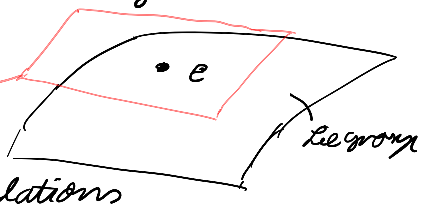
Part II: Relativistic Fields and Their Quantization

6 Poincaré Group:

Motivation:

- Special relativity: space and time in absence of gravity ⇒ flat Minkowskian space-time
- Group symmetry which leaves distances in Minkowski-space-time invariant: Poincaré group
- Lie group:
  - unifies mathematical structures of groups and manifolds
  - group elements depend continuously and differentially on certain parameters
- Poincaré group = 10-parameter, non-abelian Lie group containing rotations, boosts between inertial systems and translations in space and time
- Lie algebra: tangent plane of Lie group at identity element

tangent plane at identity element e = Lie algebra



- Poincaré algebra: generators of rotations, boosts, and translations
- Lie theorem: "Lie group $\hat{=}$ exp{ Lie algebra }"
- Casimir operators:
  - operators, which commute with all elements of the Lie algebra (Poincaré algebra)
  - eigenvalues of Casimir operators characterize irreducible representations of Poincaré group (independent)

    ⟶ mass M, spin S: elementary particles
  - each particles belongs to such an irreducible representation of Poincaré group

⇒ backbone of QFT

(Greiner, Müller: Volume V, Quantum Mechanics – Symmetries)

6.1 Special Relativity:

- Formulated by Albert Einstein in 1905: space-time in absence of gravity
- Based on two postulates:

  E1: The velocity of light is the same in all inertial systems.

  E2: The fundamental laws of physics have the same form in all inertial systems.
- Concrete physical consequences for fast moving particles, e.g. in Large Hadron Collider (LHC) at Cern;

  example: time dilatation
- But: special relativity unifies the description of space and time
- E2 ⇒ contravariant space-time four vector

$$(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, x^i) = (ct, \vec{x})$$

Latin index: $\mu = 0, 1, 2, 3$; Greek index: $i = 1, 2, 3$

- E1 ⇒ light-ray in two inertial systems $\quad (ct)^2 - \vec{x}^2 = (ct')^2 - \vec{x}'^2$
- Covariant Minkowski metric

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \Rightarrow$$

invariance of scalar product

$$g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x'^\mu x'^\nu$$

Einstein summation convention $\quad \sum_{\mu=0}^{3} \sum_{\nu=0}^{3}$

- Covariant space-time four-vector:

$$x_\mu = g_{\mu\nu} x^\nu$$

contraction

⇒ pulldown of index ⇒ $x_\mu x^\mu = x'^\mu x'_\mu$

- Identity:

$$g_{\mu\nu} \delta^\nu{}_\lambda = g_{\mu\lambda}$$

$$(\delta^\nu{}_\varkappa) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = (g^\nu{}_\varkappa) \qquad (g^{\mu\nu}) = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

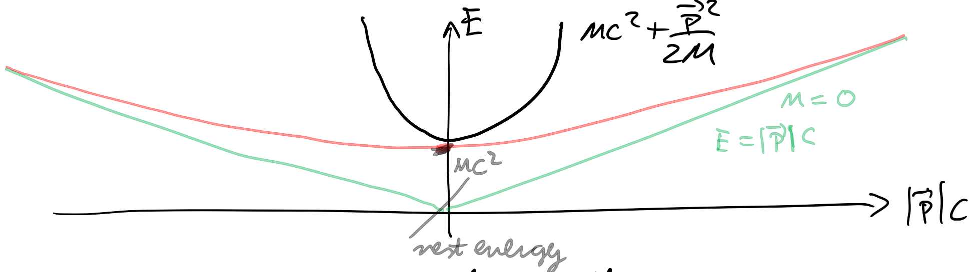
- contravariant Minkowski metric:

$$g^{\mu\nu} g_{\nu\varkappa} = \delta^\mu{}_\varkappa = g^\mu{}_\varkappa$$

$$g^{\mu\nu} x_\nu = g^{\mu\nu} g_{\nu\varkappa} x^\varkappa = g^\mu{}_\varkappa x^\varkappa = \delta^\mu{}_\varkappa x^\varkappa = x^\mu$$ pulls up indese

- concept of four-vectors is more general than space-time four vectors
- Another example: relativistic energy-momentum dispersion

$$\left.\begin{array}{l} E^2 = M^2c^4 + \vec{p}^2c^2 , \quad E'^2 = M'^2c^4 + \vec{p}'^2c^2 \\ \text{rest mass: } M = M' \text{ (Lorentz scalar)} \end{array}\right\} \boxed{\left(\frac{E}{c}\right)^2 - \vec{p}^2 = \left(\frac{E'}{c}\right)^2 - \vec{p}'^2}$$



$$E = \sqrt{M^2c^4 + |\vec{p}|^2c^2} = Mc^2\sqrt{1 + \frac{\vec{p}^2c^2}{M^2c^4}}$$

$|\vec{p}| \ll Mc$

$$\approx Mc^2\left\{1 + \frac{1}{2}\frac{\vec{p}^2c^2}{M^2c^4} + \dots\right\} = \underbrace{Mc^2 + \frac{\vec{p}^2}{2M}}_{} + \dots$$

non-relativistic energy-momentum dispersion

- Contravariant momentum four vector:

$$(p^\mu) = (p^0, p^1, p^2, p^3) = \left(\frac{E}{c}, p^i\right) = \left(\frac{E}{c}, \vec{p}\right) \Rightarrow p^\mu p^\nu g_{\mu\nu} = g_{\mu\nu} p'^\mu p'^\nu$$

- covariant momentum four vector:

$$(p_\mu) = (p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -p^i\right) = \left(\frac{E}{c}, -\vec{p}\right)$$

6.2 Defining Representation of Lorentz group:

- linear coordinate transformation between two inertial systems: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$
- Invariance:

$$g_{\mu\nu}\underline{x^\mu x^\nu} = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}(\Lambda^\mu{}_\sigma x^\sigma)(\Lambda^\nu{}_\rho x^\rho) = g_{\mu\nu}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho x^\sigma x^\rho$$

$$= g_{\sigma\rho}\Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu x^\mu x^\nu$$ holds for any x^μ, x^ν

matrix multiplication

$$\Rightarrow g_{\mu\nu} = \Lambda^\sigma{}_\mu\Lambda^\rho{}_\nu g_{\sigma\rho} = \underline{\Lambda^\sigma{}_\mu}\, g_{\sigma\rho}\,\Lambda^\rho{}_\nu = (\Lambda^T)_\mu{}^\sigma g_{\sigma\rho}\Lambda^\rho{}_\nu$$

$$\boxed{g = \Lambda^T g \Lambda} \quad (*)$$

after transformation again Minkowski metric

g is tensor of 2nd rank ⇒ needs two Λ's

- Transposed matrix:

$$\underline{(\Lambda^T)_\mu{}^\sigma} = g_{\mu\varkappa}(\Lambda^T)^{\varkappa\sigma} = g_{\mu\varkappa}\Lambda^{\sigma\varkappa} = \underline{\Lambda^\sigma{}_\mu}$$

⇒ defining relation for Lorentz transformations

- group axioms are fulfilled for all Λ in $\mathcal{L}$

1. Closedness: $\Lambda_1, \Lambda_2 \in \mathcal{L}$
$$(\Lambda_1\Lambda_2)^T g(\Lambda_1\Lambda_2) = \Lambda_2^T(\Lambda_1^T g\Lambda_1)\Lambda_2 \overset{(*)}{=} \Lambda_2^T g\Lambda_2 \overset{(*)}{=} g \Rightarrow \Lambda_1\Lambda_2 \in \mathcal{L}$$
2. Associativity: $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{L}$
$$\left.\begin{array}{l}[(\Lambda_1\Lambda_2)\Lambda_3]^T g[(\Lambda_1\Lambda_2)\Lambda_3] = \Lambda_3^T\Lambda_2^T\Lambda_1^T g\Lambda_1\Lambda_2\Lambda_3 = g \\ [\Lambda_1(\Lambda_2\Lambda_3)]^T g[\Lambda_1(\Lambda_2\Lambda_3)] \checkmark \end{array}\right\} (\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$$
3. Identity element: $\Lambda_e = I = (g^\nu{}_\varkappa) \Rightarrow \Lambda_e^T g\Lambda_e = g \Rightarrow \Lambda_e \in \mathcal{L}$
$\Lambda_e\Lambda = \Lambda\Lambda_e = \Lambda$ for $\Lambda \in \mathcal{L}$
4. Inverse element: $\Lambda \in \mathcal{L} \Rightarrow \Lambda^{-1} \in \mathcal{L}$
det (*): $-1 = \det g = \det(\Lambda^T g\Lambda) = (\det\Lambda^T)(\det g)\det(\Lambda) = (\det\Lambda)^2\det g \Rightarrow (\det\Lambda)^2 = 1$
$\Rightarrow \det\Lambda \neq 0$
$$g \overset{(*)}{=} \Lambda^T g\Lambda \Rightarrow \underbrace{(\Lambda^T)^{-1}}_{=(\Lambda^{-1})^T} g\Lambda^{-1} = g \Rightarrow \Lambda^{-1} \in \mathcal{L}$$

Set of all Lorentz transformations = Lorentz group = pseudo-orthogonal group $O(1,3)$