇒ Foldy-Wouthuysen
  - Pauli equation with Landé factor $g_S = 2$ for a point-like massive spin 1/2 particle
  - Proton and neutron, which are massive spin 1/2 particles, have Landé factor of 2.79 and -1.91 ⇒ nucleons are not point-like but are composite particle. Indeed, proton and neutron consist each of 3 quarks, which are point-like, massive spin 1/2 particles

- Dirac action: Dirac representation

$$S = \frac{A}{c}\int \underbrace{d^4x}_{= c\,dt\,d^3x} \bar\Psi_D(x)\left(i\gamma_D^\mu\partial_\mu - m\beta\right)\Psi_D(x) \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$$

$$= A\int dt\int d^3x \Big\{ \underbrace{\bar\Psi_D(\vec x,t)}_{=\Psi_D^+(\vec x,t)\gamma_D^0} \underbrace{\gamma_D^0}_{\gamma_D^0\gamma_D^0 = 1} \frac{i}{c}\frac{\partial}{\partial t} + \Psi_D^+(\vec x,t)\underbrace{\gamma_D^0\vec\gamma_D}_{=\vec\alpha = \begin{pmatrix} 0 & \vec\sigma \\ \vec\sigma & 0\end{pmatrix}}\vec D - m\beta\Big\}\Psi_D(\vec x,t)$$

$$\Psi_D(\vec x,t) = \begin{pmatrix} \tilde\zeta_D(\vec x,t)e^{-\frac{i}{\hbar}mc^2t} \\ \tilde\xi_D(\vec x,t)e^{-\frac{i}{\hbar}mc^2t}\end{pmatrix}, \quad \Psi_D^+(\vec x,t) = \left(\tilde\zeta_D^+(\vec x,t)e^{\frac{i}{\hbar}mc^2t}, \tilde\xi_D^+(\vec x,t)e^{+\frac{i}{\hbar}mc^2t}\right)$$

$$\frac{\partial\Psi_D(\vec x,t)}{\partial t} = \begin{pmatrix} \frac{\partial\tilde\zeta_D(\vec x,t)}{\partial t} - \frac{i}{\hbar}mc^2\tilde\zeta_D(\vec x,t) \\ \frac{\partial\tilde\xi_D(\vec x,t)}{\partial t} - \frac{i}{\hbar}mc^2\tilde\xi_D(\vec x,t)\end{pmatrix} e^{-\frac{i}{\hbar}mc^2}$$

$$S = A\int dt\int d^3x\Big\{\frac{i}{c}\Big[\tilde\zeta_D(\vec x,t)\frac{\partial\tilde\zeta_D(\vec x,t)}{\partial t} + \cancel{\tilde\xi_D(\vec x,t)\frac{\partial\tilde\xi_D(\vec x,t)}{\partial t}}\Big]$$
$$+2\frac{Mc}{\hbar}\Big[\cancel{\tilde\zeta_D^+(\vec x,t)\tilde\zeta_D(\vec x,t)} + \tilde\xi_D^+(\vec x,t)\tilde\xi_D(\vec x,t)\Big]$$
$$+ i\Big[\tilde\zeta_D^+(\vec x,t)\vec\sigma\cdot\vec D\tilde\xi_D(\vec x,t) + \tilde\xi_D^+(\vec x,t)\vec\sigma\cdot\vec D\tilde\zeta_D(\vec x,t)\Big]$$
$$-\underbrace{m}_{=\frac{Mc}{\hbar}}\Big[\cancel{\tilde\zeta_D^+(\vec x,t)\tilde\zeta_D(\vec x,t)} - \tilde\xi_D^+(\vec x,t)\tilde\xi_D(\vec x,t)\Big]\Big\}$$

non-relativistic limit $c\to\infty$: adiabatic elimination of $\tilde\xi_D$

$$i\hbar\frac{\partial}{\partial t}\tilde\xi_D(\vec x,t)\approx 0, \qquad \tilde\xi_D(\vec x,t) = -\frac{i\hbar c}{2Mc^2}\vec\sigma\cdot\vec D\,\tilde\zeta_D(\vec x,t)$$

$$S = A\int dt\int d^3x\left\{\frac{i}{c}\tilde{\xi}^\dagger_D(\vec{x},t)\frac{\partial\tilde{\xi}_D(\vec{x},t)}{\partial t}\right.$$

$$\cancel{+2\frac{mc}{\hbar}\frac{i\hbar c}{2mc^2}\frac{-i\hbar c}{2mc^2}\left(\vec{\nabla}\tilde{\xi}^\dagger_D(\vec{x},t)\cdot\vec{\sigma}\right)}\left(\vec{\sigma}\cdot\vec{\nabla}\tilde{\xi}_D(\vec{x},t)\right)$$

$$+i\frac{-i\hbar c}{2mc^2}\hat{\xi}^\dagger_D(\vec{x},t)\underbrace{(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\vec{\nabla})}\hat{\xi}_D(\vec{x},t)+i\frac{i\hbar c}{2mc^2}\cancel{\left(\vec{\nabla}\hat{\xi}^\dagger_D(\vec{x},t)\vec{\sigma}\cdot\right)\left(\vec{\sigma}\vec{\nabla}\hat{\xi}_D(\vec{x},t)\right)}$$

$$=\sigma^k\sigma^l\partial_k\partial_l=\frac{1}{2}\underbrace{[\sigma^k,\sigma^l]_+}_{2\delta_{kl}\cdot I}\partial_k\partial_l=\Delta$$

$$S = A\int dt\int d^3x\left\{\frac{i}{c}\hat{\xi}^\dagger_D(\vec{x},t)\frac{\partial\hat{\xi}_D(\vec{x},t)}{\partial t}+\frac{\hbar}{2m}\hat{\xi}^\dagger_D(\vec{x},t)\Delta\hat{\xi}_D(\vec{x},t)\right.$$

fix yet undetermined A: $A=\hbar c$ ⇒ Schrödinger action for $\hat{\xi}_D$

Conclusion: Dirac action in Weyl representation

$$S=\frac{1}{c}\int d^4x\,\mathcal{L},\quad \mathcal{L}=i\hbar c\,\bar{\Psi}\gamma^\mu\partial_\mu\Psi-m\bar{\Psi}\Psi$$

Remark: conserved charge: $Q=\cancel{\kappa}\int d^3x\,\Psi^\dagger(\vec{x},t)\Psi(\vec{x},t)$

→ Dirac representation → non-relativistic limit : $\kappa=1$

9.14 Plane Waves:

- Dirac equation in Weyl representation: $\left(i\gamma^\mu\partial_\mu-\frac{mc}{\hbar}\right)\Psi(x)=0$
  ⇒ plane wave solutions
- Possible solution strategy:
  - ansatz for a plane wave
  - differential eq. is converted into a homogeneous algebraic eq. for Dirac spinor amplitudes
  - → solve them with γ^μ in Weyl representation
- Here: group theoretically inspired solution method
  - determine plane wave solution in rest frame, which is straight-forwardly possible
  - boost these solutions then into a uniformly moving rest frame

9.14.1 Rest Frame:

- Ansatz: $\Psi_R(x)=\Psi(t)$ ⇒ $\left(i\gamma^0\frac{\partial}{\partial t}-\frac{mc^2}{\hbar}\right)\Psi(t)=0 \;\Big|\; \left(-i\gamma^0\frac{\partial}{\partial t}-\frac{mc^2}{\hbar}\right)\cdot$

$$\underbrace{\left(-i\gamma^0\frac{\partial}{\partial t}-\frac{mc^2}{\hbar}\right)\left(i\gamma^0\frac{\partial}{\partial t}-\frac{mc^2}{\hbar}\right)}_{\underbrace{(\gamma^0)^2}_{=I}\frac{\partial^2}{\partial t^2}+\left(\frac{mc^2}{\hbar}\right)^2}\Psi(t)=0 \Rightarrow$$

two solutions: $\Psi(t)=\Psi\,e^{\mp\frac{i}{\hbar}mc^2t}$

- algebraic eq. for spinor amplitude: $(\pm\gamma^0-I)\Psi=0$, $\gamma^0=\begin{pmatrix}0&I\\I&0\end{pmatrix}$

$$\oplus:\ \left\{\begin{pmatrix}0&I\\I&0\end{pmatrix}-\begin{pmatrix}I&0\\0&I\end{pmatrix}\right\}\Psi=\begin{pmatrix}-I&I\\I&-I\end{pmatrix}\Psi=0\quad(1)\qquad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\ominus:\ \left\{-\begin{pmatrix}0&I\\I&0\end{pmatrix}-\begin{pmatrix}I&0\\0&I\end{pmatrix}\right\}\Psi=\begin{pmatrix}-I&-I\\-I&-I\end{pmatrix}\Psi=0\quad(2)$$

- Assumption: $\chi(\pm 1/2)$ two orthonormal bi-spinors

$$\chi^\dagger(\lambda')\chi(\lambda)=\delta_{\lambda\lambda'}$$

Note: At the moment we do not specify $\chi(\pm 1/2)$ ←

- Construct bi-spinors $\chi^c(\pm 1/2)$ which are charge conjugate with respect to $\chi(\pm 1/2)$

$$\boxed{\chi^c\left(\pm\frac{1}{2}\right):=c\,\chi^*\left(\pm\frac{1}{2}\right)},\quad c=-i\sigma^2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

orthonormalization:

$$\chi^c(\lambda)^\dagger\chi^c(\lambda')\ \underline{\underline{\text{c-number}}}\ \left(\chi^c(\lambda)^\dagger\chi^c(\lambda')\right)^T=\left(\chi^T(\lambda)\underbrace{c^\dagger c}_{=I}\chi^*(\lambda')\right)^T$$

$$=\chi^\dagger(\lambda')\chi(\lambda)=\delta_{\lambda\lambda'}$$

$$(1):\quad \Psi^{(1)}=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi(1/2)\\\chi(1/2)\end{pmatrix},\quad \Psi^{(2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi(-1/2)\\\chi(-1/2)\end{pmatrix}$$

$$(2):\quad \Psi^{(3)}=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi^c(1/2)\\-\chi^c(1/2)\end{pmatrix},\quad \Psi^{(4)}=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi^c(-1/2)\\-\chi^c(-1/2)\end{pmatrix}$$

- $\Psi^{(3,4)}$ is charge conjugated of $\Psi^{(1,2)}$:

$$\overline{\psi}^{(1,2)} = \psi^{(1,2)\dagger}\gamma^0 = \frac{1}{\sqrt{2}}\left(\chi^+(\lambda), \chi^+(\lambda)\right)\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\chi^+(\lambda), \chi^+(\lambda)\right)$$

$$\psi_c^{(1,2)} = C\,\overline{\psi}^{(1,2)T} = \begin{pmatrix}c & 0\\ 0 & -c\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\chi^*(\lambda)\\ \chi^*(\lambda)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}c\chi^*(\lambda)\\ -c\chi^*(\lambda)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi^c(\lambda)\\ -\chi^c(\lambda)\end{pmatrix} = \psi^{(3,4)} \quad ✓$$

9.14.2 Boost to Uniformly Moving Reference Frame:

- Boost: $\psi^{(1,2)}e^{-\frac{i}{\hbar}mc^2t} \longrightarrow \psi^{(1,2)}_{\vec{p}}(x) = \psi^{(1,2)}_{\vec{p}}e^{-\frac{i}{\hbar}px}$

$\psi^{(3,4)}e^{+\frac{i}{\hbar}mc^2t} \longrightarrow \psi^{(3,4)}_{\vec{p}}(x) = \psi^{(3,4)}_{\vec{p}}e^{+\frac{i}{\hbar}px}$

$(p^\mu_R) = (mc, \vec{0}) \longrightarrow p = (p^0, \vec{p})$

$p^\mu_R p_{R\mu} = (mc)^2 = (p^0)^2 - \vec{p} \Rightarrow p^0 = \frac{E_{\vec{p}}}{c} = \sqrt{(\vec{p}c)^2 + m^2c^2}$

- Boosted spinor amplitudes: $\psi^{(r)}_{\vec{p}} = D(B)\,\psi^{(r)}$; $r = 1,2,3,4$

$$D(B) = \begin{pmatrix}D^{(1/2,0)}(B) & 0\\ 0 & D^{(0,1/2)}(B)\end{pmatrix} = \begin{pmatrix}e^{-\vec{\sigma}\vec{\beta}/2} & 0\\ 0 & e^{\vec{\sigma}\vec{\beta}/2}\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}} & 0\\ 0 & \sqrt{\frac{p\tilde{\sigma}}{mc}}\end{pmatrix}}_{\text{efficient shortcut notation}} = \underbrace{\begin{pmatrix}\frac{p\sigma + mc}{\sqrt{2mc(p^0+mc)}} & 0\\ 0 & \frac{p\tilde{\sigma} + mc}{\sqrt{2mc(p^0+mc)}}\end{pmatrix}}_{\text{physical expression}} \quad \text{with } p^0$$

- Formal result:

$$\boxed{\psi^{(1,2)}_{\vec{p}} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\boxed{\chi(\pm\frac{1}{2})}\\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi(\pm\frac{1}{2})\end{pmatrix}, \quad \psi^{(3,4)}_{\vec{p}} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi^c(\pm 1/2)\\ -\sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi^c(\pm 1/2)\end{pmatrix}}$$

- Side calculation:

$$(p\sigma)(p\tilde{\sigma}) = p_\mu p_\nu\sigma^\mu\tilde{\sigma}^\nu = p_\mu p_\nu\underbrace{\frac{1}{2}(\sigma^\mu\tilde{\sigma}^\nu + \sigma^\nu\tilde{\sigma}^\mu)}_{=\frac{1}{2}2\,g^{\mu\nu}I} = p^2 I = (mc)^2 I \quad (*)$$

- Proof that we have really constructed solutions of Dirac equation:

$$\gamma^\mu p_\mu\psi^{(1,2)}_{\vec{p}} = \begin{pmatrix}0 & \sigma^\mu\\ \tilde{\sigma}^\mu & 0\end{pmatrix}p_\mu\psi^{(1,2)}_{\vec{p}} = \begin{pmatrix}0 & p\sigma\\ p\tilde{\sigma} & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi(\lambda)\\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi(\lambda)\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}mc\begin{pmatrix}\frac{p\sigma}{mc}\sqrt{\frac{p\tilde{\sigma}}{mc}}\,\chi(\lambda)\\ \frac{p\tilde{\sigma}}{mc}\sqrt{\frac{p\sigma}{mc}}\,\chi(\lambda)\end{pmatrix} = \frac{1}{\sqrt{2}}mc\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\cdot\boxed{\sqrt{\frac{p\sigma}{mc}\frac{p\tilde{\sigma}}{mc}}}\,\chi(\lambda)\\ \sqrt{\frac{p\tilde{\sigma}}{mc}}\cdot\boxed{\sqrt{\frac{p\tilde{\sigma}}{mc}\frac{p\sigma}{mc}}}\,\chi(\lambda)\end{pmatrix} = mc\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\psi^{(1,2)}_{\vec{p}} \quad ✓$$

$=1$

analogously: $\psi^{(3,4)}_{\vec{p}}$

- $\psi^{(3,4)}_{\vec{p}}$ are charge conjugated of $\psi^{(1,2)}_{\vec{p}}$: see nice proof in lecture notes

9.14.3 Orthonormality Relations:

- Lecture notes: orthonormality relations between spinor amplitudes $\psi^{(r)}_{\vec{p}}$; $r = 1,2$ and $\psi^{(r)}_{-\vec{p}}$; $r = 3,4$

$$\psi^{(r)+}_{\varepsilon_r\vec{p}}\,\psi^{(r')}_{\varepsilon_{r'}\vec{p}} = \frac{E_{\vec{p}}}{mc^2}\delta_{r,r'} \quad ; \quad \varepsilon_r = \begin{cases}+1 ; & r = 1,2\\ -1 ; & r = 3,4\end{cases}$$

- Fundamental solutions: $\psi^{(r)}_{\vec{p}}(\vec{x},t) = \psi^{(r)}_{\vec{p}}\,e^{-\frac{i}{\hbar}\varepsilon_r(E_{\vec{p}}t - \vec{p}\vec{x})}$
- check for orthonormality:

$$\int d^3x\,\psi^{(r)+}_{\vec{p}}(\vec{x},t)\psi^{(r')}_{\vec{p}'}(\vec{x},t) = \psi^{(r)+}_{\vec{p}}\psi^{(r')}_{\vec{p}'}\,e^{\frac{i}{\hbar}(\varepsilon_{r'}E_{\vec{p}'} - \varepsilon_r E_{\vec{p}})t}\underbrace{\int d^3x\,e^{\frac{i}{\hbar}(\varepsilon_r\vec{p} - \varepsilon_{r'}\vec{p}')\vec{x}}}_{=(2\pi\hbar)^3\underbrace{\delta(\varepsilon_r\vec{p} - \varepsilon_{r'}\vec{p}')}_{=\delta(\vec{p}' - \varepsilon_r\varepsilon_{r'}\vec{p})}}$$

$$= \underbrace{\psi^{(r)+}_{\vec{p}}\psi^{(r')}_{\varepsilon_r\varepsilon_{r'}\vec{p}}}_{=\frac{E_{\vec{p}}}{Mc^2}\delta_{rr'}}\;\underbrace{e^{\underbrace{\frac{i}{\hbar}(\varepsilon_{r'}E_{\underbrace{\varepsilon_{r'}\varepsilon_r}_{=1}\vec{p}} - \varepsilon_r E_{\vec{p}})t}_{=0}}}_{=1}(2\pi\hbar)^3\,\delta(\vec{p}' - \underbrace{\varepsilon_r\varepsilon_{r'}}_{=1}\vec{p})$$

$$= \frac{E_{\vec{p}}}{Mc^2}(2\pi\hbar)^3\,\delta_{rr'}\,\delta(\vec{p} - \vec{p}')$$

- Proper rescaling:

$$\boxed{\psi^{(r)}_{\vec{p}}(\vec{x},t) = \sqrt{\frac{Mc^2}{E_{\vec{p}}(2\pi\hbar)^3}}\,\psi^{(r)}_{\vec{p}}\,e^{-\frac{i}{\hbar}\varepsilon_r(E_{\vec{p}}t - \vec{p}\vec{x})}}, \quad \int d^3x\,\psi^{(r)+}_{\vec{p}}(\vec{x},t)\psi^{(r')}_{\vec{p}'}(\vec{x},t) = \delta_{rr'}\,\delta(\vec{p} - \vec{p}')$$

9.14.4 Dirac Representation:

see lecture notes: plane wave solutions in Dirac representation agree with the ones obtained in problem sheet 9 (Foldy - Wouthuysen transformation)

9.15 Helicity Spinors:

So far: $\chi(\pm 1/2)$ not specified $\Longrightarrow$ we make now a particular choice

9.15.1 Rest Frame:

- spin is quantized with respect to z-axis
- choice for orthonormal bi-spinors:

$$\chi(+\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi(-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Eigenvectors of operator $D(L_3) = \sigma^3/2$ for a rotation around z-axis

$$\frac{1}{2}\sigma^3 \chi(\pm\tfrac{1}{2}) = \pm \frac{1}{2} \chi(\pm\tfrac{1}{2})$$

- charge conjugated bi-spinors:

$$\left.\begin{aligned} \chi^c(+\tfrac{1}{2}) &= c\,\chi^*(+\tfrac{1}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \chi^c(-\tfrac{1}{2}) &= c\,\chi^*(-\tfrac{1}{2}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{aligned}\right\} \quad \frac{1}{2}\sigma^3 \chi^c(\pm\tfrac{1}{2}) = \mp \frac{1}{2} \chi^c(\pm\tfrac{1}{2})$$

9.15.2 Helicity Operator:

- aim: spin 1/2 quantized with respect to momentum direction
- helicity operator: $D(\vec{L}) = \vec{\sigma}/2$

$$h(\vec{p}) = \frac{D(\vec{L})\cdot\vec{p}}{p} = \frac{\vec{\sigma}\vec{p}}{2p} = \frac{1}{2p}\left\{p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} = \frac{1}{2p}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

- helicity spinor: $h(\vec{p})\,\chi_h(\vec{p}, \pm\tfrac{1}{2}) = \pm \frac{1}{2} \chi_h(\vec{p}, \pm\tfrac{1}{2})$
- special case: $\vec{p} = p\vec{e}_z$

$$\left.\begin{aligned} h(p\vec{e}_z)\chi(+\tfrac{1}{2}) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\chi(+\tfrac{1}{2}) \\ h(p\vec{e}_\varphi)\chi(-\tfrac{1}{2}) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{1}{2}\chi(-\tfrac{1}{2}) \end{aligned}\right\} \chi_h(p\vec{e}_z, \pm\tfrac{1}{2}) = \chi(\pm\tfrac{1}{2})$$

9.15.3 Uniformly Moving Reference Frame:

- spherical coordinates: $\vec{p} = p\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$

$$\left.\begin{aligned} &\vec{p} = p\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \\ &R(\vartheta,\varphi) = R_z(\varphi)\,R_y(\vartheta) \quad \text{(see Chapter 8)} \end{aligned}\right\} \quad \vec{p} = R(\vartheta,\varphi)\,p\vec{e}_z$$

- Representation in bi-spinor space:

$$D(R(\vartheta,\varphi)) = D(R_z(\varphi))\,D(R_y(\vartheta))$$

$$\chi_h(\vec{p}, \pm\tfrac{1}{2}) = D(R(\vartheta,\varphi))\,\chi_h(p\vec{e}_z, \pm\tfrac{1}{2})$$