4.6 Poisson Brackets

two functionals: $F = F[\pi(\cdot,\cdot);\psi(\cdot,\cdot)]$, $G = G[\pi(\cdot,\cdot);\psi(\cdot,\cdot)]$

$$\{F,G\}_- = \int d^3x \left(\frac{\delta F}{\delta\psi(\vec{x},t)} \frac{\delta G}{\delta\pi(\vec{x},t)} - \frac{\delta F}{\delta\pi(\vec{x},t)} \frac{\delta G}{\delta\psi(\vec{x},t)} \right) = -\{G,F\}_-$$

$$\left.\begin{aligned} \{\psi(\vec{x},t), H\}_- &= \frac{\delta H}{\delta\pi(\vec{x},t)} = \frac{\partial\psi(\vec{x},t)}{\partial t} \\ \{\pi(\vec{x},t), H\}_- &= -\frac{\delta H}{\delta\psi(\vec{x},t)} = \frac{\partial\pi(\vec{x},t)}{\partial t} \end{aligned}\right\}$$ Hamilton equations formulated in terms Poisson brackets

temporal change of an arbitrary functional F:

$$\frac{\partial F}{\partial t} = \int d^3x \left\{ \underbrace{\frac{\partial\pi(\vec{x},t)}{\partial t}}_{-\frac{\delta H}{\delta\psi(\vec{x},t)}} \frac{\delta F}{\delta\pi(\vec{x},t)} + \underbrace{\frac{\partial\psi(\vec{x},t)}{\partial t}}_{=\frac{\delta H}{\delta\pi(\vec{x},t)}} \frac{\delta F}{\delta\psi(\vec{x},t)} \right\}$$ functional chain rule

$= \{F, H\}_-$ $\Rightarrow$ get back Hamilton equations as a special case

fundamental Poisson brackets (equal times!):

$$\{\psi(\vec{x},t), \psi(\vec{x}',t)\}_- = \int d^3x'' \left(\frac{\delta\psi(\vec{x},t)}{\delta\psi(\vec{x}'',t)} \underbrace{\frac{\delta\psi(\vec{x}',t)}{\delta\pi(\vec{x}'',t)}}_{=0} - \underbrace{\frac{\delta\psi(\vec{x},t)}{\delta\pi(\vec{x}'',t)}}_{=0} \frac{\delta\psi(\vec{x}',t)}{\delta\psi(\vec{x}'',t)} \right) = 0$$

$$\{\pi(\vec{x},t), \pi(\vec{x}',t)\}_- = 0$$

$$\{\psi(\vec{x},t), \pi(\vec{x}',t)\}_- = \delta(\vec{x}-\vec{x}')$$

4.7 Canonical Field Quantization:

classical fields $\xrightarrow{\text{second quantization}}$ quantum mechanical operators

$\psi(\vec{x},t)$, $\pi(\vec{x},t)$ $\longrightarrow$ $\hat{\psi}(\vec{x},t)$, $\hat{\pi}(\vec{x},t)$

$\{F,G\}_-$ $\longrightarrow$ $\frac{1}{i\hbar}[\hat{F},\hat{G}]_-$

anti-symmetric fundamental Poisson brackets $\longrightarrow$ anti-symmetric equal-time commutation relations

$[\hat{\psi}(\vec{x},t), \hat{\psi}(\vec{x}',t)]_- = 0$, $[\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- = 0$, $[\hat{\psi}(\vec{x},t), \hat{\pi}(\vec{x}',t)]_- = \delta(\vec{x}-\vec{x}')\, i\hbar$

$\pi(\vec{x},t) = i\hbar\, \psi^*(\vec{x},t)$ $\longrightarrow$ $\hat{\pi}(\vec{x},t) = i\hbar\, \hat{\psi}^+(\vec{x},t)$

$[\hat{\psi}^+(\vec{x},t), \hat{\psi}^+(\vec{x}',t)]_- = 0$ $\qquad$ $[\hat{\psi}(\vec{x},t), \hat{\psi}^+(\vec{x}',t)]_- = \delta(\vec{x}-\vec{x}')$

$\Rightarrow$ coincides with definition (heuristic derivation) of Chapter 3

Hamiltonian equations $\longrightarrow$ Heisenberg equation

$\frac{\partial\psi(\vec{x},t)}{\partial t} = \{\psi(\vec{x},t), H\}_-$ $\longrightarrow$ $\frac{\partial\hat{\psi}(\vec{x},t)}{\partial t} = \frac{1}{i\hbar}[\hat{\psi}(\vec{x},t), \hat{H}]_- \quad \Big| \cdot i\hbar$

$i\hbar \frac{\partial\hat{\psi}(\vec{x},t)}{\partial t} = [\hat{\psi}(\vec{x},t), \hat{H}]_-$

$\frac{\partial\pi(\vec{x},t)}{\partial t} = \{\pi(\vec{x},t), H\}_-$ $\longrightarrow$ $\frac{\partial\hat{\pi}(\vec{x},t)}{\partial t} = \frac{1}{i\hbar}[\hat{\pi}(\vec{x},t), \hat{H}]_- \quad \Big| \cdot i\hbar$

and $\hat{\pi}(\vec{x},t) = i\hbar\, \hat{\psi}^+(\vec{x},t)$ $\Rightarrow$ $i\hbar \frac{\partial\hat{\psi}^+(\vec{x},t)}{\partial t} = [\hat{\psi}^+(\vec{x},t), \hat{H}]_-$

Hamilton functional H $\longrightarrow$ Hamilton operator $\hat{H}$

5. Canonical Field Quantization for Fermions:

motivation:

- derive equal-time anti-commutation relations between fermionic operators
- What is the classical field theory for fermions?
  Problem: Pauli exclusion principle not realizable by complex (real fields)

$\Rightarrow$ One has to introduce the concept of Grassmann numbers (fields) which anti-commute and therefore take into account Pauli exclusion principle on a classical level

5.1 Grassmann Fields:

5.1.1 Grassmann Numbers:

Grassmann algebra = set of Grassmann numbers of dimension 2^n → spanned by n generators $\psi_i, i=1,\dots,n$ with anti-commutation relations $[\psi_i,\psi_j]_+ = \psi_i\psi_j+\psi_j\psi_i = 0$

special case: $i=j \Rightarrow (\psi_i)^2=0$, $(\psi_i)^k=0$ $\quad k=2,3,4,\dots$

$\hat{=}$ classical analogue of Pauli exclusion principle

represent arbitrary element of Grassmann algebra

$$f(\psi_1,\dots,\psi_n)=f^{(0)}+\sum_{i=1}^n f_i^{(1)}\psi_i+\sum_{i=1}^n\sum_{j=1}^{i-1} f^{(2)}_{ij}\psi_i\psi_j$$
$$+ \dots + \sum_{i=1}^n f_i^{(n-1)}\psi_1\cdots\psi_{i-1}\psi_{i+1}\cdots\psi_n + f^{(n)}\psi_1\cdots\psi_n$$

with $f^{(0)}, f_i^{(1)}, f_{ij}^{(2)},\dots,f_i^{(n-1)}, f^{(n)}\in\mathbb{C}$

number of independent products of p generators: $\binom{n}{p}$

$n_0=1=\binom{n}{0}$, $n_1=\sum_{i=1}^n 1 = n=\binom{n}{1}$, $n_2=\sum_{i=1}^n\sum_{j=1}^{i-1}1=\frac{n}{2}(n-1)=\binom{n}{2},\dots$

$n_{n-1}=\sum_{i=1}^n 1 = n=\binom{n}{n-1}$, $n_n=1=\binom{n}{n}$

dimension of Grassmann algebra:
$$\sum_{p=0}^n n_p = \sum_{p=0}^{n} 1^p 1^{n-p}\binom{n}{p} = (1+1)^n=2^n$$

5.1.2 Grassmann functions:

Grassmann number $\xrightarrow{\text{Grassmann function}}$ Grassmann number

example: Grassmann algebra $n=2 \Rightarrow \psi_1,\psi_2$

Grassmann number: $f(\psi_1,\psi_2)=f^{(0)}+f_1^{(1)}\psi_1+f_2^{(1)}\psi_2+f^{(2)}\psi_1\psi_2$

two examples of Grassmann functions:

$$e^{\psi_1+\psi_2}=1+\psi_1+\psi_2+\frac{1}{2}(\psi_1+\psi_2)^2+\dots = 1+\psi_1+\psi_2$$
$$e^{\psi_1\psi_2}=1+\psi_1\psi_2+\frac{1}{2}(\psi_1\psi_2)^2+\dots = \underline{1+\psi_1\psi_2}$$

Parity of a Grassmann number: $\pi(f)$

- even number of generators: $\pi(f)=0$, $[f,g]_-=0$, g any Grassmann number
- odd number of generators: $\pi(f)=1$, $[f,g]_+=0$, $\pi(g)=1$
- Grassmann number with both even and odd number of generators do not have a parity

examples: $\pi(\psi_1)=\pi(\psi_2)=1$, $\pi(e^{\psi_1\psi_2})=0$, no parity for $e^{\psi_1+\psi_2}$

5.1.3 Differentiation and Integration:

abstract construction via axioms, differs considerably from usual differentiation and integration calculus with real (complex) numbers

complex variables	Grassmann variables
$\frac{\partial}{\partial x_i}1=0$	$\frac{\partial}{\partial\psi_i}1=0$
$\frac{\partial x_j}{\partial x_i}=\delta_{ij}$	$\frac{\partial\psi_j}{\partial\psi_i}=\delta_{ij}$
$\frac{\partial}{\partial x_i}(x_jx_k)=\delta_{ij}x_k+\delta_{ik}x_j$	$\frac{\partial}{\partial\psi_i}(\psi_j\psi_k)=\delta_{ij}\psi_k-\delta_{ik}\psi_j$
$\frac{\partial}{\partial x_i}[x_j f(x_1,\dots,x_n)]=\delta_{ij}f(x_1,\dots,x_n)+x_j\frac{\partial}{\partial x_i}f(x_1,\dots,x_n)$	$\frac{\partial}{\partial\psi_i}(\psi_j f(\psi_1,\dots,\psi_n))=\delta_{ij}f(\psi_1,\dots,\psi_n)-\psi_j\frac{\partial}{\partial\psi_i}f(\psi_1,\dots,\psi_n)$
$[\frac{\partial}{\partial x_i},x_j]_-=\frac{\partial}{\partial x_i}x_j-x_j\frac{\partial}{\partial x_i}=\delta_{ij}$	$[\frac{\partial}{\partial\psi_i},\psi_j]_+=\frac{\partial}{\partial\psi_i}\psi_j+\psi_j\frac{\partial}{\partial\psi_i}=\delta_{ij}$
$[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}]_-=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}=0$	$[\frac{\partial}{\partial\psi_i},\frac{\partial}{\partial\psi_j}]_+=\frac{\partial}{\partial\psi_i}\frac{\partial}{\partial\psi_j}+\frac{\partial}{\partial\psi_j}\frac{\partial}{\partial\psi_i}=0$

Example 1: $n=2$ $\quad f(\psi_1,\psi_2)=f^{(0)}+f_1^{(1)}\psi_1+f_2^{(1)}\psi_2+f^{(2)}\psi_1\psi_2$

$\frac{\partial f(\psi_1,\psi_2)}{\partial\psi_1}=f_1^{(1)}+f^{(2)}\psi_2$, $\frac{\partial f(\psi_1,\psi_2)}{\partial\psi_2}=f_2^{(1)}-f^{(2)}\psi_1$

$$\frac{\partial^2 f(\psi_1,\psi_2)}{\partial \psi_1^2} = 0 = \frac{\partial^2 f(\psi_1,\psi_2)}{\partial \psi_2^2}, \quad \frac{\partial^2 f(\psi_1,\psi_2)}{\partial \psi_1 \partial \psi_2} = -f^{(2)}, \quad \frac{\partial^2 f(\psi_1,\psi_2)}{\partial \psi_2 \partial \psi_1} = f^{(2)}$$

Integration with respect to Grassmann variables more abstract:
- no analogue of a Riemann sum as an interpretation of an area under a curve
- inversion of differentiation also not possible because integration boundaries do not exist

Grassmann algebra with $n=1$: $[\psi,\psi]_+ = 0$, $f(\psi) = a + b\psi$

	ordinary integrals	Grassmann integrals
	$\int_{-\infty}^{+\infty} dx\, f(x)$	$\int d\psi\, f(\psi)$
linearity	$\int_{-\infty}^{+\infty} dx\,(\alpha f(x) + g(x)\beta)$ $= \alpha \int_{-\infty}^{+\infty} dx\, f(x) + \beta \int_{-\infty}^{+\infty} dx\, g(x)$	$\int d\psi\,(\alpha f(\psi) + \beta g(\psi))$ $= \alpha \int d\psi\, f(\psi) + \beta \int d\psi\, g(\psi)$
translational invariance	$\int_{-\infty}^{+\infty} dx\, f(x+x_0) = \int_{-\infty}^{+\infty} dx\, f(x)$	$\int d\psi\, f(\psi+\psi_0) = \int d\psi\, f(\psi)$

Example: $\int d\psi\,[a + b(\psi+\psi_0)] \overset{\text{linearity}}{=} \int d\psi\,(a+b\psi) + b\left(\int d\psi\, 1\right)\psi_0$ $\Big\}$ $\underline{\int d\psi\, 1 = 0}$

$\overset{\text{translational invariance}}{=} \int d\psi\,(a+b\psi)$

$\int d\psi\, \psi = ?$ $\Rightarrow$ normalisation: $\underline{\int d\psi\, \psi = 1}$

observation: $\frac{\partial}{\partial \psi} 1 = 0$, $\frac{\partial \psi}{\partial \psi} = 1$

differentiation $\equiv$ integration for Grassmann variables

$\int d\psi(a+b\psi) = b$, $\frac{\partial}{\partial \psi}(a+b\psi) = b$

Grassmann algebra: n generators ψ_i, $i = 1,\dots,n$

$\int d\psi_i\, 1 = 0 \quad \hat{=} \quad \frac{\partial}{\partial \psi_i} 1 = 0$

$\int d\psi_i\, \psi_j = \delta_{ij} \quad \hat{=} \quad \frac{\partial}{\partial \psi_i} \psi_j = \delta_{ij}$

Multiple integrals: evaluate them successively by performing one-dimensional integrals

$\int d\psi_2 \int d\psi_1\,(f^{(0)} + f_1^{(1)}\psi_1 + f_2^{(1)}\psi_2 + f^{(2)}\psi_1\psi_2) = f^{(2)}$

$\int d\psi_n \cdots \int d\psi_1\, f(\psi_1,\dots,\psi_n) = f^{(n)}$

$\Rightarrow$ integration: projecting out the corresponding coefficient

5.1.4 Complex Grassmann numbers:

two distinct sets of Grassmann numbers $\psi_1,\dots,\psi_n$ and $\psi_1^*,\dots,\psi_n^*$ which anti-commute

$[\psi_i,\psi_j]_+ = 0$, $[\psi_i^*,\psi_j^*]_+ = 0$, $[\psi_i,\psi_j^*]_+ = 0$

$\Rightarrow$ $2n$ generators

interconnection between 2 sets of generators via conjugation

$(\psi_i)^* = \psi_i^*$, $(\psi_i^*)^* = \psi_i$, $(\psi_{i_1}\cdots\psi_{i_n})^* = \psi_{i_n}^*\cdots\psi_{i_1}^*$

$(\lambda\psi_i)^* = \lambda^*\psi_i^*$, $\lambda\in\mathbb{C}$

differentiation and integration defined such that both sets of generators are considered independent

5.1.5 Grassmann fields: needed for canonical field quantisation

complex Grassmann numbers ψ_i, ψ_i^* $\xrightarrow[\text{limit}]{\text{continuum}}$ complex Grassmann fields $\psi(x)$, $\psi^*(x)$

$[\psi_i,\psi_j]_+ = [\psi_i^*,\psi_j^*]_+ = 0 \longrightarrow [\psi(x),\psi(x')]_+ = 0 = [\psi^*(x),\psi^*(x')]_+$

$[\psi_i,\psi_j^*]_+ = 0 \qquad [\psi(x),\psi^*(x')]_+ = 0$

$$f[\psi^*(\cdot);\psi(\cdot)] = f^{(0)} + \int dx_1 \left\{ f_1^{(1)}(x_1)\psi^*(x_1) + f_2^{(1)}(x_1)\psi(x_1)\right\}$$
$$+ \int dx_1 \int dx_2 \left\{ f_1^{(2)}(x_1,x_2)\psi^*(x_1)\psi^*(x_2) + f_2^{(2)}(x_1,x_2)\psi^*(x_1)\psi(x_2) + f_3^{(2)}(x_1,x_2)\psi(x_1)\psi(x_2)\right\} + \cdots$$

differentiation with respect to Grassmann fields: functional derivatives

$$\frac{\delta\psi(x)}{\delta\psi(x')} = \frac{\delta\psi^*(x)}{\delta\psi^*(x')} = \delta(x-x'), \quad \frac{\delta\psi(x)}{\delta\psi^*(x')} = \frac{\delta\psi^*(x)}{\delta\psi(x')} = 0$$

5.2 Lagrange Field Theory for Fermions:

Schrödinger fields $\psi(\vec{x},t)$, $\psi^*(\vec{x},t)$: Grassmann fields