and from $[N^{\alpha\beta}, N^{\gamma\delta}]_- = \ldots \Rightarrow \quad [S_i, S_j]_- = i\,\varepsilon_{ijk} S_k$

Pauli-Lubanski vector represents in rest frame the ~~spin~~ angular momentum of the particle

$\Rightarrow \hat{w}^\alpha$ is a relativistic generalisation of the particle spin in any inertial system

6.13 Irreducible representations of Poincaré group:

"independent"

Eigenvalues of both Casimir operators of Poincaré algebra they allow to classify the irreducible representations of Poincaré group

Note: They are infinite-dimensional as momentum can be arbitrarily large ($\Leftrightarrow$ defining representation of Lorentz group which was finite-dimensional

1. Casimir operator: $\hat{p}^2 = g_{\alpha\beta}\hat{p}^\alpha \hat{p}^\beta \Rightarrow$ eigenvalue $p^2 = g_{\alpha\beta} p^\alpha p^\beta = m^2 c^2$

$\Rightarrow$ two different classes of representations depending on whether $m \neq 0$ or $m = 0$

6.13.1 Massive Representations ($m > 0$):

2. Casimir operator: $\hat{w}^2 = g_{\alpha\beta}\hat{w}^\alpha \hat{w}^\beta$ eigenvalue $w^2 = g_{\alpha\beta} w^\alpha w^\beta$ is Lorentz scalar, i.e. has the same value in all inertial system

rest frame: $w^0 = 0, \; \vec{w} = mc\vec{S} \Rightarrow w^2 = (w^0)^2 - \vec{w}^2 = -mc\, S(S+1)$

with spin quantum numbers $S = 0, 1/2, 1, 3/2, 2, \ldots$ due to $[S_i, S_j]_- = i\varepsilon_{ijk} S_k$

Massive representations are classified with respect $m > 0$ and spin S, they are the fundamental properties of elementary particles

Elementary particles $\hat{=}$ irreducible representations of Poincaré group

6.13.2 Massless Representations ($m = 0$):

$m = 0$: particle does not have a rest frame, i.e. there is no Lorentz transformation to get the particle into its rest frame, the particle moves with light velocity

If the rest frame could be reached, then we would have $p^0 = 0, \vec{p} = \vec{0}$: particle would have no energy and no momentum

$\Rightarrow$ massless particles need a different treatment

Four-vectors p^α and w^α have the following properties

1) $\hat{w}_\alpha \hat{p}^\alpha = 0 \Rightarrow w_\alpha p^\alpha = 0$, p^α and w_α are orthogonal to each other

2) $p^2 = m^2 c^2 = 0 \Rightarrow p_\alpha p^\alpha = 0$

$w^2 = -mc\,S(S+1) = 0 \Rightarrow w_\alpha w^\alpha = 0$

light-like vectors p^α, w^α

$(p^0)^2 = \vec{p}^2$; $(w^0)^2 = \vec{w}^2$

$p^\alpha \neq 0 \Rightarrow p^0 \neq 0$; $w_\alpha \neq 0 \Rightarrow w^0 \neq 0$

linear combination: $A p^\alpha + B w^\alpha = 0$

$\alpha = 0$: $B = -\frac{p^0}{w^0} A$, as $p^0 \neq 0, w^0 \neq 0$

$\Rightarrow p^\alpha, w^\alpha$ are linear dependent $\Rightarrow \hat{p}^\alpha, \hat{w}^\alpha$ are linear dependent

$\Rightarrow \hat{w}^\alpha = \hat{h}\,\hat{p}^\alpha$

"proportionality factor operator"

Check whether $\hat{h}$ is a Casimir operator:

$\underbrace{[\hat{w}^\alpha, \hat{p}^\beta]_-}_{=0 \text{ see above}} = [\hat{h}\hat{p}^\alpha, \hat{p}^\beta]_- = \hat{h}\underbrace{[\hat{p}^\alpha, \hat{p}^\beta]_-}_{=0} + [\hat{h}, \hat{p}^\beta]_- \hat{p}^\alpha \Rightarrow [\hat{h}, \hat{p}^\alpha]_- = 0$

$\underbrace{[\hat{M}^{\alpha\beta}, \hat{w}^\gamma]_-}_{= i(g^{\beta\gamma}\hat{w}^\alpha - g^{\alpha\gamma}\hat{w}^\beta)} = [\hat{M}^{\alpha\beta}, \hat{h}\hat{p}^\gamma]_- = [\hat{M}^{\alpha\beta}, \hat{h}]_- \hat{p}^\gamma + \hat{h}\,[\hat{M}^{\alpha\beta}, \hat{p}^\gamma]_- \Rightarrow [\hat{M}^{\alpha\beta}, \hat{h}]_- = 0$

$\uparrow$ see above

$= i(g^{\beta\gamma}\underbrace{\hat{p}^\alpha}_{w^\alpha} - g^{\alpha\gamma}\underbrace{\hat{p}^\beta}_{w^\beta})$

$\uparrow$ see above

$\Rightarrow \hat{h}$ is a Casimir operator

eigenvalue: $w^\alpha = h p^\alpha \quad \xrightarrow{\alpha=0} \quad h = \frac{w^0}{p^0}$

$w_0 = w^0 = \vec{p}\cdot\vec{S}, \quad \underbrace{p^0 = |\vec{p}|}_{p^\alpha \text{ light-like}} \quad \Rightarrow \quad h = \frac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$

h: projection of particle spin upon the direction of motion (either pos./neg. helicity)

$\Rightarrow \hat{h}$: helicity operator

Note: Helicity can also be introduced for massive particles. But then it is no Casimir operator. For massive particles there could be a Lorentz transformation to transform the particle in a state with pos. helicity into a state with neg. helicity. Thus for massive particles the helicity characterizes the property of a state of an elementary particle but not an elementary particle itself. This is only the case for massless particles.

6.13.3 Other Representations:

mathematically also other unitary representations of the Poincaré group exist:

- $p_\mu p^\mu = 0$ and continuous spin
- $p_\mu p^\mu < 0$ for particles moving with a velocity larger than light (tachyons)

physically no experiment so far indicates the existence of these other representations of Poincaré group in nature

speculation: maybe dark matter is given by other representations of Poincaré group