Now back to general case: $\psi^{\sigma}(x^{\mu})$

$$\psi'^{\sigma}(x^{\mu}) = \left\{ g^{\sigma}{}_{\tau} - \frac{i}{2} \omega_{\alpha\beta} (\hat{M}^{\alpha\beta})^{\sigma}{}_{\tau} \right\} \psi^{\tau}(x^{\mu})$$

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + \underbrace{N^{\alpha\beta}}_{\text{representation of Lorentzalgebra in tensorial / spinorial component space}}$$

$$[N^{\alpha\beta}, N^{\gamma\delta}]_{-} = i(g^{\alpha\delta} N^{\beta\gamma} + g^{\beta\gamma} N^{\alpha\delta} - g^{\alpha\gamma} N^{\beta\delta} - g^{\beta\delta} N^{\alpha\gamma})$$

$$[\hat{L}^{\alpha\beta}, N^{\gamma\delta}]_{-} = 0$$

$\Rightarrow$ $\hat{M}^{\alpha\beta}$ is a representation of Lorentzalgebra

6.10 Defining Representation of Poincaré Group:

Poincaré transformation: Lorentztransformation Λ + shift a

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + \boxed{a^{\mu}} \rightarrow (\text{"inhomogeneous"})$$

Lorentztransformation: scalar product of a four-vector is invariant

Poincaré " : leaves distances between four-vectors invariant

$$g_{\mu\nu}(x^{\mu} - y^{\mu})(x^{\nu} - x^{\nu}) = g_{\mu\nu}(x'^{\mu} - y'^{\mu})(x'^{\nu} - y'^{\nu})$$

set $\mathcal{P}$ of all Poincaré transformations: group

(Λ, a): element of $\mathcal{P}$

- Closedness: $(\Lambda_1, a_1), (\Lambda_2, a_2) \in \mathcal{P}$

$$x_2^{\mu} = \Lambda_2^{\mu}{}_{\nu} \underbrace{x_1^{\nu}}_{\Lambda^{\nu}{}_{\chi} x^{\chi} + a_1^{\nu}} + a_2^{\mu} = \underbrace{\Lambda^{\mu}_{2}{}_{\nu} \Lambda_1{}^{\nu}{}_{\chi}}_{=\Lambda^{\mu}{}_{\chi}} x^{\chi} + \underbrace{\Lambda^{\mu}_{2}{}_{\nu} a^{\nu}_{1} + a^{\mu}_{2}}_{=a^{\mu}}$$

short-hand notation: $(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \boxed{\Lambda_2} a_1 + a_2) \in \mathcal{P}$

semi-direct product of Lorentzgroup $\mathcal{L}$ and translation group $\mathcal{T}$

- associativity: $(\Lambda_1, a_1), (\Lambda_2, a_2), (\Lambda_3, a_3) \in \mathcal{P}$

$$(\Lambda_1, a_1)\big((\Lambda_2, a_2)(\Lambda_3, a_3)\big) = \big((\Lambda_1, a_1)(\Lambda_2, a_2)\big)(\Lambda_3, a_3)$$

- unity element of $\mathcal{P}$: $(\Lambda_e, a_e) = (I, 0)$

$(\Lambda, a) \in \mathcal{P}$: $(I, 0)(\Lambda, a) = (\Lambda, a) = (\Lambda, a)(I, 0)$

- inverse element of $\mathcal{P}$: $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$

$$(\Lambda, a)^{-1}(\Lambda, a) = (I, 0) = (\Lambda_e, a_e)$$

Remark: $\mathcal{P} = \underbrace{\mathcal{P}_1}_{\text{restrict durelles}} \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \mathcal{P}_4$

6.11 Tensor / Spinor Representation of Poincaré Algebra:

translation with four-vector a^{μ}: $x'^{\mu} = x^{\mu} + a^{\mu} \Leftrightarrow x^{\mu} = x'^{\mu} - a^{\mu}$

passive interpretation: x'^{μ} and x^{μ} describe one and the same space-time point in S' and S

invariance of tensor/spinor field: $\psi'^{\sigma}(x'^{\mu}) = \psi^{\sigma}(\underbrace{x^{\mu}}_{=x'^{\mu} - a^{\mu}})$

$$\psi'^{\sigma}(x'^{\mu}) = \psi^{\sigma}(x'^{\mu} - a^{\mu}) \xrightarrow[\text{as simplification}]{\text{omit '}} \psi'^{\sigma}(x^{\mu}) = \psi^{\sigma}(x^{\mu} - a^{\mu})$$

infinitesimal translation: $a^{\mu} = \varepsilon^{\mu}$ $\rightarrow$ Taylor expansion

$$\psi'^{\sigma}(x^{\mu}) = \Big\{ 1 + \frac{i}{\hbar} \underbrace{\varepsilon^{\mu} \cdot \hat{p}_{\mu}}_{=\varepsilon_{\alpha} \hat{p}^{\alpha}} \Big\} \psi^{\sigma}(x^{\mu}) \quad ; \quad \hat{p}^{\alpha} = i\hbar \partial_{\alpha} \quad \text{momentum operators}$$

Result: bases generators of translation are momentum operators

Extension: infinitesimal Poincaré transformation

$$\psi'^{\sigma}(x^{\mu}) = \Big\{ 1 - \frac{i}{2} \underbrace{\hat{M}^{\alpha\beta}}_{\substack{\text{total angular momentum operators} \\ \text{infinitesimal rotation / boost}}} \omega_{\alpha\beta} + \frac{i}{\hbar} \varepsilon_{\alpha} \underbrace{\hat{p}^{\alpha}}_{\substack{\text{momentum operators} \\ \text{infinitesimal translation}}} \Big\} \psi^{\sigma}(x^{\mu})$$

$$\hat{M}^{\alpha\beta} = \underbrace{\hat{L}^{\alpha\beta}}_{\text{representation of Lorentz algebra in space time}} + \underbrace{N^{\alpha\beta}}_{\text{representation of Lorentzalgebra in tensor / spinor space}}$$

orbital angular momentum operator: $\hat{L}^{\alpha\beta} = \frac{1}{\hbar}(x^{\alpha} \hat{p}^{\beta} - x^{\beta} \hat{p}^{\alpha})$

It remains to determine commutation relations between basis generators of Poincaré algebra

$[\hat p^\alpha, \hat p^\beta]_- = 0$ $\hat{=}$ translations are a subgroup of Poincaré group

$[N^{\alpha\beta}, \hat p^\gamma]_- = 0 \;\Rightarrow\; [\hat M^{\alpha\beta}, \hat p^\gamma]_- = [\hat L^{\alpha\beta}, \hat p^\gamma]_- \underset{\substack{\uparrow \\ \text{last lecture}}}{=} i(g^{\beta\gamma}\hat p^\alpha - g^{\alpha\gamma}\hat p^\beta)$

$[\hat M^{\alpha\beta}, \hat M^{\gamma\delta}]_- \underset{\uparrow}{=} i(g^{\alpha\delta}\hat M^{\beta\gamma} + g^{\beta\gamma}\hat M^{\alpha\delta} - g^{\alpha\gamma}\hat M^{\beta\delta} - g^{\beta\delta}\hat M^{\alpha\gamma})$

as discussed: commutation relations of Lorentz algebra $\hat{=}$ Lorentz group is a subgroup of Poincaré group

6.12 Casimir Operators of Poincaré Algebra:

Casimir operator of a Lie algebra is an operator, which commutes with all operators of Lie algebra

1. Casimir operator of Poincaré algebra: $\hat p^2 = g_{\alpha\beta}\hat p^\alpha \hat p^\beta$

$$[\hat p^2, \hat p^\alpha]_- = g_{\beta\gamma}[\hat p^\beta \hat p^\gamma, \hat p^\alpha]_- = g_{\beta\gamma}\{\hat p^\beta \underbrace{[\hat p^\gamma, \hat p^\alpha]_-}_{=0} + \underbrace{[\hat p^\beta, \hat p^\alpha]_-}_{=0} \} \equiv 0$$

$\hat p^2$ is manifestly by construction a Lorentz scalar

$$[\hat p^2, \hat M^{\alpha\beta}]_- = g_{\gamma\delta}[\hat p^\gamma \hat p^\delta, \hat M^{\alpha\beta}]_- = g_{\gamma\delta}\{\hat p^\gamma \underbrace{[\hat p^\delta, \hat M^{\alpha\beta}]_-}_{=i(g^{\alpha\delta}\hat p^\beta - g^{\beta\delta}\hat p^\alpha)} + \boxed{\underbrace{[\hat p^\gamma, \hat M^{\alpha\beta}]_-}_{i(g^{\alpha\gamma}\hat p^\beta - g^{\beta\gamma}\hat p^\alpha)}} \hat p^\delta \}$$

four-"vector" definition

$$= i(\hat p^\alpha \hat p^\beta - \hat p^\beta \hat p^\alpha + \hat p^\beta \hat p^\alpha - \hat p^\alpha \hat p^\beta) \equiv 0$$

In view of 2. Casimir operator we consider Pauli-Lubanski operator: $\hat w_\alpha = \frac{1}{2}\underbrace{\varepsilon_{\alpha\beta\gamma\delta}} \hat p^\beta \hat M^{\gamma\delta}$

four-dimensional, anti-symmetric unity tensor

$\hat{=}$ relativistic extension of Levi-Civita tensor

$\varepsilon_{0123} = 1, \quad \varepsilon_{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\delta\gamma} = -\varepsilon_{\alpha\delta\gamma\beta} = -\varepsilon_{\alpha\gamma\beta\delta} = -\varepsilon_{\delta\beta\gamma\alpha} = \ldots$

scalar product of $\hat w_\alpha$ and $\hat p^\alpha$ vanishes:

$$\hat w_\alpha \hat p^\alpha = \frac{1}{2}\underbrace{\varepsilon_{\alpha\beta\gamma\delta}}_{\substack{\text{anti-symmetric in } \alpha,\beta \\ \text{by definition}}} \underbrace{\hat p^\alpha \hat p^\beta}_{\text{symmetric in } \alpha,\beta \text{ due to } [\hat p^\alpha, \hat p^\beta]_- = 0} \hat M^{\gamma\delta} = 0$$

$$[\hat w^\alpha, \hat p^\sigma]_- = g^{\alpha\alpha'}[\hat w_{\alpha'}, \hat p^\sigma]_- = \frac{1}{2} g^{\alpha\alpha'}\varepsilon_{\alpha'\beta\gamma\delta}\underbrace{[\hat p^\beta \hat M^{\gamma\delta}, \hat p^\sigma]_-}$$

$$= \hat p^\beta \underbrace{[\hat M^{\gamma\delta}, \hat p^\sigma]_-}_{=i(g^{\delta\sigma}\hat p^\gamma - g^{\gamma\sigma}\hat p^\delta)} + \underbrace{[\hat p^\beta, \hat p^\sigma]_-}_{=0} \hat M^{\gamma\delta}$$

$$= \frac{i}{2} g^{\alpha\alpha'} \underbrace{\varepsilon_{\alpha'\beta\gamma\delta}}_{\substack{\text{anti-symmetric in} \\ \text{both } \beta,\gamma \text{ and } \beta,\delta}} (g^{\delta\sigma} \underbrace{\hat p^\beta \hat p^\gamma}_{\text{symmetric in } \beta,\gamma} - g^{\gamma\sigma} \underbrace{\hat p^\beta \hat p^\delta}_{\text{symmetric in } \beta,\delta \text{ due to } [\hat p^\alpha, \hat p^\beta]_- = 0}) \equiv 0$$

$$[\hat M^{\alpha\beta}, \hat w^\rho]_- = g^{\rho\delta}\frac{1}{2}\varepsilon_{\delta\varrho\sigma\tau}\underbrace{[\hat M^{\alpha\beta}, \hat p^\varrho \hat M^{\sigma\tau}]_-}$$

$$= \underbrace{[\hat M^{\alpha\beta}, \hat p^\varrho]_-}_{=i(g^{\beta\varrho}\hat p^\alpha - g^{\alpha\varrho}\hat p^\beta)} \hat M^{\sigma\tau} + \hat p^\varrho \underbrace{[\hat M^{\alpha\beta}, \hat M^{\sigma\tau}]_-}_{=i(g^{\alpha\tau}\hat M^{\beta\sigma} + g^{\beta\sigma}\hat M^{\alpha\tau} - g^{\beta\tau}\hat M^{\alpha\sigma} - g^{\alpha\sigma}\hat M^{\beta\tau})}$$

$$= \frac{i}{2} g^{\rho\delta}\varepsilon_{\delta\varrho\sigma\tau}\{ g^{\beta\varrho}\hat p^\alpha \hat M^{\sigma\tau} - g^{\alpha\varrho}\hat p^\beta \hat M^{\sigma\tau}$$
$$+ \underbrace{g^{\alpha\tau}\hat p^\varrho \hat M^{\beta\sigma}}_{\substack{\varrho\,\sigma\;\tau \\ \text{cyclic}}} + g^{\beta\sigma}\hat p^\varrho \hat M^{\alpha\tau} - \underbrace{g^{\beta\tau}\hat p^\varrho \hat M^{\alpha\sigma}}_{\varrho\;\sigma\;\tau} + \underbrace{g^{\alpha\sigma}\hat p^\varrho \hat M^{\beta\tau}}_{\varrho\;\sigma} \}$$

anti-symmetry of σ, ϱ — cyclic permutation — anti-symmetry

$$= \frac{i}{2} g^{\gamma\delta}\{ \underbrace{g^{\beta\varrho}\cdot\varepsilon_{\delta\varrho\sigma\tau}(\hat p^\alpha \hat M^{\sigma\tau} - 2\hat p^\sigma \hat M^{\alpha\tau}) - g^{\alpha\varrho}\varepsilon_{\delta\varrho\sigma\tau}(\hat p^\beta \hat M^{\beta\tau} - 2\hat p^\sigma \hat M^{\beta\tau})} \} \quad (I)$$

What is this? Must be related to W in order to have $\hat w_\alpha$ as a "vector"

$$\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha'\beta'\gamma'\delta} = \delta_\alpha^{\alpha'}\delta_\beta^{\beta'}\delta_\gamma^{\gamma'} + \delta_\alpha^{\beta'}\delta_\beta^{\gamma'}\delta_\gamma^{\alpha'} + \delta_\alpha^{\gamma'}\delta_\beta^{\alpha'}\delta_\gamma^{\beta'}$$
$$- \delta_\alpha^{\beta'}\delta_\beta^{\alpha'}\delta_\gamma^{\gamma'} - \delta_\alpha^{\alpha'}\delta_\beta^{\gamma'}\delta_\gamma^{\beta'} - \delta_\alpha^{\gamma'}\delta_\beta^{\beta'}\delta_\gamma^{\alpha'} \quad (*)$$

useful in order to invert $\hat w_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta'\gamma'\delta'}\hat p^{\beta'}\hat M^{\gamma'\delta'}$

$$\hat w_\alpha \varepsilon^{\alpha\beta\gamma\delta} = \frac{1}{2}\underbrace{\varepsilon^{\beta\gamma\delta\alpha}\varepsilon_{\beta'\gamma'\delta'\alpha}}_{\text{apply!}}\hat p^{\beta'}\hat M^{\gamma'\delta'}$$

$$= \frac{1}{2}(\hat p^\beta \hat M^{\gamma\delta} + \hat p^\delta \hat M^{\beta\gamma} + \hat p^\gamma \hat M^{\delta\beta} - \hat p^\gamma \underbrace{\hat M^{\beta\delta}}_{=-\hat M^{\delta\beta}} - \hat p^\delta \underbrace{\hat M^{\gamma\beta}}_{=-\hat M^{\beta\gamma}} - \hat p^\beta \underbrace{\hat M^{\delta\gamma}}_{=-\hat M^{\gamma\delta}})$$

$$= \hat p^\beta \hat M^{\gamma\delta} + \hat p^\delta \hat M^{\beta\gamma} + \hat p^\gamma \hat M^{\delta\beta} \quad (***)$$

special case of (*)

$$\varepsilon_{\alpha\beta\gamma\delta}\,\varepsilon^{\alpha'\beta'\gamma\delta} = \underbrace{(4-1-1)}_{=+2}\delta_\alpha{}^{\alpha'}\delta_\beta{}^{\beta'} + \underbrace{(1+1\ \ -4)}_{=-2}\delta_\alpha{}^{\beta'}\delta_\beta{}^{\alpha'} \qquad (**)$$

$$\left(\hat{w}_\alpha\,\varepsilon^{\alpha\beta\gamma\delta}\right)\varepsilon_{\sigma\tau\gamma\delta}$$

$$1)\ \overset{(**)}{=}\ 2\,\hat{w}_\alpha\left(\delta^\alpha{}_\sigma\,\delta^\beta{}_\tau-\delta^\alpha{}_\tau\,\delta^\beta{}_\sigma\right) = 2\left(\hat{w}_\sigma\,\delta^\beta{}_\tau-\hat{w}_\tau\,\delta^\beta{}_\sigma\right)$$

$$2)\ \overset{(***)}{=}\ \varepsilon_{\sigma\tau\gamma\delta}\Big(\hat{p}^\beta\hat{M}^{\gamma\delta}+\underbrace{\hat{p}^\delta\hat{M}^{\beta\gamma}}_{=-\hat{p}^\gamma\hat{M}^{\beta\delta}}+\hat{p}^\gamma\underbrace{\hat{M}^{\delta\beta}}_{-\hat{M}^{\beta\delta}}\Big) = \varepsilon_{\sigma\tau\gamma\delta}\left(\hat{p}^\beta\hat{M}^{\gamma\delta}-2\,\hat{p}^\gamma\hat{M}^{\beta\delta}\right)$$

anti symmetry of ε antisymmetric of $\hat{M}$ identity (II)

(II) in (I):

$$[\hat{M}^{\alpha\beta},\hat{w}^\gamma]_- = i\left(g^{\beta\gamma}\hat{w}^\alpha-g^{\alpha\gamma}\hat{w}^\beta\right) \underset{\uparrow}{=} -\left(L^{\alpha\beta}\right)^\gamma{}_\delta\,\hat{w}^\delta$$

$$\left(L^{\alpha\beta}\right)^\gamma{}_\delta = i\left(g^{\alpha\gamma}g^\beta{}_\delta-g^{\beta\delta}g^\alpha{}_\delta\right)$$

=> Pauli-Lubanski operator transforms like a tensor of rank $n=1$

2. Casimir operator of Poincaré algebra: $\hat{w}^2 = g_{\alpha\beta}\hat{w}^\alpha\hat{w}^\beta$

$[\hat{p}^\alpha,\hat{w}^2]_- = 0$ due $[\hat{w}^\alpha,\hat{p}^\beta]_- = 0$

$$[\hat{M}^{\alpha\beta},\hat{w}^2]_- = g_{\gamma\delta}[\hat{M}^{\alpha\beta},\hat{w}^\gamma\hat{w}^\delta]_- = \text{abc-rule} \underset{\uparrow}{=} 0$$

$$[\hat{M}^{\alpha\beta},\hat{w}^\gamma]_- = \ldots$$

Finally: Physical interpretation of both Casimir operators of Poincaré algebra?
describe a particle with fixed four-momentum $p=(p^\mu)$ via some tensor/spinor field $\psi^\sigma(x^\mu)$
and the eigenvalue problem $\hat{p}^\mu\,\psi^\sigma(x) = p^\mu\psi^\sigma(x)$

1. Casimir operator: $\hat{p}^2\psi^\sigma(x) = p^2\psi^\sigma(x)$ with $p^2 = g_{\mu\nu}p^\mu p^\nu = (Mc)^2$
=> determined by rest mass of the particle

2. Casimir operator:
interpret physically Pauli-Lubansky operator $\hat{w}_\alpha$

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta}+N^{\alpha\beta},\quad \hat{L}^{\alpha\beta} = \frac{1}{\hbar}\left(x^\alpha\hat{p}^\beta-x^\beta\hat{p}^\alpha\right)$$

$$\hat{w}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta\hat{M}^{\gamma\delta} = \frac{1}{6}\varepsilon_{\alpha\beta\gamma\delta}\left(\hat{p}^\beta\hat{L}^{\gamma\delta}+\hat{p}^\gamma\hat{L}^{\delta\beta}+\hat{p}^\delta\hat{L}^{\beta\gamma}\right)+\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta N^{\gamma\delta}$$

$$\underbrace{=\frac{1}{\hbar}\hat{p}^\beta\left(x^\gamma\hat{p}^\delta-x^\delta\hat{p}^\gamma\right)+\frac{1}{\hbar}\hat{p}^\gamma\left(x^\delta\hat{p}^\beta-x^\beta\hat{p}^\delta\right)+\frac{1}{\hbar}\hat{p}^\delta\left(x^\beta\hat{p}^\gamma-x^\gamma\hat{p}^\beta\right)}_{\equiv 0}$$

$[\hat{p}^\alpha, x^\beta]_- = i\hbar g^{\alpha\beta}$

$$\Rightarrow\ \hat{w}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta N^{\gamma\delta}$$

- no orbital angular momentum
- only representation of Lorentz algebra in tensor/spinor space contributes ≙ spin

eigenvalue problem of Pauli-Lubansky operator: $\hat{w}_\alpha\psi^\sigma(x) = w_\alpha\psi^\sigma(x)$

$$\Rightarrow\ w_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\,p^\beta N^{\gamma\delta}$$

$N^{\alpha\beta}$

spatial indices

$S_k = \frac{1}{2}\varepsilon_{klm}N^{lm}$

$\vec{S} = (S_1,S_2,S_3) = (N^{23},N^{31},N^{12})$

spatio-temporal indices

$K_k = N^{0k}$

$\vec{K} = (K_1,K_2,K_3) = (N^{01},N^{02},N^{03})$

$$w_0 = \frac{1}{2}\underbrace{\varepsilon_{0ijk}}_{=\varepsilon_{ijk}}p^iN^{jk} = p^i\,\frac{1}{2}\varepsilon_{ijk}N^{jk} = p^iS_i = \boxed{\vec{p}\cdot\vec{S} = w_0}$$

$$w_i = \frac{1}{2}\Big(\underbrace{\varepsilon_{i0jk}}_{\substack{=-\varepsilon_{0ijk}\\=-\varepsilon_{ijk}}}p^0N^{jk}+\underbrace{\varepsilon_{ij0k}}_{\substack{=-\varepsilon_{ijk0}\\=-\varepsilon_{0ijk}\\=-\varepsilon_{ijk}}}p^j\underbrace{N^{0k}}_{=K^k}+\underbrace{\varepsilon_{ijk0}\,p^jN^{k0}}_{-(\vec{p}\times\vec{K})_i}\Big)$$

$$\underbrace{}_{-p^0\cdot 2S_i}\qquad\underbrace{}_{-(\vec{p}\times\vec{K})_i}$$

$$= -\left\{p^0S_i+(\vec{p}\times\vec{K})_i\right\}\ \Rightarrow\ (w_\alpha) = (w_0,-\vec{w})\ \Rightarrow\ \boxed{\vec{w} = p^0\vec{S}+\vec{p}\times\vec{K}}$$

rest frame of particle: $p^0 = Mc$, $\vec{p} = \vec{0}$ $\Rightarrow$ $w_0 = 0$, $\vec{w} = Mc\,\vec{S}$