$$\mathcal{A} = \int dt\, L\left(\psi^*(\cdot,t), \frac{\partial \psi^*(\cdot,t)}{\partial t}; \psi(\cdot,t), \frac{\partial \psi(\cdot,t)}{\partial t}\right)$$

$$L = \int d^3x\, \mathcal{L}\left(\psi^*(\vec{x},t), \vec\nabla\psi^*(\vec{x},t), \frac{\partial \psi^*(\vec x,t)}{\partial t}; \psi(\vec x,t), \vec\nabla\psi(\vec x,t), \frac{\partial\psi(\vec x,t)}{\partial t}\right)$$

$$\mathcal{L} = i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m}\vec\nabla\psi^*\cdot\vec\nabla\psi - V_1\, \psi^*\psi$$

fermionic Hamilton principle of Lagrange field theory:

$$\delta\mathcal{A} = \int dt \int d^3x \left\{ \delta\psi^*(\vec x,t)\frac{\delta\mathcal{A}}{\delta\psi^*(\vec x,t)} + \delta\psi(\vec x,t)\frac{\delta\mathcal{A}}{\delta\psi(\vec x,t)}\right\} \overset{!}{=} 0$$

$$\Rightarrow \frac{\delta\mathcal{A}}{\delta\psi^*(\vec x,t)} = 0, \quad \frac{\delta\mathcal{A}}{\delta\psi(\vec x,t)} = 0$$

like for bosons

$$\frac{\delta L}{\delta\psi^*(\vec x,t)} - \frac{\partial}{\partial t}\frac{\delta L}{\delta\frac{\partial\psi^*(\vec x,t)}{\partial t}} = 0$$

functional derivatives: like for bosons

$\Rightarrow$ Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\psi^*(\vec x,t)} - \vec\nabla\frac{\partial\mathcal{L}}{\partial\vec\nabla\psi^*(\vec x,t)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi^*(\vec x,t)}{\partial t}\right)} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi(\vec x,t)} - \vec\nabla\frac{\partial\mathcal{L}}{\partial\vec\nabla\psi(\vec x,t)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi(\vec x,t)}{\partial t}\right)} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\frac{\partial\psi}{\partial t} - V_1\psi, \quad \frac{\partial\mathcal{L}}{\partial\vec\nabla\psi^*} = -\frac{\hbar^2}{2m}\vec\nabla\psi, \quad \frac{\partial\mathcal{L}}{\partial\frac{\partial\psi^*}{\partial t}} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = +V_1\psi^*, \quad \frac{\partial\mathcal{L}}{\partial\vec\nabla\psi} = +\frac{\hbar^2}{2m}\vec\nabla\psi^*, \quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = -i\hbar\,\psi^*$$

$\Rightarrow$ Schrödinger equations for Grassmann fields

$$i\hbar\frac{\partial}{\partial t}\psi(\vec x,t) = \left\{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\right\}\psi(\vec x,t)$$

like for bosons

$$-i\hbar\frac{\partial}{\partial t}\psi^*(\vec x,t) = \left\{-\frac{\hbar^2}{2m}\Delta + V_1(\vec x)\right\}\psi^*(\vec x,t)$$

5.3 Hamilton Field Theory for Fermions:

$$\pi^*(\vec x,t) = \frac{\delta L}{\delta\frac{\partial\psi^*(\vec x,t)}{\partial t}} = \frac{\partial\mathcal{L}}{\partial\frac{\partial\psi^*(\vec x,t)}{\partial t}} = 0$$

$$\pi(\vec x,t) = \frac{\delta L}{\delta\frac{\partial\psi(\vec x,t)}{\partial t}} = \frac{\partial\mathcal{L}}{\partial\frac{\partial\psi(\vec x,t)}{\partial t}} = -i\hbar\,\psi^*$$

Legendre transformation:

$$L = \int d^3x\left\{\underbrace{\frac{\partial\psi^*(\vec x,t)}{\partial t}\pi^*(\vec x,t)}_{=0} + \frac{\partial\psi(\vec x,t)}{\partial t}\pi(\vec x,t)\right\} - H[\pi(\cdot,t);\psi(\cdot,t)]$$

order consistent with definition of canonical momentum

$$H = \int d^3x\left\{\underbrace{\frac{\partial\psi(\vec x,t)}{\partial t}\pi(\vec x,t) + \underbrace{(-i\hbar\,\psi^*(\vec x,t))}_{=\pi(\vec x,t)}\frac{\partial\psi(\vec x,t)}{\partial t}}_{=0} + \frac{\hbar^2}{2m}\frac{1}{-i\hbar}\vec\nabla\pi\cdot\vec\nabla\psi + \frac{V_1}{-i\hbar}\pi\,\psi\right\}$$

$$H = \int d^3x\,\mathcal{H}\left(\pi(\vec x,t), \vec\nabla\pi(\vec x,t); \psi(\vec x,t), \vec\nabla\psi(\vec x,t)\right)$$

$$\mathcal{H} = -\frac{\hbar}{2mi}\vec\nabla\pi\,\vec\nabla\psi - \frac{V_1}{i\hbar}\pi\cdot\psi$$

Hamilton principle of classical field theory in Hamilton formulation:

$$0 = \delta\mathcal{A} = \int dt\int d^3x\left\{\delta\pi(\vec x,t)\underbrace{\frac{\delta\mathcal{A}}{\delta\pi(\vec x,t)}}_{=0} + \delta\psi(\vec x,t)\underbrace{\frac{\delta\mathcal{A}}{\delta\psi(\vec x,t)}}_{=0}\right\}$$

$\Rightarrow$

$$\mathcal{A} = \int dt\int d^3x\,\frac{\partial\psi(\vec x,t)}{\partial t}\pi(\vec x,t) - H[\pi(\cdot,t);\psi(\cdot,t)]$$

$$\frac{\delta A}{\delta \pi(\vec{x},t)} = -\frac{\partial \psi(\vec{x},t)}{\partial t} - \frac{\delta H}{\delta \pi(\vec{x},t)} = 0, \quad \frac{\delta A}{\delta \psi(\vec{x},t)} = -\frac{\partial \pi(\vec{x},t)}{\partial t} - \frac{\delta H}{\delta \psi(\vec{x},t)} = 0$$

$$\frac{\partial \psi(\vec{x},t)}{\partial t} = -\frac{\partial \mathcal{L}}{\partial \pi(\vec{x},t)} + \vec{\nabla}\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\pi(\vec{x},t)} \quad (3); \quad \frac{\partial \pi(\vec{x},t)}{\partial t} = -\frac{\partial \mathcal{L}}{\partial \psi(\vec{x},t)} - \vec{\nabla}\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\psi(\vec{x},t)} \quad (4)$$

$\Rightarrow$ Hamilton equations for Grassmann Hamilton field theory

$$\frac{\partial \mathcal{L}}{\partial \pi} = -\frac{V_1}{i\hbar}\psi, \quad \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\pi} = -\frac{\hbar}{2mi}\vec{\nabla}\psi \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{V_1}{i\hbar}\pi, \quad \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\psi} = \frac{\hbar}{2mi}\vec{\nabla}\pi \quad (2)$$

$\Rightarrow$ Insert (1), (2) in (3), (4): Schrödinger equations!

5.4 Poisson Brackets:

$F = F[\pi(\cdot,\cdot);\psi(\cdot,\cdot)]$, $G = G[\pi(\cdot,\cdot);\psi(\cdot,\cdot)]$

$$\{F,G\}_+ = (-1)^{\pi(F)}\int d^3x\left(\frac{\delta F}{\delta\psi(\vec{x},t)}\frac{\delta G}{\delta\pi(\vec{x},t)} + \frac{\delta F}{\delta\pi(\vec{x},t)}\frac{\delta G}{\delta\psi(\vec{x},t)}\right)$$

$\pi(F)$: parity of F; examples: $\pi(\psi)=1$, $\pi(H)=0$

symmetry of Poisson brackets: 3 cases

1. case: $\pi(F)=\pi(G)=0$

$$\{F,G\}_+ = \int d^3x\left(\underbrace{\frac{\delta F}{\delta\psi(\vec{x},t)}}_{\pi=1}\underbrace{\frac{\delta G}{\delta\pi(\vec{x},t)}}_{\pi=1} + \underbrace{\frac{\delta F}{\delta\pi(\vec{x},t)}}_{\pi=1}\underbrace{\frac{\delta G}{\delta\psi(\vec{x},t)}}_{\pi=1}\right)$$

$$= -\int d^3x\left(\frac{\delta G}{\delta\psi(\vec{x},t)}\frac{\delta F}{\delta\pi(\vec{x},t)} + \frac{\delta G}{\delta\pi(\vec{x},t)}\frac{\delta F}{\delta\psi(\vec{x},t)}\right) = -\{G,F\}_+$$

2. case: $\pi(F)=0$, $\pi(G)=1$

$$\{F,G\}_+ = \int d^3x\left(\underbrace{\frac{\delta F}{\delta\psi(\vec{x},t)}}_{\pi=1}\underbrace{\frac{\delta G}{\delta\pi(\vec{x},t)}}_{\pi=0} + \underbrace{\frac{\delta F}{\delta\pi(\vec{x},t)}}_{\pi=1}\underbrace{\frac{\delta G}{\delta\psi(\vec{x},t)}}_{\pi=0}\right)$$

$$= \int d^3x\left(\frac{\delta G}{\delta\psi(\vec{x},t)}\frac{\delta F}{\delta\pi(\vec{x},t)} + \frac{\delta G}{\delta\pi(\vec{x},t)}\frac{\delta F}{\delta\psi(\vec{x},t)}\right) = -\{G,F\}_+$$

analogue: $\pi(F)=1$, $\pi(G)=0$

Poisson brackets anti-symmetric

3. case: $\pi(F)=\pi(G)=1$

$$\{F,G\}_+ = -\int d^3x\left(\underbrace{\frac{\delta F}{\delta\psi(\vec{x},t)}}_{\pi=0}\underbrace{\frac{\delta G}{\delta\pi(\vec{x},t)}}_{\pi=0} + \underbrace{\frac{\delta F}{\delta\pi(\vec{x},t)}}_{\pi=0}\underbrace{\frac{\delta G}{\delta\psi(\vec{x},t)}}_{\pi=0}\right)$$

$$= -\int d^3x\left(\frac{\delta G}{\delta\psi(\vec{x},t)}\frac{\delta F}{\delta\pi(\vec{x},t)} + \frac{\delta G}{\delta\pi(\vec{x},t)}\frac{\delta F}{\delta\psi(\vec{x},t)}\right) = +\{G,F\}_+$$

Poisson bracket symmetric

Hamilton equations:

$$\{\psi(\vec{x},t), H\}_+ = -\frac{\delta H}{\delta\pi(\vec{x},t)} = \frac{\partial\psi(\vec{x},t)}{\partial t}$$

$$\{\pi(\vec{x},t), H\}_+ = -\frac{\delta H}{\delta\psi(\vec{x},t)} = \frac{\partial\pi(\vec{x},t)}{\partial t}$$

Time dependence of a Grassmann functional:

$$\frac{\partial F}{\partial t} = \int d^3x\left(\underbrace{\frac{\partial\psi(\vec{x},t)}{\partial t}}_{=-\underbrace{\frac{\delta H}{\delta\pi(\vec{x},t)}}_{\pi=1}}\frac{\delta F}{\delta\psi(\vec{x},t)} + \underbrace{\frac{\partial\pi(\vec{x},t)}{\partial t}}_{=-\underbrace{\frac{\delta H}{\delta\psi(\vec{x},t)}}_{\pi=1}}\frac{\delta F}{\delta\pi(\vec{x},t)}\right)$$

$$= (-1)^{\pi(F)}\int d^3x\left(\frac{\delta F}{\delta\psi(\vec{x},t)}\frac{\delta H}{\delta\pi(\vec{x},t)} + \frac{\delta F}{\delta\pi(\vec{x},t)}\frac{\delta H}{\delta\psi(\vec{x},t)}\right) = \{F,H\}_+$$

Consider two cases $\pi(F)=0$, $\pi(F)=1$

fundamental Poisson brackets (equal time):

$$\{\psi(\vec{x},t), \psi(\vec{x}',t)\}_+ = 0 = \{\pi(\vec{x},t), \pi(\vec{x}',t)\}_+, \quad \{\psi(\vec{x},t), \pi(\vec{x}',t)\}_+ = -\delta(\vec{x}-\vec{x}')$$