7. Klein-Gordon Theory:

Motivation:

- First relativistic QFT in this lecture
- Free scalar field which describes massive particles of spin zero
  - Higgs particle (H): electrically neutral, interaction of H with other particles gives them a mass
  - Pions: $\pi^{\pm}$, π^{0} introduced by Hideki Yukawa as exchange particles giving rise to nuclear force
    But today no longer "elementary particles" as they are the lightest mesons ($\hat{=}$ composed of two quarks)
    π^0: real-valued wave function $\Psi(\vec{x},t) = \Psi^*(\vec{x},t)$
    $\pi^{\pm}$: complex-valued wave function $\Psi(\vec{x},t)$
- Minimally couple pions to electromagnetic field: scalar QED
  - quantised form: electromagnetic interaction described in terms of exchange of photons
  - scalar QED is similar to QED and easier: spinor degrees of freedom left out
  - Klein-Gordon theory is a relativistic extension of the Ginzburg-Landau theory of superconductivity

  $\Rightarrow$ Not discussed here due to time constraints

7.1 Action and Equations of Motion:

Schrödinger is not invariant under Lorentz transformation: space and time derivatives not dealt with on equal footing $\Rightarrow$ demand to set up a relativistic extension

charged pions $\pi^{\pm}$: Klein-Gordon fields $\Psi^*(x^\lambda)$, $\Psi(x^\lambda)$

$$A = A[\Psi^*(\cdot), \Psi(\cdot)]$$

$$A = \frac{1}{c}\int \underbrace{d^4x}_{=c\,dt\,d^3x} \; \mathcal{L}\left(\Psi^*(x^\lambda), \partial_\mu \Psi^*(x^\lambda); \Psi(x^\lambda), \partial_\mu \Psi(x^\lambda)\right)$$

"local" field theory

condition: $\mathcal{L}$ must be a real Lorentz invariant

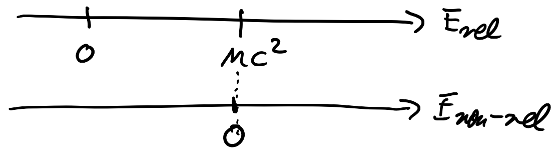
$$\Rightarrow \mathcal{L} = A\, g^{\mu\nu}\, \partial_\mu \Psi^*(x^\lambda)\, \partial_\nu \Psi(x^\lambda) + B\, \Psi^*(x^\lambda)\, \Psi(x^\lambda)$$

A, B to be determined by the condition: non-relativistic limit $\Rightarrow$ Schrödinger theory

$$\mathcal{L} = A\left\{\frac{1}{c^2}\frac{\partial \Psi^*(\vec{x},t)}{\partial t}\frac{\partial \Psi(\vec{x},t)}{\partial t} - \vec{\nabla}\Psi^*(\vec{x},t)\cdot\vec{\nabla}\Psi(\vec{x},t)\right\} + B\,\Psi^*(\vec{x},t)\,\Psi(\vec{x},t)$$

non-relativistic limit: $E_{rel} = Mc^2 + E_{non\text{-}rel}$

$$\underbrace{E_{\vec{p}} = \sqrt{\vec{p}^{\,2}c^2 + M^2c^4}}_{\text{relat. dispersion}} \; \underset{Mc^2 \gg |\vec{p}|c}{\overset{c\to\infty}{=}} \; \underbrace{Mc^2}_{\text{rest energy}} + \underbrace{\frac{\vec{p}^{\,2}}{2M}}_{\text{non-relat. dispersion}} + \ldots$$



separation ansatz:

$$\Psi(\vec{x},t) = e^{-\frac{i}{\hbar}Mc^2 t}\,\psi(\vec{x},t), \quad \frac{\partial \Psi(\vec{x},t)}{\partial t} = e^{-\frac{i}{\hbar}Mc^2 t}\left\{\frac{\partial \psi(\vec{x},t)}{\partial t} - \frac{i}{\hbar}Mc^2\,\psi(\vec{x},t)\right\}$$

$$\Psi^*(\vec{x},t) = e^{+\frac{i}{\hbar}Mc^2 t}\,\psi^*(\vec{x},t), \quad \frac{\partial \Psi^*(\vec{x},t)}{\partial t} = e^{\frac{i}{\hbar}Mc^2 t}\left\{\frac{\partial \psi^*(\vec{x},t)}{\partial t} + \frac{i}{\hbar}Mc^2\,\psi^*(\vec{x},t)\right\}$$

$$\mathcal{L} = \frac{A}{c^2}\left\{\cancel{\frac{\partial \psi^*(\vec{x},t)}{\partial t}\frac{\partial \psi(\vec{x},t)}{\partial t}} \to 0 + \frac{i}{\hbar}Mc^2\left[\psi^*(\vec{x},t)\frac{\partial \psi(\vec{x},t)}{\partial t} - \psi(\vec{x},t)\frac{\partial \psi^*(\vec{x},t)}{\partial t}\right]\right.$$

$$\left. - A\,\vec{\nabla}\psi^*(\vec{x},t)\cdot\vec{\nabla}\psi(\vec{x},t) + \left(B + \frac{M^2c^2}{\hbar^2}A\right)\psi^*(\vec{x},t)\,\psi(\vec{x},t)\right)$$

$$\xrightarrow{c\to\infty} \frac{i\hbar}{\cancel{2}}\left[\psi^*(\vec{x},t)\frac{\partial \psi(\vec{x},t)}{\partial t} - \cancel{\psi(\vec{x},t)\frac{\partial \psi^*(\vec{x},t)}{\partial t}}\right] - \frac{\hbar^2}{2M}\vec{\nabla}\psi^*(\vec{x},t)\cdot\vec{\nabla}\psi(\vec{x},t) + \text{"}\,0\,\text{"}$$

partial integral in time

Schrödinger Lagrange density

1) $A = \frac{\hbar^2}{2M} \Rightarrow \frac{A}{c^2}\cdot\frac{iMc^2}{\hbar} = \frac{\hbar^2}{2M}\frac{1}{c^2}\frac{iMc^2}{\hbar} = \frac{i\hbar}{2}$

2) $B = -\frac{M^2c^2}{\hbar^2}\cdot A = -\frac{M^2c^2}{\hbar^2}\cdot\frac{\hbar^2}{2M} = -\frac{1}{2}Mc^2$

$$S = \int dt \int d^3x\, \mathcal{L}\left(\Psi^*(\vec x,t), \vec\nabla\Psi^*(\vec x,t), \frac{\partial\Psi^*(\vec x,t)}{\partial t}; \Psi(\vec x,t), \vec\nabla\Psi(\vec x,t), \frac{\partial\Psi(\vec x,t)}{\partial t}\right)$$

$$\mathcal{L} = \frac{\hbar^2}{2Mc^2}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t} - \frac{\hbar^2}{2M}\vec\nabla\Psi^*\cdot\vec\nabla\Psi - \frac12 Mc^2\Psi^*\Psi$$

Hamilton principle: Euler-Lagrange equation

$$\frac{\delta S}{\delta\Psi^*(\vec x,t)} = 0 \Rightarrow \frac{\partial\mathcal L}{\partial\Psi^*} - \vec\nabla\frac{\partial\mathcal L}{\partial\vec\nabla\Psi^*} - \frac{\partial}{\partial t}\frac{\partial\mathcal L}{\partial\frac{\partial\Psi^*}{\partial t}} = 0$$

$$\frac{\partial\mathcal L}{\partial\Psi^*} = -\frac12 Mc^2\Psi, \quad \frac{\partial\mathcal L}{\partial\vec\nabla\Psi^*} = -\frac{\hbar^2}{2M}\vec\nabla\Psi, \quad \frac{\partial\mathcal L}{\partial\frac{\partial\Psi^*}{\partial t}} = \frac{\hbar^2}{2Mc^2}\frac{\partial\Psi}{\partial t}$$

Klein-Gordon equation

$$\underbrace{\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi}_{\text{wave equation}} + \underbrace{\left(\frac{Mc}{\hbar}\right)^2\Psi}_{\text{"mass term"}} = 0$$

$\Rightarrow$ wave equation with mass term = KG equation

Compton wave length: $k_c = \frac{Mc}{\hbar} = \frac{2\pi}{\lambda_c} \Rightarrow \lambda_c = 2\pi\frac{\hbar}{Mc}$

$\pi^\pm$: $Mc^2 = 139.6$ MeV $\Rightarrow \lambda_c = 9$ fm, 1 fm $= 10^{-15}$ m $\hat{=}$ order of magnitude of size of atomic nucleus

Physical interpretation:

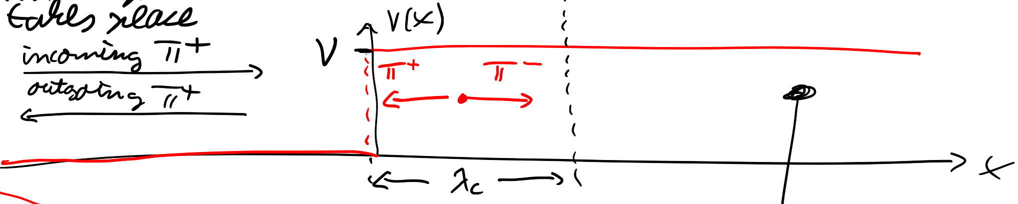
relativistic particle: $\Delta p = Mc$

Heisenberg uncertainty principle: $\Delta x\cdot\Delta p = \hbar$

$\Delta x = \frac{\hbar}{Mc}$

relativistic particle confined in a region of Compton wave length: particle-antiparticle production out of vacuum takes place

This phenomenon is best illustrated by the "Klein paradoxon"



$V \sim 2Mc^2$:

- The reflectivity coefficient > 1
- deep in threshold: negative charge

$\Rightarrow$ Relativistic quantum theory can never be restricted to a one-particle theory!

$\Rightarrow$ Need to be extended via second quantisation to a relativistic QFT!

Non-relativistic limit of Klein-Gordon equation:

$$\Psi(\vec x,t) = e^{-\frac{i}{\hbar}Mc^2t}\psi(\vec x,t), \quad \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi + \left(\frac{Mc}{\hbar}\right)^2\Psi = 0$$

$$\underbrace{\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}_{\to 0,\ c\to\infty} - \frac{2iM}{\hbar}\frac{\partial\psi}{\partial t} - \cancel{\frac{M^2c^4}{\hbar^2}\frac{1}{c^2}\psi} - \Delta\psi + \cancel{\left(\frac{Mc}{\hbar}\right)^2\psi} = 0 \Rightarrow i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\Delta\psi$$

Historic remark: Schrödinger discovered before Klein/Gordon this relativistic wave equation.

7.2 Continuity equation:

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi + \frac{M^2c^2}{\hbar^2}\Psi = 0 \quad \Big|\cdot\Psi^*$$

$$-\left(\frac{1}{c^2}\frac{\partial^2\Psi^*}{\partial t^2} - \Delta\Psi^* + \frac{M^2c^2}{\hbar^2}\Psi^* = 0\right) \quad \Big|\cdot\Psi$$

$$\underbrace{\frac{1}{c^2}\left(\Psi^*\frac{\partial^2\Psi}{\partial t^2} - \Psi\frac{\partial^2\Psi^*}{\partial t^2}\right)}_{=\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right)} + \underbrace{\left(\Psi\Delta\Psi^* - \Psi^*\Delta\Psi\right)}_{=\vec\nabla\left(\Psi\vec\nabla\Psi^* - \Psi\vec\nabla\Psi\right)} = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \operatorname{div}\vec j = 0$$

density: $\rho(\vec x,t) = \frac{K}{c^2}\left\{\Psi^*(\vec x,t)\frac{\partial\Psi(\vec x,t)}{\partial t} - \Psi(\vec x,t)\frac{\partial\Psi^*(\vec x,t)}{\partial t}\right\}$

current density: $\vec j(\vec x,t) = K\left\{\Psi(\vec x,t)\vec\nabla\Psi^*(\vec x,t) - \Psi^*(\vec x,t)\vec\nabla\Psi(\vec x,t)\right\}$

K: yet to be determined constant

$\Psi = \psi\, e^{-\frac{i}{\hbar}Mc^2t}$, $\Psi^* = \psi^* e^{\frac{i}{\hbar}Mc^2t}$

$$\rho = \frac{K}{c^2}\Big\{\underbrace{\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}}_{\to 0} - 2\frac{i}{\hbar}Mc^2\psi^*\psi\Big\}, \quad \vec j = K\left\{\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi\right\}$$

$\downarrow c\to\infty$: $\rho = \psi^*\psi$

$\downarrow c\to\infty$: $\vec j = \frac{i\hbar}{2M}\left\{\psi\vec\nabla\psi^* - \psi^*\vec\nabla\psi\right\}$

$$\frac{\kappa}{c^2}(-2)\frac{i}{\hbar}mc^2 = \frac{i\hbar}{2mc^2}(-2)\frac{i}{\hbar}mc^2 \stackrel{!}{=} 1 \qquad \kappa = \frac{i\hbar}{2m}$$

$$\Rightarrow \rho(\vec{x},t) = \frac{i\hbar}{2mc^2}\left\{\Psi^*(\vec{x},t)\frac{\partial \Psi(\vec{x},t)}{\partial t} - \Psi(\vec{x},t)\frac{\partial \Psi^*(\vec{x},t)}{\partial t}\right\}, \quad \vec{j}(\vec{x},t) = \frac{i\hbar}{2m}\left\{\Psi(\vec{x},t)\vec{\nabla}\Psi^*(\vec{x},t) - \Psi^*(\vec{x},t)\vec{\nabla}\Psi(\vec{x},t)\right\}$$

conserved quantity: $Q = \int d^3x\, \rho(\vec{x},t)$

$$\frac{\partial Q}{\partial t} = \int d^3x \frac{\partial \rho(\vec{x},t)}{\partial t} = -\int d^3x\, \mathrm{div}\, \vec{j}(\vec{x},t) \stackrel{\text{Gauß}}{=} -\oint d\vec{f}\cdot\vec{j}(\vec{x},t) \equiv 0$$

Scalar product: $\langle \Psi_1, \Psi_2\rangle = \frac{i\hbar}{2mc^2}\int d^3x \left\{\Psi_1^*(\vec{x},t)\frac{\partial \Psi_2(\vec{x},t)}{\partial t} - \Psi_2(\vec{x},t)\frac{\partial \Psi_1^*(\vec{x},t)}{\partial t}\right\}$

not positive definite

e.g. $\Psi_1 = \Psi_2 = N e^{\frac{i}{\hbar}mc^2 t} \Rightarrow \langle \Psi_1, \Psi_2\rangle = -N^2 < 0$

non-relativistic limit:

$$\langle \Psi_1, \Psi_2\rangle = \frac{i\hbar}{2mc^2}\int d^3x \left\{\psi_1^*\frac{\partial \psi_2}{\partial t} - \psi_2\frac{\partial \psi_1^*}{\partial t} - 2\frac{i}{\hbar}mc^2\psi_1^*\psi_2\right\} \xrightarrow{c\to\infty} \int d^3x\, \psi_1^*\psi_2$$

positive definite scalar product of Schrödinger

Moral: Each theory has its own "natural" scalar product

Result: $Q = \int d^3x \langle \Psi, \Psi\rangle$ {can be either pos. or neg.}

Q or better $eQ \mathrel{\hat=}$ electric charge of Klein-Gordon field

$\Rightarrow$ complex-valued Klein-Gordon field

Conversely: $\Psi^*(\vec{x},t) = \Psi(\vec{x},t) \mathrel{\hat=}$ electrically neutral, $Q \equiv 0$

7.3 Canonical Field Quantization:

$$\mathcal{L} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial t} - \frac{\hbar^2}{2m}\vec{\nabla}\Psi^*\cdot\vec{\nabla}\Psi - \frac{1}{2}mc^2\Psi^*\Psi$$

Canonically conjugated momentum fields:

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^*}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi}{\partial t}, \qquad \pi = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial t}} = \frac{\hbar^2}{2mc^2}\frac{\partial \Psi^*}{\partial t}$$

Hamilton density:

$$\mathcal{H} = \pi^*\frac{\partial \Psi^*}{\partial t} + \pi\frac{\partial \Psi}{\partial t} - \mathcal{L} = \frac{2mc^2}{\hbar^2}\pi^*\pi + \frac{\hbar^2}{2m}\vec{\nabla}\Psi^*\vec{\nabla}\Psi + \frac{1}{2}mc^2\Psi^*\Psi$$

Hamilton function: $H = \int d^3x\, \mathcal{H} \longrightarrow \hat{H}$

$\Psi^*(\vec{x},t),\ \Psi(\vec{x},t),\ \pi^*(\vec{x},t),\ \pi(\vec{x},t)$ classical fields

$\downarrow$ second quantization

$\hat{\Psi}^+(\vec{x},t),\ \hat{\Psi}(\vec{x},t),\ \hat{\pi}^+(\vec{x},t),\ \hat{\pi}(\vec{x},t)$ operators