8.6 Hamilton Function:

- specialization: free electromagnetic field, $\rho(\vec{x},t)=0$, $\vec{j}(\vec{x},t)=\vec{0}$ $\quad$ (I)
- choice: Coulomb gauge, i.e. $\operatorname{div}\vec{A}(\vec{x},t)=0$ $\quad$ (II)

  ≙ standard formulation of second quantized Maxwell theory as it is discussed in quantum optics

$$\varphi(\vec{x},t) \overset{(15)}{=} \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \overset{(I)}{\equiv} 0 \qquad (III)$$

(II)+(III): radiation gauge

$$\xrightarrow{(14),\ (I)} \quad \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{x},t)}{\partial t^2} - \Delta\vec{A}(\vec{x},t)=\vec{0} \qquad (IV)$$

vector potential determined from wave equation (IV) under implementation of Coulomb gauge (II),

- known vector potential:

  magnetic induction: $\vec{B}(\vec{x},t) \overset{(6)}{=} \operatorname{rot}\vec{A}(\vec{x},t)$

  electric field: $\vec{E}(\vec{x},t) \overset{(7)}{=} -\operatorname{grad}\varphi(\vec{x},t) - \frac{\partial\vec{A}(\vec{x},t)}{\partial t} \overset{(III)}{\equiv} -\frac{\partial\vec{A}(\vec{x},t)}{\partial t}$
- Lagrange density with (I)

$$\mathcal{L} = \alpha F^{\mu\nu}F_{\mu\nu} = 2\alpha\left(\vec{B}^2 - \frac{1}{c^2}\vec{E}^2\right) = 2\alpha\left\{\left[\vec{\nabla}\times\vec{A}(\vec{x},t)\right]^2 - \frac{1}{c^2}\left[\frac{\partial\vec{A}(\vec{x},t)}{\partial t}\right]^2\right\}$$

- Momentum field $\vec{\pi}(\vec{x},t)$ which is canonically conjugated to vector potential $\vec{A}$:

$$\vec{\pi}(\vec{x},t) = \frac{\delta A}{\delta\frac{\partial\vec{A}(\vec{x},t)}{\partial t}} = \frac{\partial\mathcal{L}}{\partial\frac{\partial\vec{A}(\vec{x},t)}{\partial t}} = -\frac{4\alpha}{c^2}\frac{\partial\vec{A}(\vec{x},t)}{\partial t} \Rightarrow \frac{\partial\vec{A}(\vec{x},t)}{\partial t} = -\frac{c^2}{4\alpha}\vec{\pi}(\vec{x},t)$$

- Legendre transformation:

$$\mathcal{H} = \vec{\pi}\cdot\frac{\partial\vec{A}}{\partial t} - \mathcal{L} = -\frac{c^2}{4\alpha}\vec{\pi}^2 - 2\alpha(\vec{\nabla}\times\vec{A})^2 + \frac{2\alpha}{c^2}\cdot\frac{c^4}{4\alpha^2}\vec{\pi}^2 = -\frac{c^2}{8\alpha}\vec{\pi}^2 - 2\alpha(\vec{\nabla}\times\vec{A})^2$$

- Comparison to known energy density of Maxwell theory:

$$\mathcal{H} = \frac{\varepsilon_0}{2}\vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 = \frac{\varepsilon_0}{2}\frac{c^4}{16\alpha^2}\vec{\pi}^2 + \frac{1}{2\mu_0}(\vec{\nabla}\times\vec{A})^2$$

$$\left.\begin{array}{lll} (\vec{\nabla}\times\vec{A})^2: & \frac{1}{2\mu_0} = -2\alpha \Rightarrow & \alpha = -\frac{1}{4\mu_0} \\ \vec{\pi}^2: & \frac{\varepsilon_0}{2}\frac{c^4}{16\alpha^2} = -\frac{c^2}{8\alpha} \Rightarrow & \alpha = -\frac{1}{4\mu_0} \end{array}\right\} \checkmark$$

- Remark:

$$\vec{\pi}(\vec{x},t) = \underbrace{-\frac{4\alpha}{c^2}}\frac{\partial\vec{A}(\vec{x},t)}{\partial t} = \underbrace{\varepsilon_0}_{\text{"Mass"}}\cdot\frac{\partial\vec{A}(\vec{x},t)}{\partial t} \qquad (\text{"}\vec{p} = m\cdot\vec{v}\text{"})$$

$$-4\cdot\varepsilon_0\cdot\mu_0\frac{-1}{4\mu_0} = \varepsilon_0$$

$$\Rightarrow \quad \vec{\pi}(\vec{x},t) \sim \vec{E}(\vec{x},t)$$

- Hamilton function:

$$H = \int d^3x\,\mathcal{H} = \frac{1}{2}\int d^3x\left\{\frac{1}{\varepsilon_0}\vec{\pi}^2(\vec{x},t) + \frac{1}{\mu_0}\cdot\left[\vec{\nabla}\times\vec{A}(\vec{x},t)\right]^2\right\}$$

- simplified:

$$(\vec{\nabla}\times\vec{A})^2 = \varepsilon_{jkl}\partial_k A_l\,\varepsilon_{jmn}\partial_m A_n = \underbrace{\varepsilon_{klj}\varepsilon_{mnj}}_{=\delta_{km}\delta_{ln}-\delta_{kn}\delta_{lm}}\partial_k A_l\,\partial_m A_n$$

$$= \partial_k A_l\,\partial_k A_l \underbrace{-\,\partial_k A_l\,\partial_l A_k}_{=-\underbrace{\partial_k(A_l\,\partial_l A_k)}_{=0\text{ due to Gauß theorem}} + A_l\,\partial_l\underbrace{\partial_k A_k}_{\operatorname{div}\vec{A}=0\text{ Coulomb gauge}}}$$

$$\Rightarrow \quad H = \frac{1}{2}\int d^3x\left\{\frac{1}{\varepsilon_0}\pi_k(\vec{x},t)\cdot\pi_k(\vec{x},t) + \frac{1}{\mu_0}\partial_k A_l(\vec{x},t)\,\partial_k A_l(\vec{x},t)\right\}$$

8.7 Canonical Field Quantization:

$$\begin{array}{ccc} \text{classical fields} & & \text{field operators} \\ A_j(\vec{x},t) & \xrightarrow[\text{quantization}]{\text{second}} & \hat{A}_j(\vec{x},t) \\ \pi_j(\vec{x},t) & & \hat{\pi}_j(\vec{x},t) \end{array}$$

How to define the equal-time commutation relations?

- Field operator $\hat{A}_j(\vec{x},t)$ and momentum operators $\hat{\pi}_j(\vec{x},t)$ commute among themselves, respectively:

$$[\hat{A}_k(\vec{x},t), \hat{A}_l(\vec{x}',t)]_- = 0 = [\hat{\pi}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- \qquad (*)$$

- commutation relations between $\hat{A}_j(\vec{x},t)$ and $\hat{\pi}_j(\vec{x},t)$ are more intriguing due to demanding compatibility with Coulomb gauge

$$\partial_k A_k(\vec{x},t) = 0 \longrightarrow \partial_k \hat{A}_k(\vec{x},t) = 0$$

- First trial: naive commutation relation WRONG

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar \boxed{\delta_{kl} \quad \delta(\vec{x}-\vec{x}')}$$

leads to a contradiction:

1) $\partial_k [\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = [\partial_k \hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = 0$

2) $i\hbar\, \partial_k\, \delta_{kl}\, \delta(\vec{x}-\vec{x}') = i\hbar\, \partial_l\, \delta(\vec{x}-\vec{x}') \neq 0$

- second trial:

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar \underbrace{\delta^T_{kl}(\vec{x}-\vec{x}')}_{\text{transversal delta function}} = i\hbar \int \frac{d^3k}{(2\pi)^3}\, \delta^T_{kl}(\vec{k})\, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$\partial_k [\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = 0 = i\hbar \int \frac{d^3k}{(2\pi)^3}\, i \boxed{k_k\, \delta^T_{kl}(\vec{k})}\, e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$k_k \delta^T_{kl}(\vec{k}) = 0$
transversality condition

$$\delta^T_{kl}(\vec{k}) = \underbrace{\delta_{kl} \cdot 1}_{\text{stems from naive approach above}} + \underbrace{\Delta_{kl}(\vec{k})}_{\text{correction terms needed}}$$

$$= k_k k_l \cdot \underbrace{f(\vec{k})}_{\text{undetermined}}$$

$$k_k\, \delta^T_{kl}(\vec{k}) = k_l + \underbrace{k_k k_k}_{= \vec{k}^2} k_l \cdot f(\vec{k}) = k_l \underbrace{(1 + \vec{k}^2 \cdot f(\vec{k}))}_{} \overset{!}{=} 0$$

$\Rightarrow f(\vec{k}) = -\frac{1}{\vec{k}^2}$

$$\delta^T_{kl}(\vec{x}-\vec{x}') = \int \frac{d^3k}{(2\pi)^3} \left\{ \delta_{kl} - \frac{k_k k_l}{\vec{k}^2} \right\} e^{i\vec{k}(\vec{x}-\vec{x}')}$$

$$= \delta_{kl} \underbrace{\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}(\vec{x}-\vec{x}')}}_{= \delta(\vec{x}-\vec{x}')} + \partial'_k \partial'_l \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{\vec{k}^2}\, e^{i\vec{k}(\vec{x}-\vec{x}')}}_{= \frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{x}'|}}$$

Greens function of Poisson equation in electrostatics

$$\Longrightarrow \delta^T_{kl}(\vec{x}-\vec{x}') = \delta_{kl}\, \delta(\vec{x}-\vec{x}') + \frac{\partial'_k \partial'_l}{4\pi} \frac{1}{|\vec{x}-\vec{x}'|}$$

$$[\hat{A}_k(\vec{x},t), \hat{\pi}_l(\vec{x}',t)]_- = i\hbar\, \delta^T_{kl}(\vec{x}-\vec{x}')$$

- Caveat:
  - One tries hard to get such fundamental commutation relations — but are they correct?
  - Derivation verified by deducing consequences and checking them against experimental measurements
  - Later on: consistent description of second quantized Maxwell field and its quanta, i.e. the photons → problem sheet 7 + exercises + lecture on 4.1.21

8.8 Heisenberg Equations:

- Hamilton operator:

$$\hat{H} = \frac{1}{2} \int d^3x \left\{ \frac{1}{\varepsilon_0} \underbrace{\hat{\pi}_k(\vec{x},t)\, \hat{\pi}_k(\vec{x},t)}_{\text{operator ordering no problem due to } (*)} + \frac{1}{\mu_0} \underbrace{\partial_k \hat{A}_l(\vec{x},t)\, \partial_k \hat{A}_l(\vec{x},t)}_{} \right\}$$

- Heisenberg equation for $\hat{A}_j(\vec{x},t)$:

$$i\hbar \frac{\partial \hat{A}_j(\vec{x},t)}{\partial t} = [\hat{A}_j(\vec{x},t), \hat{H}]_- \overset{\uparrow}{=} \frac{i\hbar}{\varepsilon_0} \int d^3x'\, \delta^T_{jk}(\vec{x}-\vec{x}')\, \hat{\pi}_k(\vec{x},t)$$

1) abc rule 2) commutation relations

$$\underset{\uparrow}{=} \frac{i\hbar}{\varepsilon_0}\left\{\hat{\pi}_j(\vec{x},t) - \frac{1}{4\pi}\int d^3x' \left(\partial_j' \frac{1}{|\vec{x}-\vec{x}'|}\right) \underbrace{\partial_k' \hat{\pi}_k(\vec{x}',t)}\right\}$$

1) transversal delta function 2) partial integration

$$(*)\ \vec{\pi}(\vec{x},t) = \varepsilon_0 \cdot \frac{\partial \vec{A}(\vec{x},t)}{\partial t} \longrightarrow \boxed{\hat{\vec{\pi}}(\vec{x},t) = \varepsilon_0 \frac{\partial \hat{\vec{A}}(\vec{x},t)}{\partial t}}$$

$$= \varepsilon_0 \frac{\partial}{\partial t} \underbrace{\partial_k \hat{A}_k(\vec{x}',t)}_{=0}$$

coulomb gauge

- Heisenberg equation for momentum operators:

$$i\hbar \frac{\partial \hat{\pi}_j(\vec{x},t)}{\partial t} = [\hat{\pi}_j(\vec{x},t), \hat{H}]_- \underset{\uparrow}{=} -\frac{i\hbar}{\mu_0}\int d^3x' \partial_k' \partial_k \delta^T_{jl}(\vec{x}-\vec{x}') \cdot \partial_k' \hat{A}_l(\vec{x}',t)$$

1) abc rule 2) commutation relations

$$\underset{\uparrow}{=} -\frac{i\hbar}{\mu_0}\left\{\partial_k \partial_k \hat{A}_j(\vec{x},t) - \int d^3x' \left(\frac{\partial_j' \partial_k'}{4\pi} \frac{1}{|\vec{x}-\vec{x}'|}\right) \partial_k' \underbrace{\partial_l' \hat{A}_l(\vec{x}',t)}_{\equiv 0 \text{ coulomb gauge}}\right\}$$

1) transversal delta function

$$\Rightarrow \frac{\partial \vec{\pi}(\vec{x},t)}{\partial t} = -\frac{1}{\mu_0} \Delta \hat{\vec{A}}(\vec{x},t) \quad (**)$$

$(*)$ and $(**)$:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\vec{A}}(\vec{x},t) - \Delta \hat{\vec{A}}(\vec{x},t) = 0$$

$\Rightarrow$ solve wave equation by taking into account coulomb gauge!

8.9 Decomposition in Plane:

ansatz:
$$\hat{\vec{A}}(\vec{x},t) = \int d^3k\, \hat{\vec{A}}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

insertion in wave equation:
$$\frac{\partial^2}{\partial t^2}\hat{\vec{A}}(\vec{k},t) + \omega_{\vec{k}}^2 \hat{\vec{A}}(\vec{k},t) = 0$$

harmonic oscillator differential equation

dispersion: $\omega_{\vec{k}} = c|\vec{k}|$

$$\hat{\vec{A}}(\vec{k},t) = \hat{\vec{A}}^{(1)}(\vec{k})\, e^{-i\omega_{\vec{k}} t} + \hat{\vec{A}}^{(2)}(\vec{k})\, e^{+i\omega_{\vec{k}} t}$$

$$\Rightarrow \hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \underline{\hat{\vec{A}}^{(1)}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}} + \underbrace{\hat{\vec{A}}^{(2)}(\vec{k})\, e^{i(\vec{k}\vec{x}+\omega_{\vec{k}}t)}}_{\hat{\vec{A}}^{(2)}(-\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}} \right\}$$

$$\vec{k} \rightarrow -\vec{k}$$

$$\omega_{\vec{k}} = \omega_{-\vec{k}}$$

$$\hat{\vec{A}}^+(\vec{x},t) = \int d^3k \left\{ \underline{\hat{\vec{A}}^{(2)+}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}} + \underline{\hat{\vec{A}}^{(1)+}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}} \right\}$$

$$\Rightarrow \hat{\vec{A}}(\vec{x},t) = \hat{\vec{A}}^+(\vec{x},t) \quad \Rightarrow \quad \hat{\vec{A}}^{(1)}(\vec{k}) = \hat{\vec{A}}^{(2)+}(-\vec{k})$$

$$\hat{\vec{A}}(\vec{k}) := \hat{\vec{A}}^{(1)}(\vec{k}), \quad \hat{\vec{A}}^{(2)}(-\vec{k}) = \hat{\vec{A}}^{(1)+}(+\vec{k}) = \hat{\vec{A}}^+(\vec{k})$$

$$\Rightarrow \hat{\vec{A}}(\vec{x},t) = \int d^3k \left\{ \hat{\vec{A}}(\vec{k})\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} + \hat{\vec{A}}^+(\vec{k})\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} \right\}$$

8.10 Construction of Polarization Vectors:

- Needed: better understand of plane waves
- Two linearly polarized plane wave:

$$\vec{A}_1(\vec{x},t) = \underbrace{A_1}_{\text{amplitude}} \underbrace{\vec{\varepsilon}_1}_{\text{polarization vector}} e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}, \qquad \vec{A}_2(\vec{x},t) = A_2 \vec{\varepsilon}_2\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}$$

normalization: $\vec{\varepsilon}_1 \cdot \vec{\varepsilon}_1^* = \vec{\varepsilon}_2 \cdot \vec{\varepsilon}_2^* = 1$

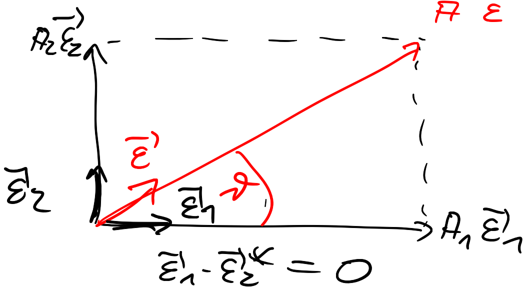
- Addition: $\vec{A}(\vec{x},t) = \vec{A}_1(\vec{x},t) + \vec{A}_2(\vec{x},t) = \{\underbrace{A_1 \vec{\varepsilon}_1 + A_2 \vec{\varepsilon}_2}_{=?}\}\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}$
- Special case: $A_1 = |A_1|\, e^{i\varphi}$, $A_2 = |A_2|\, e^{i\varphi}$

sum is also linearly polarized

$$\vec{A}(\vec{x},t) = A\, \vec{\varepsilon}\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}$$

$$A = \sqrt{|A_1|^2 + |A_2|^2}\, e^{i\varphi}$$

$$\vartheta = \arctan \frac{|A_2|}{|A_1|}$$



$$\vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2^* = 0$$

- general case: $A_1 = |A_1| e^{i\varphi_1}$, $A_2 = |A_2| e^{i\varphi_2}$, $\varphi_1 \neq \varphi_2$
  ⇒ sum is elliptically polarised plane wave
- Illustrative example: circularly polarised plane wave

$$A_1 = \frac{A_0}{\sqrt{2}}, \quad A_2 = \pm i \frac{A_0}{\sqrt{2}}$$

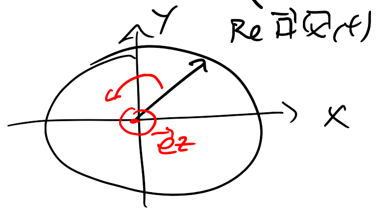
same amplitude, but phase difference 90°

$$\Rightarrow \quad \vec{A}(\vec{x},t) = \frac{A_0}{\sqrt{2}} \left(\vec{\varepsilon}_1 \pm i \vec{\varepsilon}_2\right) e^{i(\vec{k}\vec{x} - \omega_{k} t)}$$

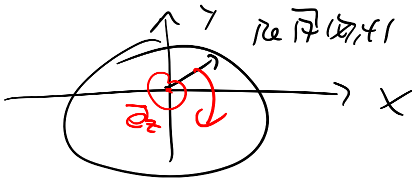
specification: $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$\vec{A}(\vec{x},t) = \boxed{\frac{A_0}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}} e^{i(kz - \omega_{k\vec{e}_z} t)}$$

$$\mathrm{Re}\,\vec{A}(\vec{x},t) = \frac{A_0}{\sqrt{2}} \begin{pmatrix} \cos(kz - \omega_{k\vec{e}_z} t) \\ \mp \sin(kz - \omega_{k\vec{e}_z} t) \\ 0 \end{pmatrix}$$



− sign (upper sign)
anti-clockwise rotation
left-circular (*)
positive helicity



+ sign (lower sign)
clockwise rotation
right-circular (*) ⟵ optics
negative helicity ⟵ elementary particle physics

(*) observer look into the coming light beam

- Helicity (Chapter 6!): $\hat{h}(\vec{k}) = \frac{\vec{k} \cdot \vec{S}}{|\vec{k}|}$, $\vec{S} = \underbrace{\begin{pmatrix} N^{23} \\ N^{31} \\ N^{12} \end{pmatrix}}_{\text{representation of Lorentz algebra in field space}} \overset{N^{\alpha\beta} = L^{\alpha\beta}}{\underline{\underline{=}}} \underbrace{\vec{L}}_{\text{generators of rotation}}$

$$\hat{h}(\vec{k}) = \frac{-i}{k} \left\{ k_x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + k_y \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + k_z \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$= \frac{i}{k} \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \quad \text{helicity operator}$$

$$\hat{h}(k\vec{e}_z)\, \vec{\varepsilon}(k\vec{e}_z, \underbrace{\lambda}_{=\pm 1}) = \lambda \underbrace{\vec{\varepsilon}(k\vec{e}_z, \lambda)}_{= \begin{pmatrix} 1 \\ i\lambda \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}}$$

$$\underbrace{i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{= \hat{h}(k\vec{e}_z)} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i\lambda \\ 0 \end{pmatrix}}_{= \vec{\varepsilon}(k\vec{e}_z, \lambda)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda \\ i \\ 0 \end{pmatrix} \underset{\lambda^2 = 1}{=} \lambda \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda i \\ 0 \end{pmatrix} = \lambda \cdot \vec{\varepsilon}(k\vec{e}_z, \lambda)$$

- Plane wave propagating with wave vector $\vec{k}$ and helicity λ:

$$\vec{A}(\vec{x},t) = A \underbrace{\vec{\varepsilon}(\vec{k},\lambda)}_{\text{defined by}} e^{i(\vec{k}\vec{x} - \omega t)}$$

$$\hat{h}(\vec{k})\, \vec{\varepsilon}(\vec{k},\lambda) = \underbrace{\lambda}_{= \pm 1} \vec{\varepsilon}(\vec{k},\lambda)$$

- Construct $\vec{\varepsilon}(\vec{k},\lambda)$:

$$k\vec{e}_z \xrightarrow[R(\Theta,\phi)]{\text{rotation}} \vec{k} \quad \Longrightarrow \quad \vec{\varepsilon}(k\vec{e}_z,\lambda) \xrightarrow[R(\Theta,\phi)]{\text{rotation}} \vec{\varepsilon}(\vec{k},\lambda)$$

- Rotation: $R(\Theta,\phi) = R_z(\phi)\, R_y(\Theta)$

$$R_z(\phi) = e^{-iL_3\phi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\Theta) = e^{-iL_z\Theta} = \begin{pmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R(\Theta,\phi) = \begin{pmatrix} \cos\Theta\cos\phi & -\sin\phi & \sin\Theta\cos\phi \\ \cos\Theta\sin\phi & \cos\phi & \sin\Theta\sin\phi \\ -\sin\Theta & 0 & \cos\Theta \end{pmatrix}$$

$$\vec k = R(\Theta,\phi)\underbrace{k\begin{pmatrix}0\\0\\1\end{pmatrix}}_{k\vec e_z} = k\begin{pmatrix}\sin\Theta\cos\phi\\ \sin\Theta\sin\phi\\ \cos\Theta\end{pmatrix} \quad ✓$$

$$\vec\varepsilon(\vec k,\lambda) = R(\Theta,\phi)\,\vec\varepsilon(k\vec e_z,\lambda) = \dots = \frac{1}{\sqrt2}\underline{\begin{pmatrix}\cos\Theta\cos\phi - \lambda i\sin\phi\\ \cos\Theta\sin\phi + \lambda i\cos\phi\\ -\sin\Theta\end{pmatrix}}$$

$$\hat h(\vec k)\,\vec\varepsilon(\vec k,\lambda) = \dots = \lambda\,\vec\varepsilon(\vec k,\lambda) \quad ✓$$

$$\vec\varepsilon(\vec k,\lambda)\Big|_{\substack{\Theta=0\\ \phi=0}} = \vec\varepsilon(k\vec e_z,\lambda) = \frac{1}{\sqrt2}\begin{pmatrix}1\\ \lambda i\\ 0\end{pmatrix} \quad ✓$$

8.11 Properties of Polarization Vectors:

- Solution of wave equation:

$$\hat{\vec A}(\vec x,t) = \int d^3k \left\{ \hat{\vec A}(\vec k)\, e^{i(\vec k\vec x - \omega_{\vec k}t)} + \hat{\vec A}^{+}(\vec k)\, e^{-i(\vec k\vec x - \omega_{\vec k}t)} \right\}$$

but: div $\vec A(\vec x,t) = 0$ Coulomb gauge $\Leftarrow$ $\vec k\cdot\hat{\vec A}(\vec k) = 0$ transversality condition

- Fourier operator vector has two transversal degrees of freedom:

$$\hat{\vec A}(\vec k) = \underbrace{N_{\vec k}}_{\text{normalization constants}} \sum_{\lambda=\pm1} \underbrace{\vec\varepsilon(\vec k,\lambda)}\; \hat a_{\vec k,\lambda}$$

transversality fulfilled provided $\vec k\cdot\vec\varepsilon(\vec k,\lambda) = 0$

Check:

$$k\begin{pmatrix}\cos\Theta\cos\phi\\ \sin\Theta\sin\phi\\ \cos\Theta\end{pmatrix}\cdot\frac{1}{\sqrt2}\begin{pmatrix}\cos\Theta\cos\phi - i\lambda\sin\phi\\ \cos\Theta\sin\phi + \lambda i\cos\phi\\ -\sin\Theta\end{pmatrix} = 0$$

$$\vec k \cdot \vec\varepsilon(\vec k,\lambda) = 0$$

- Orthonormalization relations:

$$\left.\begin{aligned} \vec\varepsilon(\vec k,\lambda)\,\vec\varepsilon(\vec k,\lambda)^* &= \dots = 1 \\ \vec\varepsilon(\vec k,\lambda)\,\vec\varepsilon(\vec k,-\lambda)^* &= \dots = 0 \end{aligned}\right\} \quad \vec\varepsilon(\vec k,\lambda)\cdot\vec\varepsilon(\vec k,\lambda') = \delta_{k,k'}$$

- Behaviour under inversion $\vec k \to -\vec k$:

$\phi \to \phi + \pi$: $\quad \sin\phi \to -\sin\phi$, $\cos\phi \to -\cos\phi$

$\Theta \to \Theta - \pi$: $\quad \sin\Theta \to \sin\Theta$, $\cos\Theta \to -\cos\Theta$

$$\Rightarrow \vec\varepsilon(-\vec k,\lambda) = \dots = \vec\varepsilon(\vec k,-\lambda) = \vec\varepsilon(\vec k,\lambda)^*$$

Result: first formula on problem sheet 7

$$\hat{\vec A}(\vec x,t) = \sum_{\lambda=\pm1}\int d^3k\, \underbrace{N_{\vec k}}_{\text{determine such that}} \left\{ \vec\varepsilon(\vec k,\lambda)\, e^{i(\vec k\vec x - \omega_{\vec k}t)}\, \underset{\substack{\downarrow\\ \text{annihilation operator}\\ \text{of a photon of wave vector } k\\ \text{and helicity } \lambda}}{\hat a_{\vec k,\lambda}} + \vec\varepsilon(\vec k,\lambda)^*\, e^{-i(\vec k\vec x - \omega_{\vec k}t)}\, \underset{\substack{\downarrow\\ \text{creation operator}\\ '' \\ ''}}{\hat a^{+}_{\vec k,\lambda}} \right\}$$