equal-time commutation relations:

$[\hat\Psi(\vec x,t), \hat\Psi(\vec x',t)]_- = 0 = [\hat\Psi^+(\vec x,t), \hat\Psi^+(\vec x',t)]_-$ $\quad [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)]_- = \delta(\vec x-\vec x')$

$\to$ $\hat\Psi^+(\vec x,t)$, $\hat\Psi(\vec x,t)$ represent creation / annihilation operator of boson at $\vec x$ at time t

$\hat H_S = \int d^3x\, \hat a^+_{\vec x} \{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\} \hat a_{\vec x} + \frac{1}{2}\int d^3x \int d^3x'\, V_2(\vec x-\vec x')\, \hat a^+_{\vec x} \hat a^+_{\vec x'} \hat a_{\vec x'} \hat a_{\vec x}$

$\hat H_H(t) = e^{\frac{i}{\hbar}\hat H_S t}\, \hat H_S\, e^{-\frac{i}{\hbar}\hat H_S t}$

$= \int d^3x\, \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a^+_{\vec x}\, e^{-\frac{i}{\hbar}\hat H_S t}}_{=\hat\Psi^+(\vec x,t)} \{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\} \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a_{\vec x}\, e^{-\frac{i}{\hbar}\hat H t}}_{=\hat\Psi(\vec x,t)}$

$= \frac{1}{2}\int d^3x \int d^3x'\, V_2(\vec x-\vec x')\, \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a^+_{\vec x}\, e^{-\frac{i}{\hbar}\hat H_S t}}_{\hat\Psi^+(\vec x,t)} \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a^+_{\vec x'}\, e^{-\frac{i}{\hbar}\hat H_S t}}_{\hat\Psi^+(\vec x',t)} \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a_{\vec x'}\, e^{-\frac{i}{\hbar}\hat H_S t}}_{\hat\Psi(\vec x',t)} \underbrace{e^{\frac{i}{\hbar}\hat H_S t}\, \hat a_{\vec x}\, e^{-\frac{i}{\hbar}\hat H_S t}}_{\hat\Psi(\vec x,t)}$

Heisenberg equation:

$i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t) = [\hat\Psi(\vec x,t), \hat H_H(t)]_-$

$= \int d^3x' \int d^3x''\, \delta(\vec x'-\vec x'')\{-\frac{\hbar^2}{2m}\Delta'' + V_1(\vec x'')\}\, [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)]_-$ $= \delta(\vec x-\vec x')\hat\Psi(\vec x'',t)$

$+ \frac{1}{2}\int d^3x' \int d^3x''\, V_2(\vec x-\vec x')\, [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_-$

$[\hat A, \hat B\hat C]_- = [\hat A, \hat B]_-\hat C + \hat B[\hat A, \hat C]_-$

$= \{\delta(\vec x-\vec x')\hat\Psi^+(\vec x'',t) + \hat\Psi^+(\vec x',t)\delta(\vec x-\vec x'')\}\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)$

$i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t) = \{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\}\hat\Psi(\vec x,t) + \int d^3x'\, V_2(\vec x-\vec x')\hat\Psi^+(\vec x',t)\hat\Psi(\vec x',t)\hat\Psi(\vec x,t)$

non-linear integro-differential equation for $\hat\Psi(\vec x,t)$

$\to$ in general not solvable analytically

$\to$ approximative solutions

3.5 Creation and Annihilation Operators for Fermions:

second quantisation bosons: guarantee symmetry of wave functions
fermions: " anti- " " " "

basis states: $|\vec x_1,\dots,\vec x_n\rangle^{-1} = \hat a^+_{\vec x_n}\cdots\hat a^+_{\vec x_1}|0\rangle$

anti-commutation relations: $[\hat a_{\vec x}, \hat a_{\vec x'}]_+ = 0 = [\hat a^+_{\vec x}, \hat a^+_{\vec x'}]_+$, $[\hat a_{\vec x}, \hat a^+_{\vec x'}]_+ = \delta(\vec x-\vec x')$

$[\hat A, \hat B]_+ = \hat A\hat B + \hat B\hat A$, vacuum state: $\hat a_{\vec x}|0\rangle = 0$, $\langle 0|\hat a^+_{\vec x} = 0$

$\underline{n=1}$: ${}^{-1}\langle\vec x_1|\vec x_1'\rangle^{-1} = \langle\hat a^+_{\vec x_1}0|\hat a^+_{\vec x_1'}0\rangle = \langle 0|\hat a_{\vec x_1}\hat a^+_{\vec x_1'}|0\rangle = \langle 0|-\cancel{\hat a^+_{\vec x_1'}\hat a_{\vec x_1}} + \delta(\vec x_1-\vec x_1')|0\rangle = \delta(\vec x_1-\vec x_1')$

$\underline{n=2}$: ${}^{-1}\langle\vec x_1,\vec x_2|\vec x_1',\vec x_2'\rangle^{-1} = \langle\hat a^+_{\vec x_1}\hat a^+_{\vec x_2}0|\hat a^+_{\vec x_1'}\hat a^+_{\vec x_2'}0\rangle = \langle 0|\hat a_{\vec x_2}\underbrace{\hat a_{\vec x_1}\hat a^+_{\vec x_1'}}_{=-\hat a^+_{\vec x_1'}\hat a_{\vec x_1}+\delta(\vec x_1-\vec x_1')}\hat a^+_{\vec x_2'}|0\rangle$

$= \delta(\vec x_1-\vec x_1')\langle 0|\hat a_{\vec x_2}\hat a^+_{\vec x_2'}|0\rangle - \langle 0|\underbrace{\hat a_{\vec x_2}\hat a^+_{\vec x_1'}}_{=\delta(\vec x_2-\vec x_1')-\cancel{\hat a^+_{\vec x_1'}\hat a_{\vec x_2}}}\underbrace{\hat a_{\vec x_1}\hat a^+_{\vec x_2'}}_{=\delta(\vec x_1-\vec x_2')-\cancel{\hat a^+_{\vec x_2'}\hat a_{\vec x_1}}}|0\rangle$

$= \delta(\vec x_1-\vec x_1')\delta(\vec x_2-\vec x_2') - \delta(\vec x_1-\vec x_2')\delta(\vec x_2-\vec x_1') = \delta^-(\vec x_1,\vec x_2;\vec x_1',\vec x_2')$ as defined in Chapter 2

Consequence of anti-commutation relations:

$[\hat a^+_{\vec x_1}, \hat a^+_{\vec x_2}]_+ = 0 \overset{\vec x_1=\vec x_2=\vec x}{\Longrightarrow} (\hat a^+_{\vec x})^2 = 0$

$\Rightarrow |\vec x_1,\dots,\vec x_n\rangle^{-1} = 0$ if $\vec x_i = \vec x_j$ for $i\neq j$ $\Rightarrow$ anti-commutation relations automatically include Pauli exclusion principle

Hamilton operator: looks formally identical to fermions as for bosons

$\hat H = \int d^3x\, \hat a^+_{\vec x}\{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\}\hat a_{\vec x} + \frac{1}{2}\int d^3x\int d^3x'\, V_2(\vec x-\vec x')\,\hat a^+_{\vec x}\hat a^+_{\vec x'}\hat a_{\vec x'}\hat a_{\vec x}$

you can prove also here: projection to $|\vec x_1,\dots,\vec x_n\rangle^{-1}$ yield Schrödinger equation for fermionic wave function $\Psi^-(\vec x_1,\dots,\vec x_n,t) = \langle\vec x_1,\dots,\vec x_n|\Psi(t)\rangle$ from general " " : $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat H|\Psi(t)\rangle$

Schrödinger picture $\to$ Heisenberg picture:

$\hat\Psi^+(\vec x,t) = e^{\frac{i}{\hbar}\hat H t}\,\hat a^+\, e^{-\frac{i}{\hbar}\hat H t}$, $\quad \hat\Psi(\vec x,t) = e^{\frac{i}{\hbar}\hat H t}\,\hat a\, e^{-\frac{i}{\hbar}\hat H t}$

$[\hat\Psi(\vec x,t), \hat\Psi(\vec x',t)]_+ = 0 = [\hat\Psi^+(\vec x,t), \hat\Psi^+(\vec x',t)]_+$, $\quad [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)]_+ = \delta(\vec x-\vec x')$

Hamiltonian in Heisenberg picture: the same form for fermions as for bosons

$\hat H_H(t) = \int d^3x\, \hat\Psi^+(\vec x,t)\{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\}\hat\Psi(\vec x,t) + \frac{1}{2}\int d^3x\int d^3x'\, V_2(\vec x-\vec x')\hat\Psi^+(\vec x,t)\hat\Psi^+(\vec x',t)\hat\Psi(\vec x',t)\hat\Psi(\vec x,t)$

Heisenberg equations for field operators

$i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t) = [\hat\Psi(\vec x,t), \hat H_H(t)]_-$ $\leftarrow$ commutator although fermionic field operators are defined via canonical anti-commutation relations!

$= \int d^3x'\int d^3x''\, \delta(\vec x'-\vec x'')\{-\frac{\hbar^2}{2m}\Delta'' + V_1(\vec x'')\}\, [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)]_-$ ①

$+ \frac{1}{2}\int d^3x'\int d^3x''\, V_2(\vec x'-\vec x'')\, [\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_-$ ②

$[\hat A, \hat B\hat C]_- = [\hat A, \hat B]_+\hat C - \hat B[\hat A, \hat C]_+$

① $\overset{ABC}{=}$ $\underbrace{[\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)]_+}_{=\delta(\vec x-\vec x')}\hat\Psi(\vec x'',t) + \hat\Psi^+(\vec x',t)\underbrace{[\hat\Psi(\vec x,t), \hat\Psi(\vec x'',t)]_+}_{=0}$

② $\overset{ABC}{=}$ $\underline{[\hat\Psi(\vec x,t), \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)]_-}\,\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t) + \hat\Psi^+(\vec x',t)\hat\Psi^+(\vec x'',t)\,\underline{[\hat\Psi(\vec x,t), \hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)]_-}$

ABC $\quad$ ABC $\quad$ 0

$= \delta(\vec x-\vec x')\hat\Psi^+(\vec x'',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t) \pm \delta(\vec x-\vec x'')\hat\Psi^+(\vec x',t)\hat\Psi(\vec x'',t)\hat\Psi(\vec x',t)$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\hat\Psi(\vec x,t)=\left\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\right\}\hat\Psi(\vec x,t)+\int d^3x'\,V_2(\vec x-\vec x')\,\hat\Psi^+(\vec x',t)\hat\Psi(\vec x',t)\,\hat\Psi(\vec x,t)$$

$\Rightarrow$ same structure as for bosons

3.6 Occupation Number Representation:

$V_2(\vec x-\vec x')\equiv 0$ (like at the end of Chapter 2)

$$\hat H=\int d^3x\,\hat a^+_{\vec x}\left\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\right\}\hat a_{\vec x}$$

$$[\hat a_{\vec x},\hat a_{\vec x'}]_\mp=0\quad[\hat a^+_{\vec x},\hat a^+_{\vec x'}]_\mp,\quad[\hat a_{\vec x},\hat a^+_{\vec x'}]_\mp=\delta(\vec x-\vec x')$$

$$\left\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\right\}\Psi_{E_\alpha}(\vec x)=E_\alpha\Psi_{E_\alpha}(\vec x)$$

$$\int d^3x\,\Psi^*_{E_\alpha}(\vec x)\,\Psi_{E_{\alpha'}}(\vec x)=\delta_{\alpha\alpha'},\qquad\sum_\alpha\Psi^*_{E_\alpha}(\vec x)\,\Psi_{E_\alpha}(\vec x')=\delta(\vec x-\vec x')$$

Complete: $\hat a_{\vec x}=\sum_{\alpha'}\Psi_{E_{\alpha'}}(\vec x)\,\hat a_{\alpha'}$, $\hat a^+_{\vec x}=\sum_{\alpha'}\Psi^*_{E_{\alpha'}}(\vec x)\,\hat a^+_{\alpha'}$

$$\int d^3x\,\hat a_{\vec x}\Psi^*_{E_\alpha}(\vec x)=\sum_{\alpha'}\hat a_{\alpha'}\underbrace{\int d^3x\,\Psi^*_{E_\alpha}(\vec x)\Psi_{E_{\alpha'}}(\vec x)}_{=\delta_{\alpha\alpha'}}=\hat a_\alpha,\qquad\hat a^+_\alpha=\int d^3x\,\hat a^+_{\vec x}\,\Psi_{E_\alpha}(\vec x)$$

$$[\hat a_\alpha,\hat a_{\alpha'}]_\mp=0=[\hat a^+_\alpha,\hat a^+_{\alpha'}]_\mp,\qquad[\hat a_\alpha,\hat a^+_{\alpha'}]_\mp=\delta_{\alpha\alpha'}$$

$\hat a^+_\alpha$, $\hat a_\alpha$: creation (annihilation) operator for creating (annihilation) a particle in state α

$$\hat H=\int d^3x\sum_{\alpha'}\Psi^*_{E_{\alpha'}}(\vec x)\,\hat a^+_{\alpha'}\left\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\right\}\sum_\alpha\Psi_{E_\alpha}(\vec x)\,\hat a_\alpha$$

$$=\sum_\alpha\sum_{\alpha'}\hat a^+_{\alpha'}\hat a_\alpha\underbrace{\int d^3x\,\Psi^*_{E_{\alpha'}}(\vec x)\underbrace{\left\{-\frac{\hbar^2}{2m}\Delta+V_1(\vec x)\right\}\Psi_{E_\alpha}(\vec x)}_{=E_\alpha\Psi_{E_\alpha}(\vec x)}}_{=E_\alpha\delta_{\alpha\alpha'}}=\sum_\alpha E_\alpha\underbrace{\overbrace{\hat a^+_\alpha\hat a_\alpha}^{=\hat n_\alpha}}_{\text{particle number operator}}$$

= how many particles are in state α

$$[\hat A\hat B,\hat C]_-=\hat A[\hat B,\hat C]_\mp\pm[\hat A,\hat C]_\mp\hat B$$

$$[\hat n_\alpha,\hat n_{\alpha'}]_-=[\hat a^+_\alpha\hat a_\alpha,\hat a^+_{\alpha'}\hat a_{\alpha'}]_-=\;\overset{\text{A,B,C}=\text{all}}{\text{----}}\;\equiv 0$$

$$[\hat H,\hat n_\alpha]_-=\sum_{\alpha'}E_{\alpha'}[\hat n_{\alpha'},\hat n_\alpha]_-=0$$

$\hat n_\alpha|\ldots,n_\alpha,\ldots,0\rangle=n_\alpha|\ldots,n_\alpha,\ldots,0\rangle$

$\Rightarrow\hat H|\ldots,n_\alpha,\ldots,0\rangle=\sum_\alpha E_\alpha n_\alpha|\ldots,n_\alpha,\ldots,0\rangle$

bosons: $n_\alpha=0,1,2,3,\ldots$ (section 3.1)

fermions: anti-commutation relations

$$\hat n^2_\alpha=\hat a^+_\alpha\underbrace{\hat a_\alpha\hat a^+_\alpha}_{=-\hat a^+_\alpha\hat a_\alpha+1}\hat a_\alpha=\hat a^+_\alpha\hat a_\alpha-\underbrace{\hat a^+_\alpha\hat a^+_\alpha}_{=0}\underbrace{\hat a_\alpha\hat a_\alpha}_{=0}=\hat n_\alpha$$

apply: $|\ldots,n_\alpha,\ldots\rangle$: $n^2_\alpha=n_\alpha\Rightarrow n_\alpha(n_\alpha-1)=0$ $\begin{cases}n_\alpha=0\\ n_\alpha=1\end{cases}$

fermions: $n_\alpha=0,1$ (Pauli exclusion principle)

each state can at most be occupied with one fermion.

Plan for next week: Justify this "hand-on" introduction of second quantisation by canonical field quantisation.