$${}_p\langle 0|\hat b(0)\rangle_p = 0$$

$$:\hat b: = \hat b = \sum_{r=1}^{2}\int d^3p\left\{\frac{(-1)^{r+1}}{2}\,\hat b^{(r)+}_{\vec p}\,\hat a^{(r)}_{\vec p} + \frac{(-1)^{r+1}}{2}\,\hat d^{(r)+}_{\vec p}\,\hat d^{(r)}_{\vec p}\right\}$$

$\Rightarrow$ Investigation of particle content in second quantized Dirac theory

9.20 Propagator as Green Function:

Define Dirac propagator: $S_{\alpha\beta}(\vec x,t;\vec x',t') = {}_p\langle 0|\hat T\left(\hat\psi_\alpha(\vec x,t)\,\hat{\bar\psi}_\beta(\vec x',t')\right)|0\rangle_p$

$$\hat T\left(\hat A(t)\hat B(t')\right) = \Theta(t-t')\,\underbrace{\hat A(t)\hat B(t')}_{\text{bosons}} \pm \Theta(t'-t)\,\hat B(t')\hat A(t)$$

$\uparrow$ fermions

$$S_{\alpha\beta}(\vec x,t;\vec x',t') = \Theta(t-t')\,{}_p\langle 0|\hat\psi_\alpha(\vec x,t)\,\hat{\bar\psi}_\beta(\vec x',t')|0\rangle_p - \Theta(t'-t)\,{}_p\langle 0|\hat{\bar\psi}_\beta(\vec x',t')\,\hat\psi_\alpha(\vec x,t)|0\rangle_p$$

Equation of motion:

$$i\hbar\frac{\partial}{\partial t}S_{\alpha\beta}(\vec x,t;\vec x',t') = i\hbar\,\delta(t-t')\,{}_p\langle 0|\left[\hat\psi_\alpha(\vec x,t),\hat{\bar\psi}_\beta(\vec x',t)\right]_+|0\rangle$$
$$+\Theta(t-t')\,{}_p\langle 0|\,\underbrace{i\hbar\frac{\partial\hat\psi_\alpha(\vec x,t)}{\partial t}}\,\hat{\bar\psi}_\beta(\vec x',t')|0\rangle_p - \Theta(t'-t)\,{}_p\langle 0|\hat{\bar\psi}_\beta(\vec x',t')\left(i\hbar\frac{\partial}{\partial t}\hat\psi_\alpha(\vec x,t)\right)|0\rangle_p$$

$$\sum_{\gamma=1}^{4}\gamma^0_{\gamma\beta}\,\underbrace{\left[\hat\psi_\alpha(\vec x,t),\hat\psi^+_\gamma(\vec x',t)\right]_+}_{=\delta_{\alpha\gamma}\,\delta(\vec x-\vec x')} = \gamma^0_{\alpha\beta}\,\delta(\vec x-\vec x')$$

Heisenberg equation

$$= \sum_{\gamma=1}^{4}\left(-i\hbar c\,\vec\alpha_{\alpha\gamma}\cdot\vec\nabla + mc^2\beta_{\alpha\gamma}\right)\hat\psi_\gamma(\vec x,t)$$

$=$ dito

$$i\hbar\frac{\partial}{\partial t}S_{\alpha\beta}(\vec x,t;\vec x',t') = \sum_{\gamma=1}^{4}\left(-i\hbar c\,\underbrace{\vec\alpha_{\alpha\gamma}}_{\gamma^0\vec\gamma}\cdot\vec\nabla + mc^2\underbrace{\beta_{\alpha\gamma}}_{\gamma^0}\right)S_{\gamma\beta}(\vec x,t;\vec x',t') = i\hbar\,\gamma^0_{\alpha\beta}\,\delta(t-t')\,\delta(\vec x-\vec x')$$

multiply with $\frac{\gamma^0}{c}$ with taking into account $(\gamma^0)^2 = 1$

$$\left\{i\hbar\left(\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + \vec\gamma\cdot\vec\nabla\right) - mc\right\}S(\vec x,t;\vec x',t') = \frac{i\hbar}{c}\,\delta(t-t')\,\delta(\vec x-\vec x')$$

$$\Rightarrow\quad \left(i\hbar\,\gamma^\mu\partial_\mu - mc\right)S(x;x') = i\hbar\,\delta(x-x')$$

$\Rightarrow$ Dirac propagator $\hat=$ Green function of Dirac equation

9.21 Propagator Calculation:

$$\hat\psi(\vec x,t) = \sum_{r=1}^{4}\int d^3p\,\psi^{(r)}_{\vec p}(\vec x,t)\,\hat a^{(r)}_{\vec p}$$

$$= \int d^3p\left\{\sum_{r=1}^{2}\underbrace{\psi^{(r)}_{\vec p}(\vec x,t)}_{=U^{(r)}_{\vec p}(\vec x,t)}\,\underbrace{\hat a^{(r)}_{\vec p}}_{=\hat b^{(r)}_{\vec p}} + \underbrace{\sum_{r=3}^{4}}_{\sum_{r=1}^{2}}\underbrace{\psi^{(r)}_{\vec p}(\vec x,t)}_{=V^{(r)}_{\vec p}(\vec x,t)}\,\underbrace{\hat a^{(r)}_{\vec p}}_{\hat d^{(r)+}_{\vec p}}\right\}$$

$$\hat b^{(r)}_{\vec p}|0\rangle_p = 0$$
$$\hat d^{(r)}_{\vec p}|0\rangle_p = 0$$

$$\hat{\bar\psi}(\vec x,t) = \sum_{r=1}^{2}\int d^3p\left\{\bar u^{(r)}_{\vec p}(\vec x,t)\,\hat b^{(r)+}_{\vec p} + \bar v^{(r)}_{\vec p}(\vec x,t)\,\hat d^{(r)}_{\vec p}\right\}$$

$$S_{\alpha\beta}(\vec x,t;\vec x',t') = \sum_{r=1}^{2}\sum_{r'=1}^{2}\int d^3p\int d^3p'$$
$$\Big[{}_p\langle 0|\left\{u^{(r)}_{\vec p\alpha}(\vec x,t)\,\hat b^{(r)}_{\vec p} + \cancel{v^{(r)}_{\vec p\alpha}(\vec x,t)\,\hat d^{(r)+}_{\vec p}}\right\}\left\{\bar u^{(r')}_{\vec p'\beta}(\vec x',t')\,\hat b^{(r')+}_{\vec p'} + \cancel{\bar v^{(r')}_{\vec p'\beta}(\vec x',t')\,\hat d^{(r')}_{\vec p'}}\right\}|0\rangle_p\,\Theta(t-t')$$
$$-{}_p\langle 0|\left\{\cancel{\hat b^{(r')+}_{\vec p'}}\ \ \hat d^{(r')}_{\vec p'}\right\}\left\{\cancel{\hat b^{(r)}_{\vec p}}\ \ \hat d^{(r)+}_{\vec p}\right\}|0\rangle_p\,\Theta(t'-t)\Big]$$

$$= \sum_{r=1}^{2}\sum_{r'=1}^{2}\int d^3p\int d^3p'\Big\{\Theta(t-t')\,u^{(r)}_{\vec p\alpha}(\vec x,t)\,\bar u^{(r')}_{\vec p'\beta}(\vec x',t')\,{}_p\langle 0|\underbrace{\hat b^{(r)}_{\vec p}\,\hat b^{(r')+}_{\vec p'}}_{=+\delta_{rr'}\,\delta(\vec p-\vec p') - \cancel{\hat b^{(r')+}_{\vec p'}\hat b^{(r)}_{\vec p}}}|0\rangle_p$$
$$-\Theta(t'-t)\,\bar v^{(r')}_{\vec p'\beta}(\vec x',t')\,v^{(r)}_{\vec p\alpha}(\vec x,t)\,{}_p\langle 0|\underbrace{\hat d^{(r')}_{\vec p'}\,\hat d^{(r)+}_{\vec p}}_{=\delta_{rr'}\,\delta(\vec p-\vec p') - \cancel{\hat d^{(r)+}_{\vec p}\hat d^{(r')}_{\vec p'}}}|0\rangle_p\Big\}$$

$$= \sum_{r=1}^{2}\int d^3p\Big\{\Theta(t-t')\,\underbrace{u^{(r)}_{\vec p\alpha}(\vec x,t)}\,\bar u^{(r)}_{\vec p\beta}(\vec x',t') - \Theta(t'-t)\,\underbrace{v^{(r)}_{\vec p\alpha}(\vec x,t)}\,\bar v^{(r)}_{\vec p\beta}(\vec x',t')\Big\}$$

$$= u^{(r)}_{\vec p\alpha}\,e^{-\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec p}}} \qquad = v^{(r)}_{\vec p\alpha}\,e^{+\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\sqrt{\frac{mc^2}{(2\pi\hbar)^3E_{\vec p}}}$$

$$S_{\alpha\beta}(\vec x,t;\vec x',t') = \int d^3p\,\frac{mc^2}{(2\pi\hbar)^3E_{\vec p}}$$
$$\cdot\left\{\Theta(t-t')\,e^{-\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\,\underbrace{P^u_{\alpha\beta}(p)}_{\text{polarization sum of particles}} - \Theta(t'-t)\,e^{+\frac{i}{\hbar}\left[E_{\vec p}(t-t') - \vec p(\vec x-\vec x')\right]}\,P^v_{\alpha\beta}(p)\right\}$$

$= -P^u_{\alpha\beta}(-p)$

polarization sum of antiparticles

have to evaluate polarization sums: auxiliary calculations

$$P^u_{\alpha\beta}(\vec p) = \sum_{r=1}^{2} u^{(r)}_{\vec p\alpha}\,\bar u^{(r)}_{\vec p\beta} = \sum_{r=1}^{2} \psi^{(r)}_{\vec p\alpha}\,\bar\psi^{(r)}_{\vec p\beta}$$

$$P^v_{\alpha\beta}(\vec p) = \sum_{r=1}^{2} v^{(r)}_{\vec p\alpha}\,\bar v^{(r)}_{\vec p\beta} = \sum_{r=3}^{4} \psi^{(r)}_{\vec p\alpha}\,\bar\psi^{(r)}_{\vec p\beta}$$

$$\underset{r=1,2}{\psi^{(r)}_{\vec p}} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi\left(\frac{(-1)^{r+1}}{2}\right)\\ \sqrt{\frac{p\bar\sigma}{mc}}\,\chi\left(\frac{(-1)^{r+1}}{2}\right)\end{pmatrix};\quad \underset{r=3,4}{\psi^{(r)}_{\vec p}} = \frac{1}{\sqrt2}\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\,\chi_c\left(\frac{(-1)^{r+1}}{2}\right)\\ -\sqrt{\frac{p\bar\sigma}{mc}}\,\chi_c\left(\frac{(-1)^{r+1}}{2}\right)\end{pmatrix}$$

$$\bar\psi^{(r)}_{\vec p} = \frac{1}{\sqrt2}\left(\chi^+\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\sigma}{mc}},\ \chi^+\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\bar\sigma}{mc}}\right)\underbrace{\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}}_{=\gamma^0}$$

$$= \frac{1}{\sqrt2}\left(\chi^+\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\bar\sigma}{mc}},\ \chi^+\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\sigma}{mc}}\right)\qquad r=1,2$$

$$\bar\psi^{(r)}_{\vec p} = \frac{1}{\sqrt2}\left(-\chi^+_c\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\bar\sigma}{mc}},\ \chi^+_c\left(\frac{(-1)^{r+1}}{2}\right)\sqrt{\frac{p\sigma}{mc}}\right)$$

completeness of Pi-spinors:

$$\sum_{r=1}^{2}\chi\left(\frac{(-1)^{r+1}}{2}\right)\chi^+\left(\frac{(-1)^{r+1}}{2}\right) = \chi\left(\tfrac12\right)\chi^+\left(\tfrac12\right)+\chi\left(-\tfrac12\right)\chi^+\left(-\tfrac12\right) = I$$

can be shown for helicity spinors, see manuscript

completeness of charge conjugated Pi-spinors: $\chi^c_h(\vec p,\lambda) = c\,\chi^*_h(\vec p,\lambda)$

$$\sum_{r=1}^{2}\chi^c_h\left(\frac{(-1)^{r+1}}{2}\right)\chi^{c+}_h\left(\frac{(-1)^{r+1}}{2}\right) = \sum_{r=1}^{2} c\,\chi^*_h\left(\vec p,\frac{(-1)^{r+1}}{2}\right)\chi^T_h\left(\vec p,\frac{(-1)^{r+1}}{2}\right)c^+$$

$$= c\underbrace{\left\{\sum_{r=1}^{2}\chi_h\left(\vec p,\frac{(-1)^{r+1}}{2}\right)\chi^+_h\left(\vec p,\frac{(-1)^{r+1}}{2}\right)\right\}^*}_{=I} c^+ = cc^+ = I,\quad c=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

---

polarization sum of particles:

$$P^u(\vec p) = \frac12\begin{pmatrix}\sqrt{\frac{p\sigma}{mc}}\\ \sqrt{\frac{p\bar\sigma}{mc}}\end{pmatrix}\underbrace{\sum_{r=1}^{2}\chi\left(\frac{(-1)^{r+1}}{2}\right)\chi^+\left(\frac{(-1)^{r+1}}{2}\right)}_{=I}\left(\sqrt{\frac{p\bar\sigma}{mc}},\sqrt{\frac{p\sigma}{mc}}\right)$$

$$= \frac12\begin{pmatrix}\underbrace{\sqrt{\frac{p\sigma}{mc}\frac{p\bar\sigma}{mc}}}_{=I} & \frac{p\sigma}{mc}\\ \frac{p\bar\sigma}{mc} & \underbrace{\sqrt{\frac{p\bar\sigma}{mc}\frac{p\sigma}{mc}}}_{=I}\end{pmatrix} = \frac12\left\{\frac{p_\mu}{mc}\underbrace{\begin{pmatrix}0 & \sigma^\mu\\ \tilde\sigma^\mu & 0\end{pmatrix}}_{=\gamma^\mu}+\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix}\right\} = \frac{p_\mu\gamma^\mu+mc}{2mc}$$

see last time

polarization sum of antiparticles, see manuscript: $P^v(\vec p) = \frac{p_\mu\gamma^\mu - mc}{2mc}$

observation: $P^v(p) = -P^u(-p)$

Intermediate result for Dirac propagator:

$$S_{\alpha\beta}(\vec x,t;\vec x',t') = \int d^3p\,\frac{mc}{(2\pi\hbar)^3E_p}$$

$$\left\{\Theta(t-t')\,e^{-\frac{i}{\hbar}\underbrace{\left[E_p(t-t')-\vec p(\vec x-\vec x')\right]}_{=p(x-x')}}P^u_{\alpha\beta}(p)+\Theta(t'-t)\,e^{+\frac{i}{\hbar}\underbrace{\left[E_p(t-t')-\vec p(\vec x-\vec x')\right]}_{=p(x-x')}}P^u_{\alpha\beta}(-p)\right\}$$

Note: This form of the Dirac propagator appears analogous to the propagators of other massive particles, where the differences in the spin degrees is taken care of by the polarization sum.

Example: Recover Klein-Gordon propagator by setting $P^u_{\alpha\beta}(p)\equiv 1$

9.22 Four-dimensional Fourier Representation:

Insert explicit result for polarization sum

$$\frac{p_\mu\gamma^\mu_{\alpha\beta}+mc\,\delta_{\alpha\beta}}{2mc}\qquad\qquad \frac{-p_\mu\gamma^\mu_{\alpha\beta}+mc\,\delta_{\alpha\beta}}{2mc}$$

$$p_\mu\to i\hbar\partial_\mu\qquad \frac{i\hbar\partial_\mu\gamma^\mu_{\alpha\alpha}+mc\,\delta_{\alpha\beta}}{2mc}\qquad\qquad p_\mu\to i\hbar\partial_\mu\qquad \frac{i\hbar\partial_\mu\gamma^\mu_{\alpha\beta}+mc\,\delta_{\alpha\beta}}{2mc}$$

$$S_{\alpha\beta}(\vec x,t;\vec x',t') = \frac{i\hbar\partial_\mu\gamma^\mu_{\alpha\beta}+mc\,\delta_{\alpha\beta}}{2mc}\int d^3p\,\frac{mc^2}{(2\pi\hbar)^3E_p}$$

$$\cdot\left\{\Theta(t-t')\, e^{-\frac{i}{\hbar}\left[E_{\vec p}(t-t')-\vec p(\vec x-\vec x')\right]} + \Theta(t'-t)\, e^{\frac{i}{\hbar}\left[E_{\vec p}(t-t')-\vec p(\vec x-\vec x')\right]}\right\}$$

$\Rightarrow$ Recover Klein-Gordon propagator

$$S(x;x') = \frac{i\hbar\,\partial_\mu\gamma^\mu + mc}{2mc}\ \underbrace{G(x;x')}$$

$$i\hbar\, 2mc \lim_{\varepsilon\downarrow 0}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{1}{p^2-m^2c^2+i\varepsilon}\, e^{-\frac{i}{\hbar}p(x-x')}$$

$$= i\hbar \lim_{\varepsilon\downarrow 0}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{p_\mu\gamma^\mu + mc}{\underbrace{p^2-m^2c^2+i\varepsilon}}\, e^{-\frac{i}{\hbar}p(x-x')}$$

$$= p_\mu p_\nu \underbrace{g^{\mu\nu}} - m^2c^2 = p_\mu p_\nu \frac{1}{2}(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu) - m^2c^2 = (p_\mu\gamma^\mu)(p_\nu\gamma^\nu) - m^2c^2$$

$$= (p_\mu\gamma^\mu - mc)(p_\nu\gamma^\nu + mc)$$

clifford algebra

$$S(x;x') = \lim_{\varepsilon\downarrow 0}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i\hbar}{p_\mu\gamma^\mu - mc + i\varepsilon}\, e^{-\frac{i}{\hbar}p(x-x')}$$

$\rightarrow$ manifest covariant

$\rightarrow$ Green function of Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\underline{S(x;x')} = \cancel{\lim_{\varepsilon\downarrow 0}}\int\frac{d^4p}{(2\pi\hbar)^4}\, i\hbar\, \underbrace{\frac{\cancel{p_\mu\gamma^\mu - mc}}{\cancel{p_\mu\gamma^\mu - mc + i\varepsilon}}}_{=1}\, e^{-\frac{i}{\hbar}p(x-x')}$$

$$= i\hbar\,\delta(x-x') \checkmark$$

Part III Interacting Relativistic Fields and Their Quantization

Chapter 10: Relativistic Light-Matter Interaction:

Introduction:

- QED = relativistic quantum field theory of electrodynamics
- all phenomena where electrically charged particles interact via exchange of photons
- 3 stages of description:
  - > relativistic mechanics
  - > first quantization
  - > second quantization
- common guiding principle to introduce interactions between free fields: minimal coupling based on local gauge theory
- Result: second quantized Hamiltonian of QED
- Interactions have to be treated perturbatively
- Review Dirac interaction picture: allows to treat relativistic light-matter interaction order by order
- generic scattering problem described by scattering operator, whose matrix elements yield cross sections

10.1 Relativistic Mechanics:

10.1.1 Basics:

- trajectory described by some parameter σ:

$$(x^\mu(\sigma)) = (ct(\sigma), \vec x(\sigma)); \quad (\dot x^\mu(\sigma)) = \left(\frac{dx^\mu(\sigma)}{d\sigma}\right) = \left(c\frac{dt(\sigma)}{d\sigma}, \frac{d\vec x(\sigma)}{d\sigma}\right)$$

- Action: $\mathcal{A}[x^\lambda(\cdot)] = \int_{\sigma_i}^{\sigma_f} d\sigma\, L\left(x^\lambda(\sigma); \dot x^\lambda(\sigma)\right)$
- Hamilton principle: Euler-Lagrange equations

$$\frac{\delta\mathcal{A}}{\delta x^\lambda(\sigma)} = \frac{\partial L}{\partial x^\lambda(\sigma)} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot x^\lambda(\sigma)} = 0$$

- Mechanical gauge invariance: regauging of L

$$L'(x^\lambda;\dot x^\lambda) = L(x^\lambda;\dot x^\lambda) + \underbrace{\frac{d}{d\sigma}\chi(x^\lambda)}_{=\partial_\nu\chi(x^\lambda)\,\dot x^\nu}$$

$$\Rightarrow \mathcal{A}' = \mathcal{A} + \underbrace{\chi(\sigma_f) - \chi(\sigma_i)}_{\text{surface terms}} \rightarrow \text{irrelevant for equation of motion}$$

$$\frac{\partial L'}{\partial x^\mu} - \frac{d}{d\sigma}\frac{\partial L'}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} + \partial_\mu \partial_\nu X \dot{x}^\nu - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu} - \underbrace{\frac{d}{d\sigma}\, \partial_\mu X(x^\lambda)}_{= \partial_\nu \partial_\mu X \dot{x}^\nu} = 0$$

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) X = 0$$

(theorem of Schwarz