8. Maxwell Field:

Motivation:

- All electrodynamic processes are described by Maxwell theory
- Apparent contradiction:
  Maxwell theory $\hat{=}$ first-quantized theory (2π) no Planck constant appears
- Resolution: assume that quanta of Maxwell field, i.e. the photons, have a mass M
  - spatio-temporal derivatives: due to dimensional reasons the mass M appears in terms of the inverse Compton wave length: $\lambda_c^{-1} = \frac{Mc}{\hbar}$
  - limit $M \to 0$: $\hbar$ disappears

Plan for this Chapter:

- Relativistic covariant formulation of Maxwell theory
- Canonical field quantization
- Main obstacle: vanishing of rest mass implies local gauge symmetry
  $\to$ difficult to handle in second quantization
- Result: properties of a photon like its energy, momentum and spin ← Problem Sheet 7
- Photon propagator: important building block in Feynman diagrams of QED

8.1 Maxwell Equations:

- Forces upon electric charges which are at rest or move: electric field $\vec{E}$, magnetic induction $\vec{B}$
- Physically: generated by charge density ρ or current density $\vec{j}$
- Mathematically: described by partial differential equations set up by James Clerk Maxwell
- General structure is based on Helmholtz vector decomposition theorem:
  any vector field is determined from the knowledge of its divergence, its rotation and its boundary conditions

	electric field	magnetic field
homogeneous Maxwell equations	$\mathrm{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (1)	$\mathrm{div}\,\vec{B} = 0$ (3)
inhomogeneous Maxwell equations	$\mathrm{div}\,\vec{E} = \frac{1}{\varepsilon_0}\rho$ (2)	$\mathrm{rot}\,\vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ (4)

- Constants:
  vacuum dielectric constant ε_0
  vacuum permeability μ_0
  $\Big\}$ vacuum light velocity $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ (5)
- ρ and $\vec{j}$ are not independent from each other:
  $$\frac{\partial \rho}{\partial t} \overset{(2)}{=} \varepsilon_0\, \mathrm{div}\frac{\partial \vec{E}}{\partial t} \overset{(4)}{=} \varepsilon_0 \Big\{ c^2 \underbrace{\mathrm{div}\,\mathrm{rot}\,\vec{B}}_{=0} - \underbrace{c^2 \mu_0}_{\overset{(5)}{=}\frac{1}{\varepsilon_0}} \mathrm{div}\,\vec{j} \Big\} = -\mathrm{div}\,\vec{j}$$
  $\Rightarrow$ continuity equation $\hat{=}$ charge conservation
- International System of Unit SI (= Système international d'unités):
  used here, quite cumbersome, but results directly used for experiments
- QFT often relies on Lorentz-Heaviside system: $\varepsilon_0 = \mu_0 = c = 1$

8.2 Local Gauge Symmetry

- Consequences of homogeneous Maxwell equations:
  $$\mathrm{div}\,\vec{B} \overset{(3)}{=} 0 \quad \Rightarrow \quad \vec{B} = \mathrm{rot}\,\vec{A} \quad \text{vector potential} \qquad (6)$$
  $$(6) \text{ in } (1): \quad \mathrm{rot}\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \vec{0} \quad \Rightarrow \quad \vec{E} = -\mathrm{grad}\,\varphi - \frac{\partial \vec{A}}{\partial t} \quad \text{scalar potential} \qquad (7)$$
- Consequences of inhomogeneous Maxwell equations:
  $$\frac{1}{\varepsilon_0}\rho \overset{(2)}{=} \mathrm{div}\,\vec{E} \overset{(7)}{=} -\underbrace{\mathrm{div}\,\mathrm{grad}}_{=\Delta}\varphi - \frac{\partial}{\partial t}\mathrm{div}\,\vec{A} \qquad (8)$$
  $$\mu_0 \vec{j} \overset{(4)}{=} \mathrm{rot}\,\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \overset{(6),(7)}{=} \underbrace{\mathrm{rot}\,\mathrm{rot}}_{=\mathrm{grad}\,\mathrm{div} - \Delta}\vec{A} - \frac{1}{c^2}\left\{-\mathrm{grad}\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right\}$$
  $$\Rightarrow \quad \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \Delta \vec{A} + \mathrm{grad}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \mathrm{div}\,\vec{A}\right) = \mu_0 \vec{j} \qquad (9)$$
  coupled partial differential equation for φ, $\vec{A}$
- Local gauge symmetry: $\varphi' = \varphi + \frac{\partial \Lambda}{\partial t}$ (10) with $\Lambda = \Lambda(\vec{x}, t)$ gauge function
  $\vec{A}' = \vec{A} - \mathrm{grad}\,\Lambda$ (11)
- Change of description of electromagnetic field but the physics is not changed
  $$\vec{B}' \overset{(6)}{=} \mathrm{rot}\,\vec{A}' \overset{(11)}{=} \mathrm{rot}\,\vec{A} - \underbrace{\mathrm{rot}\,\mathrm{grad}\,\Lambda}_{=\vec{0}} = \mathrm{rot}\,\vec{A} \overset{(6)}{=} \vec{B}$$
  $$\vec{E}' \overset{(7)}{=} -\mathrm{grad}\,\varphi' - \frac{\partial \vec{A}'}{\partial t} \overset{(10),(11)}{=} -\mathrm{grad}\,\varphi - \cancel{\frac{\partial}{\partial t}\mathrm{grad}\,\Lambda} + \cancel{\frac{\partial}{\partial t}\mathrm{grad}\,\Lambda} - \frac{\partial \vec{A}}{\partial t} \overset{(7)}{=} \vec{E}$$

similarly: (8)+(9) also invariant with respect to (10), (11)

- Choice of a particular gauge allows to decouple equations of motion (8), (9)
  → discuss two prominent examples

Coulomb Gauge:

- Longitudinal part of vector potential vanishes: $\operatorname{div}\vec{A} = 0$ (12)
  (12) in (8): $\Delta\varphi = -\frac{\rho}{\varepsilon_0}$ (13)
  (12) in (9): $\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \Delta\vec{A} = \mu_0\vec{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{grad}\varphi$ (14)
- Solution of Poisson Eq. (13): $\varphi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x\,\frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$ (15)
  ⇒ $\varphi(\vec{x},t)$ is no longer a dynamical degree of freedom
- Due to (12) and (15) from the 4 degrees of freedom $\varphi, \vec{A}$ only two are relevant
  ⇒ Later on: quantization yields two types of photons
- Advantage of Coulomb Gauge: remaining 2 dynamical degrees of freedom can be physically identified with 2 transversal degrees of freedom of $\vec{A}$
  → used in quantum optics and in this lecture
- Disadvantage of Coulomb Gauge: not manifestly Lorentz invariant, valid only in a particular inertial system

Lorentz Gauge:

- Definition: $\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \operatorname{div}\vec{A} = 0$ (16)
- (16) in (8), (9):
  $\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = \frac{1}{\varepsilon_0}\rho$ (17)
  $\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \Delta\vec{A} = \mu_0\vec{j}$ (18)
- Advantage: (16)–(18) are Lorentz invariant
- Disadvantage:
  - Quantization worked out by Suraj Gupta and Konrad Bleuler
  - Apart from two physical transversal degrees of freedom of $\vec{A}$ also an unphysical longitudinal degree of freedom of $\vec{A}$ appears, which to be eliminated afterwards

8.3 Field Strength Tensors:

- Electric field $\vec{E}$ and magnetic induction $\vec{B}$: consider the components as elements of an anti-symmetric 4×4 matrix F, which is called electromagnetic field strength tensor
- Contravariant components:

$$(F^{\mu\nu}) = \left(F^{\mu\nu}(\vec{E},\vec{B})\right) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

- Covariant components: $F_{\mu\nu} = g_{\mu\lambda}g_{\nu\kappa}F^{\lambda\kappa}$

$$(F_{\mu\nu}) = \left(F^{\mu\nu}(-\vec{E},\vec{B})\right) = \begin{pmatrix} 0 & & & \\ -E_x/c & 0 & & \\ -E_y/c & B_z & 0 & \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

- Dual electromagnetic field strength tensor: ${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$

$$({}^*F^{\mu\nu}) = \left(F^{\mu\nu}\left(c\vec{B}, -\frac{\vec{E}}{c}\right)\right) = \begin{pmatrix} 0 & & & \\ B_x & 0 & & \\ B_y & -E_z/c & 0 & \\ B_z & E_y & -E_x/c & 0 \end{pmatrix}$$

$$({}^*F_{\mu\nu}) = \ldots$$

- Maxwell equations formulated in terms of F and ${}^*F$:

homogeneous Maxwell equations:	inhomogeneous Maxwell equation
$\partial_\mu {}^*F^{\mu\nu} = 0$	$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$

contravariant current density $(j^\nu) = \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$

$$\underbrace{\partial_\mu}_{=\frac{\partial}{\partial x^\mu}} {}^*F^{\mu\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} 0 & & & \\ B_x & 0 & & \\ B_y & -E_z/c & 0 & \\ B_z & E_y & -E_x/c & 0 \end{pmatrix} = \left(\operatorname{div}\vec{B}, -\frac{1}{c}\operatorname{rot}\vec{E} - \frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = (0,\vec{0}) \checkmark$$

$$\partial_\mu F^{\mu\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} 0 & & & \\ E_x/c & & & \\ E_y/c & B_z & 0 & \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} = \left(\frac{1}{c}\operatorname{div}\vec{E}, \operatorname{rot}\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right) = \mu_0\begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix} \checkmark$$

– Consistency relation: $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \quad | \cdot \partial_\nu$

$$\underbrace{\partial_\nu \partial_\mu}_{\text{symm. in } \mu,\nu} \underbrace{F^{\mu\nu}}_{\text{anti-symmetric in } \mu,\nu} = \mu_0 \partial_\nu j^\nu = 0 \Rightarrow \partial_\nu j^\nu = 0$$

Lorentz-invariant formulation of continuity eq.

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \vec\nabla\right)\begin{pmatrix} c\rho \\ \vec j \end{pmatrix} = \frac{\partial \rho}{\partial t} + \vec\nabla \vec j = 0$$

8.4 Four-Vector Potential:

– combine φ and $\vec A$ to: $(A^\lambda) = \begin{pmatrix} \varphi/c \\ \vec A \end{pmatrix}$

– Unite (6) and (7) via: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ (*)

with $(\partial^\mu) = \left(\frac{\partial}{\partial x_\mu}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec\nabla\right)$

– Examples:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial \varphi}{\partial x} = -\frac{1}{c}E_x$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = -B_z$$

– Homogeneous Maxwell equations:

$$\partial_\mu {}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}\partial_\mu \underbrace{F_{\lambda\kappa}}_{=\partial_\lambda A_\kappa - \partial_\kappa A_\lambda} = \frac{1}{2}\left(\varepsilon^{\mu\nu\lambda\kappa}\partial_\mu\partial_\lambda A_\kappa - \underbrace{\varepsilon^{\mu\nu\lambda\kappa}\partial_\mu\partial_\kappa A_\lambda}_{\text{renaming } = \varepsilon^{\lambda\nu\kappa\mu}\partial_\lambda\partial_\mu A_\kappa}\right)$$

$$\varepsilon^{\lambda\nu\kappa\mu} = -\varepsilon^{\mu\nu\kappa\lambda} = \varepsilon^{\mu\nu\lambda\kappa}$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}\left(\partial_\mu\partial_\lambda - \partial_\lambda\partial_\mu\right)A_\kappa \underset{\text{theorem of Schwarz}}{=} 0$$

→ homogeneous Maxwell equations already fulfilled

– Inhomogeneous Maxwell Equations:

$$\partial_\mu F^{\mu\nu} = \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 j^\nu \overset{?}{=} (8), (9)$$

– Local gauge transformation: $A'^\mu = A^\mu + \partial^\mu \Lambda \;\hat{=}\; (10), (11)$ (**)

– Invariance:

$$F'^{\mu\nu} \overset{(*)}{=} \partial^\mu A'^\nu - \partial^\nu A'^\mu \overset{(**)}{=} \partial^\mu A^\nu - \partial^\nu A^\mu + \underbrace{\left(\partial^\mu\partial^\nu - \partial^\nu\partial^\mu\right)\Lambda}_{=0 \text{ theorem of Schwarz}} \overset{(*)}{=} F^{\mu\nu}$$

$\Rightarrow {}^*F'^{\mu\nu} = {}^*F^{\mu\nu}$

$\Rightarrow$ Physics invariant

8.5 Euler-Lagrange Equations: (Problem Sheet 4)

$$S[A_\nu(\cdot)] = \frac{1}{c}\int \underbrace{d^4x}_{=c\,dt \cdot d^3x}\, \mathcal{L}, \quad \mathcal{L} = \mathcal{L}\left(A_\nu(x^\lambda); \partial_\mu A_\nu(x^\lambda)\right)$$

Hamilton principle: Euler-Lagrange equations

$$\frac{\delta S}{\delta A_\nu(x^\lambda)} = \frac{\partial \mathcal{L}}{\partial A_\nu(x^\lambda)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu A_\nu(x^\lambda)} = 0$$

Ansatz: $\mathcal{L} = \alpha F^{\lambda\kappa}F_{\lambda\kappa} + \beta j^\lambda A_\lambda$ (α, β: constants) $\rightarrow \partial_\mu F^{\mu\nu} = \frac{\beta}{4\alpha} j^\nu \overset{!}{=} \mu_0 j^\nu$

Lorentz-invariant Lagrange density (scalar)

$\Rightarrow \beta = 4\alpha\mu_0$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \beta j^\nu, \quad \frac{\partial \mathcal{L}}{\partial\, \partial_\mu A_\nu} = \underbrace{\ldots}_{\text{see exercises}} = 4\alpha F^{\mu\nu}$$

Result: $\mathcal{L} = \alpha F^{\mu\nu}F_{\mu\nu} + 4\alpha\mu_0 j^\mu A_\mu$ (α: not yet determined)

1. Way to determine α: Noether theorem, Problem sheet 4/5 ⇒ energy-momentum tensor
⇒ energy + momentum conservation ⇒ Appendix of Latex manuscript
2. Way to determine α: down-to earth procedure, Legendre transformation

Aim for tomorrow: provide all ingredients necessary to solve Problem sheet 7, which is supposed to be submitted until 31.12.2020