linear coordinate transformation: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ ← contravariant, covariant; row, column

Lorentz transformation: $g_{\mu\nu} = \underbrace{\Lambda^\sigma{}_\mu}_{=(\Lambda^T)_\mu{}^\sigma} \underbrace{g_{\sigma\rho}}_{\text{Minkowski metric}} \Lambda^\rho{}_\nu \Rightarrow \underbrace{g = \Lambda^T g \Lambda}_{\det} \quad (*)$

$\mathcal{L}$: set of all Lorentz transformation is a group

Classification due to two properties:

- determinant: $\det g = (\det\Lambda)^2 \cdot \det g \overset{\det g \neq 0}{\Longrightarrow} (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$

  $\det\Lambda = +1$: special Lorentz transformation

  $\det\Lambda = -1$: non-special " "

- specialize in $(*)$: $\mu = \nu = 0 \Rightarrow 1 = g_{00} = \Lambda^\sigma{}_0 g_{\sigma\rho} \Lambda^\rho{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2$

  $\Rightarrow (\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1$

  $\Lambda^0{}_0 \geq 1$: orthochronous Lorentz transformation

  $\Lambda^0{}_0 \leq -1$: non- " " "

$\Rightarrow$ Four branches of Lorentz transformation

branch	Det Λ	$\Lambda^0{}_0$	example
$\mathcal{L}_1$	$+1$	>0	identity: diag $(1,1,1,1)$
$\mathcal{L}_2$	-1	>0	space-inversion: diag $(1,-1,-1,-1)$
$\mathcal{L}_3$	-1	<0	time inversion: diag $(-1,1,1,1)$
$\mathcal{L}_4$	$+1$	<0	space-time inversion: diag $(-1,-1,-1,-1)$

Lorentz group is not connected: branche cannot be transformed into each other

$\mathcal{L}_1$: special orthochronous Lorentz transformations $\Rightarrow$ subgroup of $\mathcal{L}$ $\Rightarrow$ we will restrict ourselves first to $\mathcal{L}_1$

$\mathcal{L}_1$ is called "the Lorentz group"

6.3 Defining Representation of Lorentz Algebra

4×4 matrices Λ: $4\cdot 4 = 16$ degrees of freedom

$-(g = \Lambda^T g \Lambda$: $\frac{1}{2} 4\cdot 5 = 10$ degrees of freedom $)$

$\Rightarrow$ dimension of Lorentz group: $16 - 10 = 6$ degrees of freedom

explore elements of Lorentz group in vicinity of identity

infinitesimal deviation from identity: $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \underbrace{\omega^\mu{}_\nu}_{\text{small}}$

insert this into $(*)$: $g_{\mu\nu} = \Lambda^\sigma{}_\mu g_{\sigma\rho} \Lambda^\rho{}_\nu$

$= g_{\sigma\rho}(g^\sigma{}_\mu + \omega^\sigma{}_\mu)(g^\rho{}_\nu + \omega^\rho{}_\nu) =$ multiply out up to first order in ω

$= g_{\sigma\rho} \underbrace{g^\sigma{}_\mu}_{=\delta^\sigma_\mu} g^\rho{}_\nu + g_{\sigma\rho}\omega^\sigma{}_\mu \underbrace{g^\rho{}_\nu}_{=\delta^\rho_\nu} + g_{\sigma\rho} \underbrace{g^\sigma{}_\mu}_{=\delta^\sigma_\mu} \omega^\rho{}_\nu$

$= g_{\mu\rho} \underbrace{g^\rho{}_\nu}_{=\delta^\rho_\nu} + g_{\sigma\nu}\omega^\sigma{}_\mu + g_{\mu\rho}\omega^\rho{}_\nu = g_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} \Rightarrow \omega_{\mu\nu} + \omega_{\nu\mu} = 0$ anti-symmetric

set of all anti-symmetric 4×4 matrices $\omega^\mu{}_\nu$ as elements of Lorentz algebra

$\Rightarrow$ dimension is 6 $\hat{=}$ same dimension as of Lorentz group

represent an element of Lorentz algebra:

$$\underbrace{\omega^\mu{}_\nu}_{\text{element of Lorentz algebra}} = g^{\alpha\mu} g^\beta{}_\nu \underbrace{\omega_{\alpha\beta}}_{=-\omega_{\beta\alpha}} = \frac{1}{2}\Big(g^{\alpha\mu} g^\beta{}_\nu \omega_{\alpha\beta} + \underbrace{g^{\alpha\mu} g^\beta{}_\nu \underbrace{\omega_{\alpha\beta}}_{=-\omega_{\beta\alpha}}}_{=-g^{\beta\mu} g^\alpha{}_\nu \omega_{\alpha\beta}}\Big) = -\frac{i}{2} \underbrace{\boxed{i(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} g^\alpha{}_\nu)}}_{=(L^{\alpha\beta})^\mu{}_\nu} \underbrace{\omega_{\alpha\beta}}_{\text{coefficients}}$$

$(L^{\alpha\beta})^\mu{}_\nu$: basis elements of Lorentz algebra

α,β: characterise (enumerate) basis elements of Lorentz algebra

μ,ν: components of (4×4) matrix

anti-symmetries: $(L^{\alpha\beta})^\mu{}_\nu = -(L^{\beta\alpha})^\mu{}_\nu = -(L^{\alpha\beta})_\nu{}^\mu$

Commutator between two basis elements:

$$[L^{\alpha\beta}, L^{\gamma\delta}]_- = \underbrace{i\left(g^{\alpha\delta} L^{\beta\gamma} + g^{\beta\gamma} L^{\alpha\delta} - g^{\alpha\gamma} L^{\beta\delta} - g^{\beta\delta} L^{\alpha\gamma}\right)}_{\text{some combination of basis elements}} = i \underbrace{C^{\alpha\beta\gamma\delta}_{\varepsilon\zeta}}_{\text{structure coefficients}} L^{\varepsilon\zeta}$$

$\hat{=}$ closedness of Lorentz algebra

structure coefficients characteristic for Lorentz algebra

$$C^{\alpha\beta\gamma\delta}_{\varepsilon\zeta} = g^{\alpha\delta} g^\beta{}_\varepsilon g^\gamma{}_\zeta + g^{\beta\gamma} g^\alpha{}_\varepsilon g^\delta{}_\zeta - g^{\alpha\gamma} g^\beta{}_\varepsilon g^\delta{}_\zeta - g^{\beta\delta} g^\alpha{}_\varepsilon g^\gamma{}_\zeta$$

6.4 Classify Basis Elements of Lorentz Algebra:

basis elements: sorted into two classes

α,β: spatial indices — α,β: spatio-temporal indices

$L_k = \frac{1}{2}\underbrace{\varepsilon_{klm}}\, L^{lm}$

Levi-Civita symbol: $\begin{cases} \varepsilon_{123} = +1 \\ \varepsilon_{klm} = \varepsilon_{kml} = -\varepsilon_{mlk} = -\varepsilon_{lkm} \end{cases}$

$$L_1 = L^{23} = -i\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix} \qquad M_1 = L^{01} = i\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$L_2 = L^{23} = -i\begin{pmatrix} 0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix} \qquad M_2 = L^{02} = i\begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$$

$$L_3 = L^{12} = -i\begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix} \qquad M_3 = L^{03} = i\begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

$\Rightarrow$ properties of these basis elements

$\rightarrow$ inversion: $\underline{\varepsilon_{ijk} L_k} = \frac{1}{2}\underbrace{\varepsilon_{ijk}\,\varepsilon_{lmk}}_{=\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} L^{lm} = \frac{1}{2}\big(L^{ij} - \underbrace{L^{ji}}_{=-L^{ij}}\big) = \underline{L^{ij}}$

commutator relations:

$[L_k, L_l]_- = i\,\varepsilon_{klm} L_m \Rightarrow$ subalgebra of Lorentz algebra

$[L_k, M_l]_- = i\,\varepsilon_{klm} M_m, \qquad [M_k, M_l]_- = -i\,\varepsilon_{klm} L_m$

6.5 Lie Theorem:

Lorentz group $\xrightarrow[\text{of identity}]{\text{restrict to vicinity}}$ Lorentz algebra (elements = generators)

$\xleftarrow{\text{Lie theorem}}$

$$\underbrace{\Lambda}_{\substack{\text{element of}\\ \text{Lorentz group}}} = \underbrace{\exp}_{\substack{\text{exponential}\\ \text{function}}}\Big\{-\frac{i}{2}\underbrace{L^{\alpha\beta}}_{\text{basis generators}}\ \underbrace{\omega_{\alpha\beta}}_{\text{expansion coefficients}}\Big\}$$

$$\frac{1}{2}L^{\alpha\beta}\omega_{\alpha\beta} = \frac{1}{2}\underbrace{L^{ij}}_{=\frac{1}{2}\varepsilon_{ijk}L_k}\omega_{ij} + \underbrace{\frac{1}{2}\big(L^{0i}\omega_{0i} + \underbrace{L^{i0}}_{=-L^{0i}}\underbrace{\omega_{i0}}_{=-\omega_{0i}}\big)}_{=\underbrace{L^{0i}}_{=M_i}\underbrace{\omega_{0i}}_{=:\xi_i}\ \text{(rapidities)}}$$

$$= L_k \cdot \underbrace{\frac{1}{2}\varepsilon_{ijk}\omega_{ij}}_{=:\ \varphi_k\ \text{(rotation angles)}}$$

$$\Rightarrow\quad \Lambda = e^{-i(\vec{L}\cdot\vec{\varphi} + \vec{M}\cdot\vec{\xi})}$$

exp: matrix-valued exponential function $\Rightarrow$ Taylor expansion

$\vec{\xi} = \vec{0}$: rotations, $\qquad \vec{\varphi} = \vec{0}$: boosts

$R(\vec{\varphi}) = e^{-i\vec{L}\cdot\vec{\varphi}} \qquad B(\vec{\xi}) = e^{-i\vec{M}\cdot\vec{\xi}}$

6.6 Rotations:

Result: $$R(\vec{\varphi}) = \left(\begin{array}{c|c} 1 & 0\quad 0\quad 0 \\ \hline 0 & \\ 0 & R_{jk}(\vec{\varphi}) \\ 0 & \end{array}\right)$$

$$R_{jk}(\vec{\varphi}) = \frac{\varphi_i}{|\vec{\varphi}|}\varepsilon_{ijk}\sin|\vec{\varphi}| + \frac{\varphi_j\varphi_k}{|\vec{\varphi}|}(1-\cos|\vec{\varphi}|) + \delta_{jk}\cos|\vec{\varphi}|$$

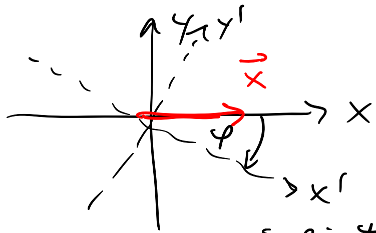
Fulfills properties characteristic for rotation

$R(\vec{\varphi})\begin{pmatrix}0\\\vec{\varphi}\end{pmatrix} = 1\cdot\begin{pmatrix}0\\\vec{\varphi}\end{pmatrix} \qquad \mathrm{Tr}\,R(\vec{\varphi}) = 2 + 2\cos|\vec{\varphi}|$

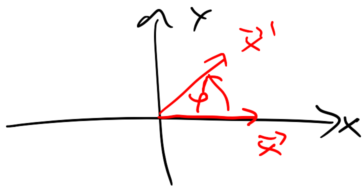
special: $$\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad \text{for } \vec{\varphi} = \varphi\,\vec{e}_z$$

$$\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{R} \vec{x}' = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}$$

passive interpretation
vector is fixed
coordinate system is rotated



active interpretation
vector is not rotated
coordinate system is fixed



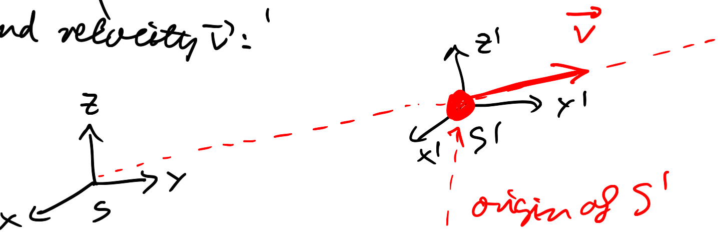
in the following: passive interpretation

6.7 Boosts:

$$B(\vec\zeta) = e^{-i\vec{M}\cdot\vec\zeta} \xrightarrow[\text{valued function}]{\text{evaluation of matrix}} \left(\begin{array}{c|c} \cosh|\vec\zeta| & \frac{\zeta_j}{|\vec\zeta|}\sinh|\vec\zeta| \\ \hline \frac{\zeta_i}{|\vec\zeta|}\sinh|\vec\zeta| & \delta_{ij} + \frac{\zeta_i\zeta_j}{|\vec\zeta|^2}(\cosh|\vec\zeta|-1)\end{array}\right)$$

relation between rapidity $\vec\zeta$ and velocity $\vec v$:
passive interpretation

$S \longrightarrow S'$



origin of S' in S: $(x^\mu) = \begin{pmatrix} ct\\ \vec v t\end{pmatrix}$ origin of S' in S': $(x'^\mu) = \begin{pmatrix} ct' \\ \vec 0\end{pmatrix}$

Boost: $x'^\mu = B^\mu{}_\nu(\vec\zeta)\, x^\nu$

(1) $t' = t\cosh|\vec\zeta| + \frac{\vec\zeta}{|\vec\zeta|}\,\frac{\vec v t}{c}\sinh|\vec\zeta|$

(2) $\vec 0 = \frac{\vec\zeta}{|\vec\zeta|}\sinh|\vec\zeta| + \frac{\vec v}{c} + \frac{\vec\zeta\cdot\vec v}{|\vec\zeta|\, c}\,\frac{\vec\zeta}{|\vec\zeta|}\cosh|\vec\zeta|$

conclusion (2): $\frac{\vec\zeta}{|\vec\zeta|} = \bullet\,\frac{\vec v}{|\vec v|}$ (3)

(3) in (2): $\xrightarrow{\text{algebra}}$ $\frac{|\vec v|}{c} = \tanh|\vec\zeta|$

$$\cosh|\vec\zeta| = \frac{1}{\sqrt{1-\tanh^2|\vec\zeta|}} = \frac{1}{\sqrt{1-\frac{\vec v^2}{c^2}}} =: \gamma$$ Lorentz factor of special relativity

$$\sinh|\vec\zeta| = \frac{\tanh|\vec\zeta|}{\sqrt{1-\tanh^2|\vec\zeta|}} = \gamma\frac{|\vec v|}{c}$$

$$\Rightarrow B(\vec v) = \left(\begin{array}{c|c}\gamma & -\frac{v_j}{c}\gamma \\ \hline -\frac{v_i}{c}\gamma & \delta_{ij}+\frac{v_iv_j}{|\vec v|^2}(\gamma-1)\end{array}\right)$$

Note from (1): $t' = t\sqrt{1-\frac{v^2}{c^2}}$ time dilation
observer in S: clock in S' ticks slower than clock in S

6.8 Scalar Field Representation:

scalar field: $\phi(x^\mu)$, tensor of rank $n=0$ due to its invariance with respect to Lorentz trafos

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \iff x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$$

passive interpretation: (x^μ) and (x'^μ) describe one and the same space-time point

invariance with respect to Lorentz trafos: $\phi(\underbrace{x^\mu}_{=(\Lambda^{-1})^\mu{}_\nu x'^\nu}) = \phi'(x'^\mu)$ $\Rightarrow \phi\left((\Lambda^{-1})^\mu{}_\nu x^{\cancel{\prime}\nu}\right) = \phi'(x^{\cancel{\prime}\mu})$
notational simplification

$$\Rightarrow \phi\left((\Lambda^{-1})^\mu{}_\nu x^\nu\right) = \phi'(x^\mu)$$

specialize to: $\Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu\,\omega_{\alpha\beta}$

$(\Lambda^{-1})^\mu{}_\nu = g^\mu{}_\nu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu\,\omega_{\alpha\beta}$