9.14 Dirac Representation:

see lecture notes: plane wave solutions in Dirac representation agree with the ones obtained in problem sheet 9 (Foldy – Wouthuysen transformation)

9.15 Helicity Spinors:

So far: $\chi(\pm 1/2)$ not specified $\Longrightarrow$ we make now a particular choice

9.15.1 Rest Frame:

- spin is quantized with respect to z-axis
- choice for orthonormal bi-spinors:

$$\chi(+\tfrac{1}{2}) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi(-\tfrac{1}{2}) = \begin{pmatrix}0\\1\end{pmatrix}$$

- Eigenvectors of operator $D(L_3) = \sigma^3/2$ for a rotation around z-axis

$$\frac{1}{2}\sigma^3\,\chi(\pm\tfrac{1}{2}) = \pm\frac{1}{2}\chi(\pm\tfrac{1}{2})$$

- charge conjugated bi-spinors:

$$\left.\begin{aligned}\chi^c(+\tfrac{1}{2}) &= c\,\chi^*(+\tfrac{1}{2}) = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\\ \chi^c(-\tfrac{1}{2}) &= c\,\chi^*(-\tfrac{1}{2}) = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}\end{aligned}\right\} \quad \frac{1}{2}\sigma^3\chi^c(\pm\tfrac{1}{2}) = \mp\frac{1}{2}\chi^c(\pm\tfrac{1}{2})$$

9.15.2 Helicity Operator:

- Aim: spin 1/2 quantized with respect to momentum direction
- helicity operator: $D(\vec{L}) = \vec{\sigma}/2$

$$h(\vec{p}) = \frac{D(\vec{L})\cdot\vec{p}}{p} = \frac{\vec{\sigma}\vec{p}}{2p} = \frac{1}{2p}\left\{p_x\begin{pmatrix}0&1\\1&0\end{pmatrix} + p_y\begin{pmatrix}0&-i\\i&0\end{pmatrix} + p_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\} = \frac{1}{2p}\begin{pmatrix}p_z & p_x - ip_y\\ p_x + ip_y & -p_z\end{pmatrix}$$

- helicity spinor: $h(\vec{p})\,\chi_h(\vec{p}, \pm\tfrac{1}{2}) = \pm\frac{1}{2}\chi_h(\vec{p},\pm\tfrac{1}{2})$
- special case: $\vec{p} = p\vec{e}_z$

$$\left.\begin{aligned}h(p\vec{e}_z)\chi(+\tfrac{1}{2}) &= \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\chi(+\tfrac{1}{2})\\ h(p\vec{e}_z)\chi(-\tfrac{1}{2}) &= \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{1}{2}\chi(-\tfrac{1}{2})\end{aligned}\right\} \quad \chi_h(p\vec{e}_z, \pm\tfrac{1}{2}) = \chi(\pm\tfrac{1}{2})$$

9.15.3 Uniformly moving reference frame:

- spherical coordinates:

$$\left.\begin{aligned}\vec{p} &= p\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}\\ R(\vartheta,\varphi) &= R_z(\varphi)\,R_y(\vartheta) \quad (\text{see Chapter } 8)\end{aligned}\right\} \quad \vec{p} = R(\vartheta,\varphi)\,p\,\vec{e}_z$$

- Representation in bi-spinor space:

$$D(R(\vartheta,\varphi)) = D(R_z(\varphi))\,D(R_y(\vartheta))$$

$$\underbrace{\chi_h(\vec{p},\pm\tfrac{1}{2})}_{\text{bi-spinors describing quantization along } \vec{p}\text{-axis}} = D(R(\vartheta,\varphi))\,\underbrace{\chi_h(p\vec{e}_z, \pm\tfrac{1}{2})}_{\text{bi-spinors describing quantization along z-axis}}$$

$$D(R_y(\vartheta)) = e^{-iD(L_2)\vartheta} \overset{\text{see above}}{=\!=} \cos(\tfrac{\vartheta}{2})\underbrace{I}_{=\begin{pmatrix}1&0\\0&1\end{pmatrix}} - i\sin(\tfrac{\vartheta}{2})\underbrace{\sigma^2}_{=\begin{pmatrix}0&-i\\i&0\end{pmatrix}} = \begin{pmatrix}\cos\vartheta/2 & -\sin\vartheta/2\\ \sin\vartheta/2 & \cos\vartheta/2\end{pmatrix}$$

$$D(R_z(\varphi)) = e^{-iD(L_3)\varphi} = \cos(\tfrac{\varphi}{2})I - i\sin(\tfrac{\varphi}{2})\underbrace{\sigma^3}_{=\begin{pmatrix}1&0\\0&-1\end{pmatrix}} = \begin{pmatrix}e^{-i\varphi/2} & 0\\ 0 & e^{+i\varphi/2}\end{pmatrix}$$

$$D(R(\vartheta,\varphi)) = D(R_z(\varphi))D(R_y(\vartheta)) = \ldots = \begin{pmatrix}\cos(\vartheta/2)e^{-i\varphi/2} & -\sin(\tfrac{\vartheta}{2})e^{-i\varphi/2}\\ \sin(\tfrac{\vartheta}{2})e^{i\varphi/2} & \cos(\vartheta/2)e^{i\varphi/2}\end{pmatrix}$$

$$\chi_h(\vec{p},+\tfrac{1}{2}) = D(R(\vartheta,\varphi))\cdot\underbrace{\chi_h(p\vec{e}_z,+\tfrac{1}{2})}_{=\begin{pmatrix}1\\0\end{pmatrix}} = \begin{pmatrix}\cos(\vartheta/2)e^{-i\varphi/2}\\ \sin(\tfrac{\vartheta}{2})e^{i\varphi/2}\end{pmatrix}$$

$$\chi_h(\vec{p},-\tfrac{1}{2}) = D(R(\vartheta,\varphi))\,\underbrace{\chi_h(p\vec{e}_z,-\tfrac{1}{2})}_{=\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}-\sin(\vartheta/2)e^{-i\varphi/2}\\ \cos(\vartheta/2)e^{i\varphi/2}\end{pmatrix}$$

$$\chi_h^c(\vec{p},+\tfrac{1}{2}) = c\,\chi_h^*(\vec{p},+\tfrac{1}{2}) = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\cos(\vartheta/2)e^{i\varphi/2}\\ \sin(\vartheta/2)e^{-i\varphi/2}\end{pmatrix} = \begin{pmatrix}-\sin\vartheta/2\, e^{-i\varphi/2}\\ \cos\vartheta/2\, e^{i\varphi/2}\end{pmatrix}$$

$$\chi^c_h(\vec p,-\tfrac12) = c\,\chi^*_h(\vec p,-\tfrac12) = \begin{pmatrix} -\cos\vartheta/2\; e^{-i\varphi/2} \\ -\sin\vartheta/2\; e^{i\varphi/2}\end{pmatrix}$$

Crosscheck: $\chi_h(\vec p,\pm\frac12) = D(R(\vartheta,\varphi))\,\chi^c_h(p\vec e_z,\pm\frac12)$

helicity operator: $h(\vec p) = \frac12\begin{pmatrix}\cos\vartheta & \sin\vartheta\, e^{-i\varphi}\\ \sin\vartheta\, e^{i\varphi} & -\cos\vartheta\end{pmatrix}$

$$h(\vec p)\,\chi_h(\vec p,\pm\tfrac12) = \pm\tfrac12\,\chi_h(\vec p,\pm\tfrac12)$$
$$h(\vec p)\,\chi^c_h(\vec p,\pm\tfrac12) = \mp\tfrac12\,\chi^c_h(\vec p,\pm\tfrac12)$$

consequences for Dirac spinors:

$$\psi^{(1,2)}_{\vec p} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi(\pm\frac12)\\ \sqrt{\frac{p\bar\sigma}{mc}}\,\chi(\pm\frac12)\end{pmatrix} \longrightarrow \psi^{(1,2)}_{\vec p} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi_h(\pm\frac12)\\ \sqrt{\frac{p\bar\sigma}{mc}}\,\chi_h(\pm\frac12)\end{pmatrix}$$

$$\psi^{(3,4)}_{\vec p} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi^c(\pm\frac12)\\ -\sqrt{\frac{p\bar\sigma}{mc}}\,\chi^c(\pm\frac12)\end{pmatrix} \longrightarrow \psi^{(3,4)}_{\vec p} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi^c_h(\pm\frac12)\\ -\sqrt{\frac{p\bar\sigma}{mc}}\,\chi^c_h(\pm\frac12)\end{pmatrix}$$

justification: helicity in space of Dirac spinors

$$H(\vec p) = \frac{D(\Sigma)\cdot\vec p}{p} = \frac{1}{2p}\begin{pmatrix}\vec\sigma\vec p & 0\\ 0 & \vec\sigma\vec p\end{pmatrix} = \begin{pmatrix} h(\vec p) & 0\\ 0 & h(\vec p)\end{pmatrix}$$

$$\sqrt{\frac{p\sigma}{mc}} = \frac{(p^0+mc)I - \vec\sigma\vec p}{\sqrt{2mc\,(p^0+mc)}}, \quad \sqrt{\frac{p\bar\sigma}{mc}} = \frac{(p^0+mc)I + \vec\sigma\vec p}{\sqrt{2mc\,(p^0+mc)}}$$

$\Rightarrow \left[\sqrt{\frac{p\sigma}{mc}}, h(\vec p)\right]_- = 0 = \left[\sqrt{\frac{p\bar\sigma}{mc}}, h(\vec p)\right]_-$ due to the algebra $[\sigma^k,\sigma^l]_- = i\epsilon_{klm}\sigma^m$

$\Rightarrow H(\vec p)\,\psi^{(r)}_{\vec p} = \xi_r\,\psi^{(r)}_{\vec p}$; $\xi_r = \frac{(-1)^{r+1}}{2}$, $r=1,2$; $\xi_r = \frac{(-1)^r}{2}$, $r=3,4$

9.16 Canonical Field Quantization:

$A = \frac1c\int d^4x\,\mathcal L$, $\mathcal L = i\hbar c\,\bar\psi\,\gamma^\mu\partial_\mu\psi - mc^2\psi$

$$\pi(\vec x,t) = \frac{\delta A}{\delta\frac{\partial\psi(\vec x,t)}{\partial t}} = \frac{\partial\mathcal L}{\partial\frac{\partial\psi(\vec x,t)}{\partial t}} = i\hbar\,\bar\psi(\vec x,t)\gamma^0 = i\hbar\,\psi^+(\vec x,t)\underbrace{(\gamma^0)^2}_{=1}$$

$$\bar\pi(\vec x,t) = \frac{\delta A}{\delta\frac{\partial\bar\psi(\vec x,t)}{\partial t}} = \frac{\partial\mathcal L}{\partial\left(\frac{\partial\bar\psi(\vec x,t)}{\partial t}\right)} \equiv 0$$

$\Rightarrow \psi(\vec x,t), \pi(\vec x,t)$ or $\psi(\vec x,t), \psi^+(\vec x,t)$ are independent field degrees of freedom

Appendix: Noether theorem, application to Dirac theory
conserved quantities $\Rightarrow$ sandwich principle

charge: $Q = \int d^3x\,\psi^+(\vec x,t)\,\psi(\vec x,t)$

energy: $E = \int d^3x\,\psi^+(\vec x,t)\underbrace{H_D(\vec x)}_{=-i\hbar c\,\vec\alpha\cdot\vec\nabla + mc^2\beta}\psi(\vec x,t)$

momentum: $\vec P = \int d^3x\,\psi^+(\vec x,t)\,\frac{\hbar}{i}\vec\nabla\,\psi(\vec x,t)$

helicity: $h = \int d^3x\,\psi^+(\vec x,t)\underbrace{\begin{pmatrix}\vec\sigma/2 & 0\\ 0 & \vec\sigma/2\end{pmatrix}}_{=D(\Sigma)}\frac{\frac{\hbar}{i}\vec\nabla}{|\frac{\hbar}{i}\vec\nabla|}\,\psi(\vec x,t)$

canonical quantization: $\psi(\vec x,t), \pi(\vec x,t) \longrightarrow \hat\psi(\vec x,t), \hat\pi(\vec x,t)$

bosonic quantization: inconsistencies, e.g. violation of microcausality

$\Rightarrow$ fermionic a: equal-time anti-commutator algebra

$$[\hat\psi_\alpha(\vec x,t), \hat\psi_\beta(\vec x',t)]_+ = [\hat\pi_\alpha(\vec x,t), \hat\pi_\beta(\vec x',t)]_+ = 0$$
$$[\hat\psi_\alpha(\vec x,t), \hat\pi_\beta(\vec x',t)]_+ = i\hbar\,\delta_{\alpha\beta}\,\delta(\vec x-\vec x')$$
$$\pi(\vec x,t) = i\hbar\,\psi^+(\vec x,t) \longrightarrow \hat\pi(\vec x,t) = i\hbar\,\hat\psi^+(\vec x,t)$$
$$[\hat\psi_\alpha(\vec x,t), \hat\psi_\beta(\vec x',t)]_+ = 0 = [\hat\psi^+_\alpha(\vec x,t), \hat\psi^+_\beta(\vec x',t)]_+$$
$$[\hat\psi_\alpha(\vec x,t), \hat\psi^+_\beta(\vec x',t)]_+ = \delta_{\alpha\beta}\,\delta(\vec x-\vec x')$$

second quantized operators for conserved quantities

$$\hat{Q} = \int d^3x\, \hat{\psi}^+(\vec{x},t)\, \hat{\psi}(\vec{x},t)$$
$$\hat{H} = \int d^3x\, \hat{\psi}^+(\vec{x},t)\, H_D(x)\, \hat{\psi}(\vec{x},t)$$
$$\hat{\vec{p}} = \int d^3x\, \hat{\psi}^+(\vec{x},t)\, \frac{\hbar}{i}\vec{\nabla}\, \hat{\psi}(\vec{x},t)$$
$$h = \int d^3x\, \hat{\psi}^+(\vec{x},t) \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix} \frac{\frac{\hbar}{i}\vec{\nabla}}{\left|\frac{\hbar}{i}\vec{\nabla}\right|}\, \hat{\psi}(\vec{x},t)$$

Heisenberg equations:

$$i\hbar\frac{\partial}{\partial t}\hat{\psi}(\vec{x},t) = [\hat{\psi}(\vec{x},t)\hat{H}]_- = H_D(\vec{x})\,\hat{\psi}(\vec{x},t) = (-i\hbar c\vec{\alpha}\vec{\nabla} + mc^2\beta)\,\hat{\psi}(\vec{x},t)$$
$$i\hbar\frac{\partial}{\partial t}\hat{\psi}^+(\vec{x},t) = [\hat{\psi}^+(\vec{x},t), \hat{H}]_- = -\{H_D(\vec{x})\hat{\psi}(\vec{x},t)\}^+ = (-i\hbar c\vec{\alpha}\vec{\nabla} - mc^2\beta)\,\hat{\psi}^+(\vec{x},t)$$

9.17 decomposition into plane waves:

$$\hat{\psi}(\vec{x},t) = \sum_{r=1}^{4}\int d^3p\, \psi^{(r)}_{\vec{p}}(\vec{x},t)\, \hat{a}^{(r)}_{\vec{p}} \Rightarrow \hat{a}^{(r)}_{\vec{p}} = \int d^3x \sum_{\alpha=1}^{4}\psi^{(r)+}_{\vec{p}\alpha}(\vec{x},t)\,\hat{\psi}_\alpha(\vec{x},t)$$
$$\hat{\psi}^+(\vec{x},t) = \sum_{r=1}^{4}\int d^3p\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\, \hat{a}^{(r)+}_{\vec{p}} \Rightarrow \hat{a}^{(r)+}_{\vec{p}} = \int d^3x \sum_{\alpha=1}^{4}\psi^{(r)}_{\vec{p}\alpha}(\vec{x},t)\,\hat{\psi}^+_\alpha(\vec{x},t)$$

$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')}_{\vec{p}'}]_+ = [\hat{a}^{(r)+}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_+ = 0$$
$$[\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r')+}_{\vec{p}'}]_+ = \int d^3x \int d^3x' \sum_{\alpha=1}^{4}\sum_{\alpha'=1}^{4} \psi^{(r)+}_{\vec{p}\alpha}(\vec{x},t)\,\psi^{(r')}_{\vec{p}'\alpha'}(\vec{x}',t)\, \underbrace{[\hat{\psi}_\alpha(\vec{x},t), \hat{\psi}^+_{\alpha'}(\vec{x}',t)]_-}_{=\delta_{\alpha\alpha'}\delta(\vec{x}-\vec{x}')}$$
$$= \int d^3x \sum_{\alpha=1}^{4} \psi^{(r)+}_{\vec{p}\alpha}(\vec{x},t)\,\psi^{(r')}_{\vec{p}'\alpha}(\vec{x},t) = \delta(\vec{p}-\vec{p}')\,\delta_{rr'}$$

$\Rightarrow \hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r)+}_{\vec{p}}$ fulfill a canonical anti-commutator algebra

For the time being $\hat{a}^{(r)}_{\vec{p}}, \hat{a}^{(r)+}_{\vec{p}}$ are interpreted as annihilation/creation operators of fermionic particles.

9.18 Second Quantized Operators:

$$\hat{Q} = \int d^3x\, \hat{\psi}^+(\vec{x},t)\,\hat{\psi}(\vec{x},t) = \sum_{r=1}^{4}\sum_{r'=1}^{4}\int d^3p \int d^3p'\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r')}_{\vec{p}'} \cdot \underbrace{\int d^3x\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\,\psi^{(r')}_{\vec{p}'}(\vec{x},t)}_{=\delta_{rr'}\delta(\vec{p}-\vec{p}')}$$
$$= \sum_{r=1}^{4}\int d^3p\, \underbrace{\hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}}}_{\text{number operator}}$$

$\Rightarrow$ All particles have the same sign of charge (positive)

$$\hat{H} = \int d^3x\, \hat{\psi}^+(\vec{x},t) H_D(x)\hat{\psi}(\vec{x},t) = \sum_{r=1}^{4}\sum_{r'=1}^{4}\int d^3p\int d^3p'\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r')}_{\vec{p}'} \cdot \int d^3x\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\, \underbrace{H_D(\vec{x})\,\psi^{(r')}_{\vec{p}'}(\vec{x},t)}$$
$$= (-i\hbar c\,\vec{\alpha}\vec{\nabla} + mc^2\beta)\,\psi^{(r')}_{\vec{p}'}(\vec{x},t) \underset{\text{Dirac equation}}{=} i\hbar\frac{\partial}{\partial t}\psi^{(r')}_{\vec{p}'}(\vec{x},t) = \varepsilon_r E_{\vec{p}}\,\psi^{(r')}_{\vec{p}'}(\vec{x},t)$$
$$\varepsilon_r = \begin{cases} +1 & ; r=1,2 \\ -1 & ; r=3,4 \end{cases}$$
$$= \sum_{r=1}^{4}\int d^3p\, \varepsilon_r E_r\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}}$$
$$= \int d^3p \left\{ \sum_{r=1}^{2} E_{\vec{p}}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} - \sum_{r=3}^{4} E_{\vec{p}}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} \right\}$$
$$\hat{\vec{p}} = \sum_{r=1}^{4}\int d^3p\, \varepsilon_r \vec{p}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} = \int d^3p \left\{ \sum_{r=1}^{2} \vec{p}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} - \sum_{r=3}^{4} \vec{p}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} \right\}$$
$$\hat{h} = \sum_{r=1}^{4}\sum_{r'=1}^{4}\int d^3p\int d^3p'\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r')}_{\vec{p}'} \int d^3x\, \psi^{(r)+}_{\vec{p}}(\vec{x},t)\, \underbrace{\underbrace{\begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix}\frac{\vec{p}}{p}}_{H(\vec{p})}\, \psi^{(r')}_{\vec{p}'}(\vec{x},t)}_{= \varepsilon_r \psi^{(r')}_{\vec{p}'}(\vec{x},t)}$$
$$= \sum_{r=1}^{4}\int d^3p\, \varepsilon_r\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}}$$
$$= \int d^3p \left\{ \sum_{r=1}^{2} \frac{(-1)^{r+1}}{2}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} + \sum_{r=3}^{4} \frac{(-1)^r}{2}\, \hat{a}^{(r)+}_{\vec{p}}\hat{a}^{(r)}_{\vec{p}} \right.$$

$\Rightarrow r = 1,4$: helicity $+\frac{1}{2}$ $\quad r = 2,3$: helicity $-\frac{1}{2}$

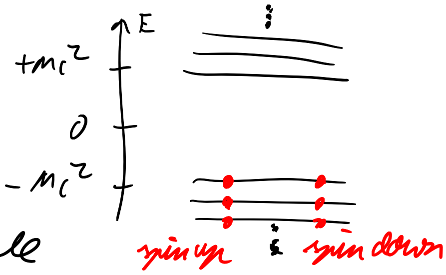
Note: $\hat{Q}, \hat{H}, \hat{\vec{p}}, \hat{h}$ turn out to not explicitly depend on time, they correspond therefore to conserved quantities

9.19 Dirac Sea:

- Vacuum state $|0\rangle_V$: $\quad \hat{a}^{(r)}_{\vec{p}}|0\rangle_V = 0$ for all $r, \vec{p}$
- ~~Problem: we have positive and negative energies? How to interpret this?~~
- Problem: we have another vacuum state $|0\rangle_P$ which is physically realized
- Paul Dirac (1930): Instead of $|0\rangle_V$

$$|0\rangle_P = \prod_{r=3,4}\underbrace{\prod_{\vec{p}}}_{\text{continuum product}} \hat{a}^{(r)+}_{\vec{p}}\,|0\rangle_V$$

All negative energy states are occupied forming the "Dirac sea"



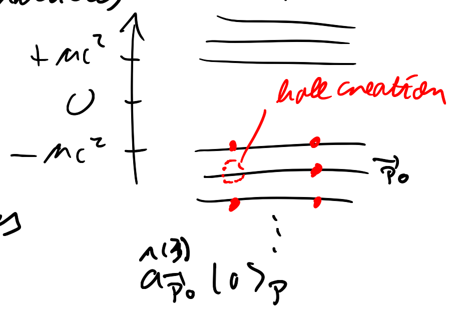
- Definition of physical vacuum $|0\rangle_P$:

$$\hat a^{(r)}_{\vec p}|0\rangle_P = 0 \quad \text{for } r=1,2, \text{ and } \vec p$$
$$\hat a^{(r)+}_{\vec p}|0\rangle_P = 0 \quad \text{for } r=3,4, \text{ and } \vec p$$
$$[\hat a^{(r)+}_{\vec p}, \hat a^{(r')+}_{\vec p'}]_+ = 0 \quad \widehat{=} \text{ Pauli exclusion principle}$$

- Reinterpretation:

$r=1,2$: $\hat a^{(r)}_{\vec p}, \hat a^{(r)+}_{\vec p}$ are annihilation, creation operators of particles

$r=3,4$: $\hat a^{(r)}_{\vec p}, \hat a^{(r)+}_{\vec p}$ are creation, annihilation operators of holes



- Dirac hole theory:

$r=1,2 \,\widehat{=}$ particles (e.g. electrons with spin up/down)

$r=3,4 \,\widehat{=}$ antiparticles (e.g. positrons with spin (down))

- Double role of $\hat a^{(r)}_{\vec p}, \hat a^{(r)+}_{\vec p}$ as creation (annihilation operators depending on r is confusing
- Introduce new notation:
  - particles: renaming
    creation operators $\hat a^{(1)+}_{\vec p} = \hat b^{(1)+}_{\vec p}$, $\hat a^{(2)+}_{\vec p} = \hat b^{(2)+}_{\vec p}$
    annihilation operators $\hat a^{(1)}_{\vec p} = \hat b^{(1)}_{\vec p}$, $\hat a^{(2)}_{\vec p} = \hat b^{(2)}_{\vec p}$
  - antiparticles:
    creation operators $\hat a^{(3)}_{\vec p} = \hat d^{(1)+}_{\vec p}$, $\hat a^{(4)}_{\vec p} = \hat d^{(2)+}_{\vec p}$
    annihilation operators $\hat a^{(3)+}_{\vec p} = \hat d^{(1)}_{\vec p}$, $\hat a^{(4)+}_{\vec p} = \hat d^{(2)}_{\vec p}$
- Anti-commutator algebra remains invariant:

$$[\hat b^{(r)}_{\vec p}, \hat b^{(r')}_{\vec p'}]_+ = [\hat b^{(r)}_{\vec p}, \hat d^{(r')}_{\vec p'}]_+ = [\hat d^{(r)}_{\vec p}, \hat d^{(r')}_{\vec p'}]_+ = [\hat b^{(r)}_{\vec p}, \hat d^{(r')+}_{\vec p}]_+ = 0$$
$$[\hat b^{(r)+}_{\vec p}, \hat b^{(r')+}_{\vec p'}]_+ = [\hat b^{(r)+}_{\vec p}, \hat d^{(r')}_{\vec p'}]_+ = [\hat d^{(r)+}_{\vec p}, \hat d^{(r')+}_{\vec p'}]_+ = [\hat b^{(r)+}_{\vec p}, \hat d^{(r)}_{\vec p}]_+ = 0 \qquad r=1,2$$
$$[\hat b^{(r)}_{\vec p}, \hat b^{(r')+}_{\vec p'}]_+ = [\hat d^{(r)}_{\vec p}, \hat d^{(r')+}_{\vec p'}]_+ = \delta_{rr'}\,\delta(\vec p-\vec p')$$

- Physical vacuum: $\hat b^{(r)}_{\vec p}|0\rangle_P = 0$, $\hat d^{(r)}_{\vec p}|0\rangle_P = 0 \qquad r=1,2$
- Consequence for $\hat H$:

$$\hat H = \sum_{r=1}^{2}\int d^3p\,\{E_{\vec p}\,\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} - E_{\vec p}\,\underbrace{\hat d^{(r)}_{\vec p}\hat d^{(r)+}_{\vec p}}_{=-\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p} + \delta(\vec 0)}\}$$
$$ E_{\vec p}$$

$${}_P\langle 0|\hat H|0\rangle_P = -\sum_{r=1}^{2}\int d^3p\, E_{\vec p}\,\delta(\vec 0)$$

negative for fermions (positive for Klein-Gordon or Maxwell)

$$:\hat H: = \hat H - {}_P\langle 0|\hat H|0\rangle_P = \sum_{r=1}^{2}\int d^3p\,\{E_{\vec p}\,\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} + E_{\vec p}\,\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p}\}$$

- Consequence for $\hat Q$:

$$\hat Q = \sum_{r=1}^{2}\int d^3p\,\{\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} + \underbrace{\hat d^{(r)}_{\vec p}\hat d^{(r)+}_{\vec p}}_{=-\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p} + \delta(\vec 0)}\}$$

$${}_P\langle 0|\hat Q|0\rangle_P = \sum_{r=1}^{2}\int d^3p\,1\,\delta(\vec 0)$$

$$:\hat Q: = \hat Q - {}_P\langle 0|\hat Q|0\rangle_P = \sum_{r=1}^{2}\int d^3p\,\{\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} - \hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p}\}$$

- Consequences for $\hat{\vec P}$:

$$\hat{\vec P} = \sum_{r=1}^{2}\int d^3p\,\{\vec p\,\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} - \vec p\,\underbrace{\hat d^{(r)}_{\vec p}\hat d^{(r)+}_{\vec p}}_{=-\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p} + \delta(\vec 0)}\}$$

$${}_P\langle 0|\hat{\vec P}|0\rangle_P \equiv 0$$

$$:\hat{\vec P}: = \hat{\vec P} = \sum_{r=1}^{2}\int d^3p\,\{\vec p\,\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} + \vec p\,\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p}\}$$

- Consequences for $\hat h$:

$$\hat h = \sum_{r=1}^{2}\int d^3p\,\{\frac{(-1)^{r+1}}{2}\,\hat b^{(r)+}_{\vec p}\hat b^{(r)}_{\vec p} + \frac{(-1)^{r}}{2}\,\underbrace{\hat d^{(r)}_{\vec p}\hat d^{(r)+}_{\vec p}}_{=-\hat d^{(r)+}_{\vec p}\hat d^{(r)}_{\vec p} + \delta(\vec 0)}\}$$

$$_P\langle 0|\hat{h}(0)|0\rangle_P = 0$$

$$:\hat{h}: = \hat{h} = \sum_{r=1}^{2} \int d^3p \left\{ \frac{(-1)^{r+1}}{2}\, \hat{b}_{\vec{p}}^{(r)\dagger}\, \hat{b}_{\vec{p}}^{(r)} + \frac{(-1)^{r+1}}{2}\, \hat{d}_{\vec{p}}^{(r)\dagger}\, \hat{d}_{\vec{p}}^{(r)} \right\}$$

$\Rightarrow$ Investigation of particle content   second quantized Dirac theory