ground state $|0\rangle$

$\hat a|0\rangle = 0 \Leftrightarrow \langle 0|\hat a^+ = 0$

vacuum state $|0\rangle$

$\hat a_{\vec x}|0\rangle = 0 \Leftrightarrow \langle 0|\hat a^+_{\vec x} = 0$

$|n\rangle = \frac{1}{\sqrt{n!}}(\hat a^+)^n|0\rangle$

basis states of Hilbert space

$|\vec x_1,\dots,\vec x_n\rangle^{+1} = \hat a^+_{\vec x_1}\cdots\hat a^+_{\vec x_n}|0\rangle$

$\vec x_i \neq \vec x_j$ for $i\neq j$

[Note: $|\vec x_1,\vec x_1\rangle^{+1} = \frac{1}{\sqrt 2}(\hat a^+_{\vec x_1})^2|0\rangle$]

n=1 boson: $|\vec x_1\rangle^{+1} = \hat a^+_{\vec x_1}|0\rangle$

$$^{+1}\langle \vec x_1|\vec x_1'\rangle^{+1} = \langle \hat a^+_{\vec x_1}0|\hat a^+_{\vec x_1'}0\rangle = \langle 0|\hat a_{\vec x_1}\hat a^+_{\vec x_1'}|0\rangle \quad \to 0$$

$$= \cancel{\hat a^+_{\vec x_1'}\hat a_{\vec x_1}} + \delta(\vec x_1-\vec x_1')$$

$$= \delta(\vec x_1-\vec x_1'), \quad \langle 0|0\rangle = 1$$

n=2 bosons: $|\vec x_1,\vec x_2\rangle^{+1} = \hat a^+_{\vec x_1}\hat a^+_{\vec x_2}|0\rangle$; $\vec x_1\neq\vec x_2$

$$^{+1}\langle \vec x_1,\vec x_2|\vec x_1',\vec x_2'\rangle^{+1} = \langle \hat a^+_{\vec x_1}\hat a^+_{\vec x_2}0|\hat a^+_{\vec x_1'}\hat a^+_{\vec x_2'}0\rangle$$

$$= \langle 0|\hat a_{\vec x_2}\underbrace{\hat a_{\vec x_1}\hat a^+_{\vec x_1'}}_{=\hat a^+_{\vec x_1'}\hat a_{\vec x_1}+\delta(\vec x_1-\vec x_1')}\hat a^+_{\vec x_2'}|0\rangle$$

$(\hat A\hat B)^+ = \hat B^+\hat A^+$

$$= \delta(\vec x_1-\vec x_1')\langle 0|\hat a_{\vec x_2}\hat a^+_{\vec x_2'}|0\rangle + \langle 0|\hat a_{\vec x_2}\hat a^+_{\vec x_1'}\hat a_{\vec x_1}\hat a^+_{\vec x_2'}|0\rangle$$

$$\hat a_{\vec x_2}\hat a^+_{\vec x_2'} = \delta(\vec x_2-\vec x_2') + \cancel{\hat a^+_{\vec x_2'}\hat a_{\vec x_2}} \qquad \hat a_{\vec x_2}\hat a^+_{\vec x_1'} = \cancel{\hat a^+_{\vec x_1'}}\hat a_{\vec x_2} + \delta(\vec x_1'-\vec x_2) \qquad \hat a_{\vec x_1}\hat a^+_{\vec x_2'} = \cancel{\hat a^+_{\vec x_2'}}\hat a_{\vec x_1} + \delta(\vec x_1-\vec x_2')$$

$$= \delta(\vec x_1-\vec x_1')\,\delta(\vec x_2-\vec x_2') + \delta(\vec x_1-\vec x_2')\,\delta(\vec x_2-\vec x_1')$$

$$= \delta^{+1}(\vec x_1,\vec x_2;\vec x_1',\vec x_2')$$

Conclusion: symmetrisation is automatically taken into account by using creation/annihilation operators.

3.3 Schrödinger Equation for Interacting Bosons

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \underbrace{\hat H}_{\text{many-boson Hamiltonian}}\ \underbrace{|\Psi(t)\rangle}_{\text{many-boson state}}, \qquad \hat H = \hat H_1 + \hat H_2$$

one-particle contribution: "sandwich" principle

$$\hat H_1 = \int d^3x\ \hat a^+_{\vec x}\underbrace{\left\{-\frac{\hbar^2}{2M}\Delta + V_1(\vec x)\right\}}_{\text{first-quantised Hamiltonian}}\hat a_{\vec x}$$

two-particle contribution:

$$\hat H_2 = \frac12\int d^3x\int d^3x'\ \hat a^+_{\vec x}\hat a^+_{\vec x'}\,V_2(\vec x-\vec x')\,\hat a_{\vec x'}\hat a_{\vec x}$$

normal ordering: creation operators are at the left, annihilation operators at the right

→ zero-point energies and physical effects emerging from them are neglected

→ vacuum energy of Hamiltonian vanishes: $\hat H|0\rangle = 0 \Leftrightarrow \langle 0|\hat H = 0$

Aim: Recover Schrödinger equation for a wave function describing a fixed number of bosons via projection to basis states $|\vec x_1,\dots,\vec x_n\rangle^{+1} = \hat a^+_{\vec x_1}\cdots\hat a^+_{\vec x_n}|0\rangle$, $\vec x_i\neq\vec x_j$ for $i\neq j$

$^{+1}\langle\vec x_1,\dots,\vec x_n| = \langle 0|\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}$; $(\hat A\hat B)^+ = \hat B^+\hat A^+$

$$i\hbar\frac{\partial}{\partial t}\underbrace{{}^{+1}\langle\vec x_1,\dots,\vec x_n|\Psi(t)\rangle}_{=\Psi^{+1}(\vec x_1,\dots,\vec x_n;t)} = {}^{+1}\langle\vec x_1,\dots,\vec x_n|\hat H|\Psi(t)\rangle = \langle 0|\underbrace{\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}\hat H}_{=[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat H]_-}|\Psi(t)\rangle$$

because $\langle 0|\hat H = 0$ $\longrightarrow$ $=[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat H]_-$

$$= \int d^3y \int d^3z\, \delta(\vec y - \vec z)\left\{-\frac{\hbar^2}{2m}\Delta_{\vec z} + V_1(\vec z)\right\} \langle 0| \underline{[\hat a_{\vec x_n}\cdots \hat a_{\vec x_1}, \hat a^+_{\vec y}\hat a_{\vec z}]_-} |\Psi(t)\rangle$$

$$+\frac12 \int d^3y_1\int d^3y_2 \int d^3z_1 \int d^3 z_2\, \delta(\vec y_1-\vec z_1)\delta(\vec y_2 - \vec z_2) V_2(\vec z_1-\vec z_2)\langle 0|\underline{[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}, \hat a^+_{\vec y_1}\hat a^+_{\vec y_2}\hat a_{\vec z_1}\hat a_{\vec z_2}]_-}|\Psi(t)\rangle$$

(1) $[\hat A\hat B,\hat C]_- = \hat A[\hat B,\hat C]_- + [\hat A,\hat C]_-\hat B$

(2) $[\hat A,\hat B\hat C]_- = [\hat A,\hat B]_-\hat C + \hat B[\hat A,\hat C]_-$

$$[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}, \hat a^+_{\vec y}\hat a_{\vec z}]_- \overset{(2)}{=} [\underbrace{\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}}_{\hat A}, \underbrace{\hat a^+_{\vec y}}_{\hat B}]_-\hat a_{\vec z} + \hat a^+_{\vec y}\underbrace{[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat a_{\vec z}]_-}_{=0}$$

$$\overset{(1)}{=} \Big\{\hat a_{\vec x_n}\underbrace{[\hat a_{\vec x_{n-1}}\cdots\hat a_{\vec x_1},\hat a^+_{\vec y}]_-}_{\text{iteratively apply (1)}} + \underbrace{[\hat a_{\vec x_n},\hat a^+_{\vec y}]_-}_{=\delta(\vec x_n-\vec y)}\hat a_{\vec x_{n-1}}\cdots\hat a_{\vec x_1}\Big\}\hat a_{\vec z} \qquad (*)$$

$$= \sum_{r=1}^n \hat a_{\vec x_n}\cdots\hat a_{r+1}\,\delta(\vec x_r-\vec y)\,\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_1}\hat a_{\vec z}$$

second commutator:

$$[\underbrace{\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}}_{=\hat A}, \underbrace{(\hat a^+_{\vec y_1}\hat a^+_{\vec y_2})}_{=\hat B}\underbrace{(\hat a_{\vec z_1}\hat a_{\vec z_2})}_{=\hat C}]_- \overset{(2)}{=} [\underbrace{\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}}_{=\hat A}, \underbrace{\hat a^+_{\vec y_1}\hat a^+_{\vec y_2}}_{=\hat B}]_-\underbrace{\hat a_{\vec z_1}\hat a_{\vec z_2}}_{=\hat C} + 0$$

$$\overset{(2)}{=}\Big\{[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat a^+_{\vec y_1}]_-\hat a^+_{\vec y_2} + \hat a^+_{\vec y_1}[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat a^+_{\vec y_2}]_-\Big\}\hat a_{\vec z_1}\hat a_{\vec z_2}$$

known from (*)

$$= \sum_{r=1}^n \delta(\vec x_r-\vec y_1)\,\hat a_{\vec x_n}\cdots\hat a_{\vec x_{r+1}}\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_1}\hat a^+_{\vec y_2}\hat a_{\vec z_1}\hat a_{\vec z_2}$$

$$+\sum_{r=1}^n\delta(\vec x_r-\vec y_2)\,\hat a^+_{\vec y_1}\hat a_{\vec x_n}\cdots\hat a_{\vec x_{r+1}}\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_1}\hat a_{\vec z_1}\hat a_{\vec z_2}$$

→ this vanishes in $\langle 0|\ldots|\Psi(t)\rangle$

second expectation value:

$$\langle 0|[\hat a_{\vec x_n}\cdots\hat a_{\vec x_1},\hat a^+_{\vec y_1}\hat a^+_{\vec y_2}\hat a_{\vec z_1}\hat a_{\vec z_2}]_-|\Psi(t)\rangle$$

$$=\sum_{r=1}^n\delta(\vec x_r-\vec y_1)\langle 0|\underbrace{\hat a_{\vec x_n}\cdots\hat a_{\vec x_{r+1}}\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_1}\hat a^+_{\vec y_2}}_{=[\hat a_{\vec x_n}\cdots\hat a_{\vec x_{r+1}}\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_1},\,\hat a^+_{\vec y_2}]_-}\hat a_{\vec z_1}\hat a_{\vec z_2}|\Psi(t)\rangle$$

$$\overset{(*)}{=}\sum_{r=1}^n\sum_{\mu=1}^n \delta(\vec x_r-\vec y_1)\,\delta(\vec x_\mu-\vec y_2)\langle 0|\hat a_{\vec x_n}\cdots\hat a_{\vec x_{r+1}}\hat a_{\vec x_{r-1}}\cdots\hat a_{\vec x_{\mu+1}}\hat a_{\vec x_{\mu-1}}\cdots\hat a_{\vec x_1}\hat a_{\vec z_1}\hat a_{\vec z_2}|\Psi(t)\rangle$$

$$i\hbar\frac{\partial}{\partial t}\langle 0|\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}|\Psi(t)\rangle = \sum_{r=1}^n\left\{-\frac{\hbar^2}{2m}\Delta_{\vec x_r} + V_1(\vec x_r)\right\}\langle 0|\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}|\Psi(t)\rangle$$

$$+\frac12\sum_{r=1}^n\sum_{\mu=1}^n V_2(\vec x_r-\vec x_\mu)\langle 0|\hat a_{\vec x_n}\cdots\hat a_{\vec x_1}|\Psi(t)\rangle \quad ✓$$

⇒ Schrödinger equation for n bosons

3.4 Field Operators in Heisenberg Picture

Schrödinger picture: $\hat a^+_{\vec x}, \hat a_{\vec x}$ time independent, $|\Psi(t)\rangle$ time dependent

Heisenberg " : operators time dependent, states are time independent

$$\begin{cases} i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle = \hat H_S|\Psi_S(t)\rangle \\ i\hbar\frac{\partial}{\partial t}\hat O_S = 0 \end{cases}$$

[assumption: no explicit time dependence of $\hat O_S$ → closed quantum system]

formal solution: $|\Psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat H_S t}\underbrace{|\Psi_S(0)\rangle}_{=:|\Psi_H\rangle}$ state in Heisenberg picture

$$|\Psi_S(t)\rangle = e^{-\frac i\hbar \hat H_S t}|\Psi_H\rangle \Leftrightarrow |\Psi_H\rangle = e^{\frac i\hbar\hat H_S t}|\Psi_S(t)\rangle \quad (*)$$

equation of motion of $|\Psi_H\rangle$:

$$i\hbar\frac{\partial}{\partial t}|\Psi_H\rangle \overset{(*)}{=} -\hat H_S e^{+\frac i\hbar \hat H_S t}|\Psi_S(t)\rangle + \underbrace{e^{\frac i\hbar\hat H_S t}\, i\hbar\frac{\partial}{\partial t}|\Psi_S(t)\rangle}_{=\hat H_S e^{\frac i\hbar \hat H_S t}|\Psi_S(t)\rangle} \equiv 0$$

Aim: determine $\hat O_H(t)$

working principle: expectation values do not change once one performs a transformation from Schrödinger to Heisenberg picture $\langle\Psi_S(t)|\hat O_S|\Psi_S(t)\rangle \overset{!}{=} \langle\Psi_H|\hat O_H(t)|\Psi_H\rangle$

$$\overset{(*)}{\Rightarrow} = \langle e^{-\frac i\hbar\hat H_S t}\Psi_H|\hat O_S|e^{-\frac i\hbar \hat H_S t}\Psi_H\rangle = \langle\Psi_H|\underbrace{\boxed{e^{\frac i\hbar\hat H_S t}\hat O_S e^{-\frac i\hbar\hat H_S t}}}_{=\hat O_H(t)}|\Psi_H\rangle \text{ for all } |\Psi_H\rangle$$

example: $\hat O_S = \hat H_S \Rightarrow \hat H_H(t) = e^{\frac i\hbar\hat H_S t}\hat H_S e^{-\frac i\hbar \hat H_S t} \equiv \hat H_S$

equation of motion for $\hat O_H(t)$: Heisenberg equation

$$i\hbar\frac{\partial}{\partial t}\hat O_H(t) = e^{\frac i\hbar\hat H_S t}\{-\hat H_S\hat O_S + \hat O_S\hat H_S\}e^{-\frac i\hbar\hat H_S t} + \underbrace{e^{\frac i\hbar \hat H_S t}\, i\hbar\frac{\partial}{\partial t}\hat O_S\, e^{-\frac i\hbar\hat H_S t}}_{=0}$$ neglected

$$= [\hat O_H(t),\hat H_S]_- = [\hat O_H(t),\hat H_H(t)]_-$$

Now: second quantization → field operators $\hat\Psi(\vec x,t) = \hat a_{\vec x H}(t) = e^{\frac i\hbar\hat H_S t}\hat a_{\vec x}e^{-\frac i\hbar\hat H_S t}$

$\hat\Psi^+(\vec x,t) = \hat a^+_{\vec x H}(t) = e^{\frac i\hbar\hat H_S t}\hat a^+_{\vec x}e^{-\frac i\hbar \hat H_S t}$, equal-time commutation relations:

$[\hat\Psi(\vec x,t), \hat\Psi(\vec x',t)]_- = 0 = [\hat\Psi^+(\vec x,t), \hat\Psi^+(\vec x',t)]_-$ $\quad [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)]_- = \delta(\vec x-\vec x')$

→ $\hat\Psi^+(\vec x,t)$, $\hat\Psi(\vec x,t)$ represent creation / annihilation operator of boson at $\vec x$ at time t

$$\hat H_S = \int d^3x\, \hat a^+_{\vec x} \{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\} \hat a_{\vec x} + \frac12 \int d^3x \int d^3x'\, V_2(\vec x-\vec x')\, \hat a^+_{\vec x} \hat a^+_{\vec x'} \hat a_{\vec x'} \hat a_{\vec x}$$

$$\hat H_H(t) = e^{\frac{i}{\hbar}\hat H_S t}\, \hat H_S\, e^{-\frac{i}{\hbar}\hat H_S t}$$

$$= \int d^3x\, \underbrace{e^{\frac i\hbar \hat H_S t}\hat a^+_{\vec x} e^{-\frac i\hbar \hat H_S t}}_{=\hat\Psi^+(\vec x,t)} \{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\} \underbrace{e^{\frac i\hbar \hat H_S t}\hat a_{\vec x} e^{-\frac i\hbar \hat H t}}_{=\hat\Psi(\vec x,t)}$$

$$= \frac12 \int d^3x \int d^3x'\, V_2(\vec x-\vec x')\, \underbrace{e^{\frac i\hbar \hat H_S t}\hat a^+_{\vec x} e^{-\frac i\hbar \hat H_S t}}_{\hat\Psi^+(\vec x,t)} \underbrace{e^{\frac i\hbar \hat H_S t}\hat a^+_{\vec x'} e^{-\frac i\hbar \hat H_S t}}_{\hat\Psi^+(\vec x',t)} \underbrace{e^{\frac i\hbar \hat H_S t}\hat a_{\vec x'} e^{-\frac i\hbar \hat H_S t}}_{\hat\Psi(\vec x',t)} \underbrace{e^{\frac i\hbar \hat H_S t}\hat a_{\vec x} e^{-\frac i\hbar \hat H_S t}}_{\hat\Psi(\vec x,t)}$$

Heisenberg equation:

$$i\hbar \frac{\partial}{\partial t}\hat\Psi(\vec x,t) = [\hat\Psi(\vec x,t), \hat H_H(t)]_-$$

$$= \int d^3x' \int d^3x''\, \delta(\vec x'-\vec x'') \{-\frac{\hbar^2}{2m}\Delta''+V_1(\vec x'')\} [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)]_- = \delta(\vec x-\vec x')\hat\Psi(\vec x'',t)$$

$$+\frac12\int d^3x'\int d^3x''\, V_2(\vec x'-\vec x'') [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_-$$

$$[\hat A, \hat B\hat C]_- = [\hat A,\hat B]_-\hat C + \hat B[\hat A,\hat C]_-$$

$$= \{\delta(\vec x-\vec x')\hat\Psi^+(\vec x'',t) + \hat\Psi^+(\vec x',t)\delta(\vec x-\vec x'')\}\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)$$

$$i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t) = \{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\}\hat\Psi(\vec x,t) + \int d^3x'\, V_2(\vec x-\vec x')\hat\Psi^+(\vec x',t)\hat\Psi(\vec x',t)\hat\Psi(\vec x,t)$$

non-linear integro-differential equation for $\hat\Psi(\vec x,t)$

→ in general not solvable analytically

→ approximative solutions

3.5 Creation and Annihilation Operators for Fermions:

second quantisation bosons: guarantee symmetry of wave functions
fermions: " anti- " " " "

basis states: $|\vec x_1,\dots,\vec x_n\rangle^{-1} = \hat a^+_{\vec x_1}\cdots\hat a^+_{\vec x_n}|0\rangle$

anti-commutation relations: $[\hat a_{\vec x},\hat a_{\vec x'}]_+ = 0 = [\hat a^+_{\vec x},\hat a^+_{\vec x'}]_+$, $\quad [\hat a_{\vec x},\hat a^+_{\vec x'}]_+ = \delta(\vec x-\vec x')$

$[\hat A,\hat B]_+ = \hat A\hat B+\hat B\hat A$, vacuum state: $\hat a_{\vec x}|0\rangle = 0$, $\langle 0|\hat a^+_{\vec x} = 0$

n=1: ${}^{-1}\langle \vec x_1|\vec x_1'\rangle^{-1} = \langle \hat a^+_{\vec x_1}0|\hat a^+_{\vec x_1'}0\rangle = \langle 0|\hat a_{\vec x_1}\hat a^+_{\vec x_1'}|0\rangle = \langle 0|-\cancel{\hat a^+_{\vec x_1'}\hat a_{\vec x_1}}+\delta(\vec x_1-\vec x_1')|0\rangle = \delta(\vec x_1-\vec x_1')$

n=2: ${}^{-1}\langle\vec x_1,\vec x_2|\vec x_1',\vec x_2'\rangle^{-1} = \langle \hat a^+_{\vec x_1}\hat a^+_{\vec x_2}0|\hat a^+_{\vec x_1'}\hat a^+_{\vec x_2'}0\rangle = \langle 0|\hat a_{\vec x_2}\underbrace{\hat a_{\vec x_1}\hat a^+_{\vec x_1'}}_{=-\hat a^+_{\vec x_1'}\hat a_{\vec x_1}+\delta(\vec x_1-\vec x_1')}\hat a^+_{\vec x_2'}|0\rangle$

$$= \delta(\vec x_1-\vec x_1')\langle 0|\hat a_{\vec x_2}\hat a^+_{\vec x_2'}|0\rangle - \langle 0|\underbrace{\hat a_{\vec x_2}\hat a^+_{\vec x_1'}}_{=\delta(\vec x_2-\vec x_1')-\cancel{\hat a^+_{\vec x_1'}\hat a_{\vec x_2}}}\underbrace{\hat a_{\vec x_1}\hat a^+_{\vec x_2'}}_{=\delta(\vec x_1-\vec x_2')-\cancel{\hat a^+_{\vec x_2'}\hat a_{\vec x_1}}}|0\rangle$$

$$= \delta(\vec x_1-\vec x_1')\delta(\vec x_2-\vec x_2') - \delta(\vec x_1-\vec x_2')\delta(\vec x_2-\vec x_1') = \delta^-(\vec x_1,\vec x_2;\vec x_1',\vec x_2')$$ as defined in Chapter 2

Consequence of anti-commutation relations:

$[\hat a^+_{\vec x_1},\hat a^+_{\vec x_2}]_+ = 0 \overset{\vec x_1=\vec x_2=\vec x}{\Longrightarrow} (\hat a^+_{\vec x})^2 = 0$ ⇒ anti-commutation relations automatically

⇒ $|\vec x_1,\dots,\vec x_n\rangle^{-1} = 0$ if $\vec x_i = \vec x_j$ for $i\neq j$

include Pauli exclusion principle

Hamilton operator: looks formally identical to fermions as for bosons

$$\hat H = \int d^3x\,\hat a^+_{\vec x}\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\}\hat a_{\vec x}+\frac12\int d^3x\int d^3x'\, V_2(\vec x-\vec x')\,\hat a^+_{\vec x}\hat a^+_{\vec x'}\hat a_{\vec x'}\hat a_{\vec x}$$

you can prove also here: projection to $|\vec x_1,\dots,\vec x_n\rangle^{-1}$ yield Schrödinger equation for fermionic wave function $\Psi^-(\vec x_1,\dots,\vec x_n,t) = \langle\vec x_1,\dots,\vec x_n|\Psi(t)\rangle$ from general " " : $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat H|\Psi(t)\rangle$

Schrödinger picture → Heisenberg picture:

$\hat\Psi^+(\vec x,t) = e^{\frac i\hbar\hat Ht}\,\hat a^+\, e^{-\frac i\hbar\hat Ht}$, $\quad \hat\Psi(\vec x,t) = e^{\frac i\hbar\hat Ht}\,\hat a\, e^{-\frac i\hbar\hat Ht}$

$[\hat\Psi(\vec x,t),\hat\Psi(\vec x',t)]_+ = 0 = [\hat\Psi^+(\vec x,t),\hat\Psi^+(\vec x',t)]_+$, $\quad [\hat\Psi(\vec x,t),\hat\Psi^+(\vec x',t)]_+ = \delta(\vec x-\vec x')$

Hamiltonian in Heisenberg picture: the same form for fermions as for bosons

$$\hat H_H(t) = \int d^3x\,\hat\Psi^+(\vec x,t)\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\}\hat\Psi(\vec x,t)+\frac12\int d^3x\int d^3x'\, V_2(\vec x-\vec x')\hat\Psi^+(\vec x,t)\hat\Psi^+(\vec x',t)\hat\Psi(\vec x',t)\hat\Psi(\vec x,t)$$

Heisenberg equations for field operators

$i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t) = [\hat\Psi(\vec x,t),\hat H_H(t)]_-$ ← commutator although fermionic field operators are defined via canonical anti-commutation relations!

$$= \int d^3x'\int d^3x''\,\delta(\vec x'-\vec x'')\{-\frac{\hbar^2}{2m}\Delta''+V_1(\vec x'')\}[\hat\Psi(\vec x,t),\hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)]_- \quad ①$$

$$+\frac12\int d^3x'\int d^3x''\, V_2(\vec x'-\vec x'')[\hat\Psi(\vec x,t),\hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_- \quad ②$$

$$[\hat A,\hat B\hat C]_- = [\hat A,\hat B]_+\hat C \pm \hat B[\hat A,\hat C]_+$$

① $\overset{ABC}{=} \underbrace{[\hat\Psi(\vec x,t),\hat\Psi^+(\vec x',t)]_+}_{=\delta(\vec x-\vec x')}\hat\Psi(\vec x'',t) + \hat\Psi^+(\vec x',t)\underbrace{[\hat\Psi(\vec x,t),\hat\Psi(\vec x'',t)]_+}_{=0}$

② $\overset{ABC}{=} [\hat\Psi(\vec x,t),\hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)]_-\,\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t) + \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\underbrace{[\hat\Psi(\vec x,t),\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_-}_{\overset{ABC}{=}0}$

$$\overset{ABC}{=} \delta(\vec x-\vec x')\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t) \pm \delta(\vec x-\vec x'')\hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)$$