$$\phi'(x^\mu) = \phi\left(\left(g^\mu{}_\nu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu \omega_{\alpha\beta}\right)x^\nu\right) = \phi\left(x^\mu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu \omega_{\alpha\beta} x^\nu\right)$$

small as $\omega_{\alpha\beta}$ are small

$$= \left\{ 1 + \frac{i}{2}\omega_{\alpha\beta}(L^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu \right\} \phi(x^\mu)$$

$$= -\frac{i}{2}\omega_{\alpha\beta}\hat{L}^{\alpha\beta} \quad \text{with} \quad \hat{L}^{\alpha\beta} = -(L^{\alpha\beta})^\mu{}_\nu x^\nu \partial_\mu$$

another representation of Lorentz algebra; defining representation of Lorentz algebra

$$(L^{\alpha\beta})^\mu{}_\nu = i(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} g^\alpha{}_\nu)$$

$$\hat{L}^{\alpha\beta} = -i(g^{\alpha\mu} g^\beta{}_\nu - g^{\beta\mu} \underbrace{g^\alpha{}_\nu}_{=\delta^\alpha{}_\nu}) x^\nu \partial_\mu = i(x^\alpha \partial^\beta - x^\beta \partial^\alpha)$$

four-momentum operator: $\hat{p}^\alpha = i\hbar \partial^\alpha$

dimensionless orbital angular momentum operator: $\hat{L}^{\alpha\beta} = \frac{1}{\hbar}(x^\alpha \hat{p}^\beta - x^\beta \hat{p}^\alpha)$

$$[\hat{p}^\alpha, x^\beta]_- = i\hbar[\partial^\alpha, x^\beta]_- = i\hbar(\partial^\alpha x^\beta) = i\hbar g^{\alpha\gamma}(\partial_\gamma x^\beta) = i\hbar g^{\alpha\beta}$$

$$\partial_\gamma x^\beta = \frac{\partial}{\partial x^\gamma} x^\beta = \delta_\gamma{}^\beta$$

commutator relations:

$$[\hat{L}^{\alpha\beta}, x^\gamma]_- = \dots = -(L^{\alpha\beta})^\gamma{}_\delta x^\delta$$

$$[\hat{L}^{\alpha\beta}, \hat{p}^\gamma]_- = \dots = -(L^{\alpha\beta})^\gamma{}_\delta \hat{p}^\delta$$

$x^\gamma, \hat{p}^\gamma$: "vector operators" ← special case

$\hat{O}^{\lambda_1 \dots \lambda_n}$: tensor operator of rank n

$$[\hat{L}^{\mu\nu}, \hat{O}^{\lambda_1 \dots \lambda_n}]_- = -\sum_{k=1}^{n} (L^{\mu\nu})^{\lambda_k}{}_\varkappa \hat{O}^{\lambda_1 \dots \lambda_{k-1} \varkappa \lambda_{k+1} \dots \lambda_n}$$

check for structure constants of Lorentz algebra: ✓

$$[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}]_- = i(g^{\alpha\delta}\hat{L}^{\beta\gamma} + g^{\beta\gamma}\hat{L}^{\alpha\delta} - g^{\alpha\gamma}\hat{L}^{\beta\delta} - g^{\beta\delta}\hat{L}^{\alpha\gamma})$$

→ $\hat{L}^{\alpha\beta}$: representation of Lorentz algebra in the Hilbert space of scalar fields

rewrite:

$$[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}]_- = -(L^{\alpha\beta})^\gamma{}_\sigma \hat{L}^{\sigma\delta} - (L^{\alpha\beta})^\delta{}_\sigma \hat{L}^{\gamma\sigma}$$

→ angular momentum operator a tensor operator of rank $n=2$

6.9 Tensor/Spinor Field Representation:

Aim: tensor/spinor field $\psi^\sigma(x^\mu)$ — σ: tensor/spinor components; x^μ: four-vector of space time

example: four-vector potential $A^\sigma(x^\mu)$

transformation properties: $A'^\sigma(x'^\mu) = \Lambda^\sigma{}_\tau A^\tau(x^\mu)$, with $x^\mu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$ — notation: neglect $'$

Λ due to "vector" property of A

$$A'^\sigma(x^\mu) = \Lambda^\sigma{}_\tau A^\tau((\Lambda^{-1})^\mu{}_\nu x^\nu)$$

$$\Lambda^\sigma{}_\tau = g^\sigma{}_\tau - \frac{i}{2}(L^{\alpha\beta})^\sigma{}_\tau \omega_{\alpha\beta}, \qquad (\Lambda^{-1})^\mu{}_\nu = g^\mu{}_\nu + \frac{i}{2}(L^{\alpha\beta})^\mu{}_\nu \omega_{\alpha\beta}$$

expand up to first order in ω_α:

$$A'^\sigma(x^\mu) = \left\{ g^\sigma{}_\tau - \frac{i}{2}\omega_{\alpha\beta} \cdot (\hat{M}^{\alpha\beta})^\sigma{}_\tau \right\} A^\tau(x^\mu)$$

total angular momentum

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + L^{\alpha\beta}$$

orbital angular momentum operator as in scalar case — comes from representation of Lorentz algebra in space time

spin angular momentum operator — comes from representation of Lorentz algebra in vector space

$$[\hat{L}^{\alpha\beta}, L^{\alpha\beta}]_- = 0 \Rightarrow \text{independence of both representations}$$

$\hat{M}^{\alpha\beta}$ is representation of Lorentz algebra:

$$[\hat{M}^{\alpha\beta}, \hat{M}^{\gamma\delta}]_- = i(g^{\alpha\delta}\hat{M}^{\beta\gamma} + g^{\beta\gamma}\hat{M}^{\alpha\delta} - g^{\alpha\gamma}\hat{M}^{\beta\delta} - g^{\beta\delta}\hat{M}^{\alpha\gamma})$$

Now back to general case: $\psi^\sigma(x^\mu)$

$$\psi'^\sigma(x^\mu) = \left\{ g^\sigma{}_\tau - \frac{i}{2}\omega_{\alpha\beta}(\hat{M}^{\alpha\beta})^\sigma{}_\tau \right\} \psi^\tau(x^\mu)$$

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + N^{\alpha\beta}$$

$N^{\alpha\beta}$: representation of Lorentz algebra in tensorial / spinorial component space

$$[N^{\alpha\beta}, N^{\gamma\delta}]_- = i\left(g^{\alpha\delta}N^{\beta\gamma} + g^{\beta\gamma}N^{\alpha\delta} - g^{\alpha\gamma}N^{\beta\delta} - g^{\beta\delta}N^{\alpha\gamma}\right)$$

$$[\hat{L}^{\alpha\beta}, N^{\gamma\delta}]_- = 0$$

$\Rightarrow \hat{M}^{\alpha\beta}$ is a representation of Lorentz algebra

6.10 Defining Representation of Poincaré Group:

Poincaré transformation: Lorentz transformation Λ + shift a

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \rightarrow \text{("inhomogeneous")}$$

Lorentz transformation: scalar product of a four-vector is invariant

Poincaré ": leaves distances between four-vectors invariant

$$g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - x^\nu) = g_{\mu\nu}(x'^\mu - y'^\mu)(x'^\nu - y'^\nu)$$

Set $\mathcal{P}$ of all Poincaré transformations: group

(Λ, a): element of $\mathcal{P}$

- Closedness: $(\Lambda_1, a_1), (\Lambda_2, a_2) \in \mathcal{P}$

$$x_2^\mu = \Lambda_2{}^\mu{}_\nu \underbrace{x_1^\nu}_{\Lambda^\nu{}_\varkappa x^\varkappa + a_1^\nu} + a_2^\mu = \underbrace{\Lambda^\mu{}_2{}_\nu \Lambda_1{}^\nu{}_\varkappa}_{=\Lambda^\mu{}_\varkappa} x^\varkappa + \underbrace{\Lambda_2^\mu{}_\nu a^\nu_1 + a_2^\mu}_{=a^\mu}$$

short-hand notation: $(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \in \mathcal{P}$

semi-direct product of Lorentz group $\mathcal{L}$ and translation group $\mathcal{T}$

- associativity: $(\Lambda_1, a_1), (\Lambda_2, a_2), (\Lambda_3, a_3) \in \mathcal{P}$

$$(\Lambda_1, a_1)\left((\Lambda_2, a_2)(\Lambda_3, a_3)\right) = \left((\Lambda_1, a_1)(\Lambda_2, a_2)\right)(\Lambda_3, a_3)$$

- unity element of $\mathcal{P}$: $(\Lambda_e, a_e) = (I, 0)$

$(\Lambda, a) \in \mathcal{P}$: $(I, 0)(\Lambda, a) = (\Lambda, a) = (\Lambda, a)(I, 0)$

- inverse element of $\mathcal{P}$: $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$

$$(\Lambda, a)^{-1}(\Lambda, a) = (I, 0) = (\Lambda_e, a_e)$$

Remark: $\mathcal{P} = \underbrace{\mathcal{P}_1}_{\text{restricted ourselves}} \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \mathcal{P}_4$