Chapter 1

Introduction

This lecture provides a hands-on insight into quantum electrodynamics, which represents an important building block of the standard model of elementary particles physics. To this end we proceed in three steps. At first, we introduce the concept of second quantization, which allows to deal with an arbitrary number of quantum particles, by the example of non-relativistic many-body theory. Then we discuss the relativistic wave equations as representations of the Poincaré symmetry of space-time. And, ultimately, we work out how to perturbatively calculate scattering cross sections of fundamental quantum electrodynamic processes by using the technique of Feynman diagrams.

1.1 Standard Model

The standard model of elementary particle physics describes quite successfully the basic structure of matter and three of the overall four fundamental interactions. All its predictions agree precisely with all experimental measurements performed so far within the respective error bars. The basic concept of the standard model is local gauge invariance. This means that the physics does not change provided that the particle wave functions acquire local phase factors, which change continuously from space-time point to space-time point. This represents a quite hard restriction. Free massive particles do not fulfill this condition, as their wave functions only allow for a global change of the phase factors. But if we postulate to have in addition for massive particles also an invariance with respect to a local change of their phase factors, we can deduce how these massive particles interact. Within such a local gauge theory it turns out that the interaction between the massive particles is mediated by an exchange of gauge bosons, which represent the quantized excitations of the corresponding gauge fields. In this way the three interactions of the standard model can be classified as is summarized in Tab. 1.1.

The first and by far most successful theory of fundamental interactions is quantum electrodynamics. Its $U(1)$ gauge theory was later on extended to the description of the other two interactions of the standard model, which lead to the electroweak theory, unifying both the

interaction	gauge symmetry	gauge bosons
electromagnetic	$U(1)$	photon
weak	$U(2)$	intermediate vector bosons
strong	$SU(3)$	gluon

Table 1.1: Overview of the three types of interactions of the standard model of elementary particle physics together with their gauge symmetries and gauge bosons.

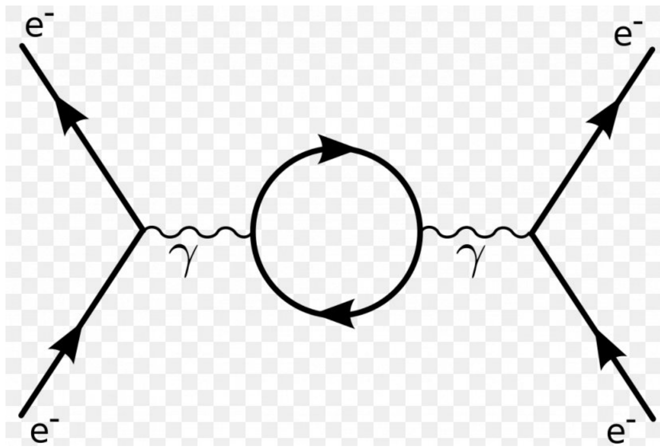

Figure 1.1: Due to the presence of the vacuum the scattering of two electrons also involves the creation and annihilation of virtual electrons and positrons.

eletrodynamic and weak interactions, as well as quantum chromodynamics, the quantum theory of the strong interaction. Furthermore, quantum electrodynamics is the theory in all natural sciences, whose predictions agree most precisely with experimental results. According to a comparison of Richard Feynman, its precision of 10 orders of magnitude corresponds to a resolution, where the thickness of a single hair is resolvable by looking from the West to the East Coast of America.

1.2 Non-Relativistic Quantum Many-Body Theory

Within a quantum electrodynamic scattering process not only real particles are involved. The physical vacuum is not empty but, instead, consists of a sea of virtual particles, which are also involved in a scattering process, see Fig. 1.1. Therefore, it is necessary to work out a quantum mechanical formalism which is capable of describing an arbitrary number of particles. The formalism of *first quantization* is not appropriate for that as there the number of particles remains conserved. With the first quantization it is possible to calculate, for instance, for the hydrogen atom the stationary energy states and the respective transition probabilities between them. But the fundamental processes of the absorption of a photon and the corresponding excitation of an electron as well as the later relaxation of the electron to the ground state

bosons	fermions
integer spin	half-integer spin
particles mediating forces	matter particles
Bose-Einstein statistics	Fermi-Dirac statistics
symmetric many-body wave function	anti-symmetric many-body wave function

Table 1.2: According to the spin-statistic theorem there exist with bosons and fermions two kinds of indistinguishable particles.

spin	0	1/2	1	3/2	2
mass > 0	Higgs	leptons, quarks	intermediate vector bosons	Δ resonances	
mass $= 0$			photon, gluon		graviton

Table 1.3: Classification of elementary particles according to their spin and mass.

and the corresponding emission of a photon are not describable within the first quantization formalism as they violate the particle number conservation.

The description of identical particles, which have exactly the same physical properties as, for instance, mass, spin and charge, turns out to be problematic in the realm of quantum mechanics. In classical mechanics identical particles are distinguishable so that the trajectory of each particle can always be identified. All experiments suggest, however, that this principle of distiguishability can no longer be maintained in quantum mechanics. Due to the Heisenberg uncertainty relation the probability densities of identical particles overlap so that the identification of a single particle is not possible. Despite of this fundamental principle of indistinguishability of identical particles in quantum mechanics one is nevertheless forced, due to calculational purposes, to enumerate the particles. But this artificial particle enumeration has to be performed in such a way that physically observable results turn out to be invariant with respect to any change of this particle labeling. From this definition of indistinguishability then follows that a many-particle wave function must obey special symmetry requirements. To this end Wolfgang Pauli derived 1940 the spin-statistic theorem of relativistic quantum field theory. By unifying the basic principles of special relativity with those of quantum mechanics he showed that there are in three dimensions exactly two kinds of indistinguishable identical particles, namely bosons and fermions. Their respective properties are summarized in Tab. 1.2.

It turns out that concrete calculations with (anti-)symmetric many-body wave functions are quite cumbersome. Therefore, one has worked out a quite elegant formalism for quantum many-body systems, which is capable of dealing with an arbitrary number of particles and is called *second quantization*. In Part I of the lecture we work out the so-called canonical field quantization which deals with creation and annihilation operators for particles. Note that the Bose-Einstein and Fermi-Dirac statistics is automatically taken into account by defining appropriate commutation relations for the creation and annihilation operators. Because of illustrative purposes and in view for applications in the realm of solid-state physics we restrict ourselves in Part I to elaborate this second quantization formalism in the realm of non-relativistic quantum many-body theory. Thus, this amounts to quantize the first quantized Schrödinger theory, which is possible to perform separately for both bosons and fermions.

1.3 Relativistic Fields and Their Quantization

In Part II we discuss at first the Poincaré group as the fundamental space-time symmetry in the absence of gravity. By the concrete examples of rotations, boosts, and translations we introduce the concepts of Lie groups and Lie algebras as well as their respective representations. In particular, the Casimir operators of the Poincaré group are of importance, i.e. those operators which commute with all generators of rotations, boosts, and translations. Namely, it turns out that all states of relativistic quantum field theory can be classified with respect to the eigenvalues of the Casimir operators of the Poincaré group, which are the spin and the mass of the elementary particles, respectively, see Tab. 1.3.

Thus, one can understand relativistic quantum field theory as the representation theory of the Poincaré group. From this group-theoretical point of view we discuss in detail the examples of both the Maxwell and the Dirac field. To this end we determine the respective free solutions with their different helicity and polarization states. But instead of directly solving the respective Maxwell and Dirac equation, we take group theory to our advantage. For the massless (massive) spin 1 (1/2) particles we solve the underlying Maxwell (Dirac) equation in a particular reference frame (the inertial frame) and rotate (boost) then the solution to an arbitrary reference (inertial) frame. Afterwards, we second quantize the Maxwell as well as the Dirac theory and construct their respective free propagators. Furthermore, we discuss the fundamental relations between symmetries and conservation laws in terms of the seminal Noether theorem. As a concrete example we deal with all conservation laws of quantum electrodynamics.

1.4 Quantum Electrodynamics

In Part III we finally turn to quantum electrodynamics. At first we derive the light-matter interaction by postulating the aforementioned local gauge invariance. Based on the formalism of second quantization we then perform a systematic perturbation theory around the free theory and expand with respect to the light-matter interaction strength. In particular, we demonstrate that, although using the non-covariant Coulomb gauge for the Maxwell field, we finally yield covariant perturbative corrections, which can be graphically represented in terms of Feynman diagrams. In order to construct all Feynman diagrams order by order we introduce a graphical recursion relation, which is based on cutting the lines of Feynman diagrams of lower orders and

Mott scattering	$e^-Z - e^-Z$
Møller scattering	$e^- - e^-$
Bhabha scattering	$e^- - e^+$
Compton scattering	$e^-\gamma - e^-\gamma$

Table 1.4: Examples of scattering processes in quantum electrodynamics.

gluing them together with new interaction vertices. Based on the Feynman diagrams we are then able to calculate the cross sections for individual scattering processes, see the examples mentioned in Tab. 1.4.

In lowest order the respective cross sections are generically finite, but in higher orders notorious infinities appear. These infinities prevent to make any concrete quantitative prediction for an experimental measurement of a cross section. In quantum electrodynamics it turns out that these infinities can be systematically removed order by order with a so-called renormalization scheme. In a first step one regularizes the infinite integrals, i.e. one introduces an additional calculational degree of freedom in such a way that these integrals become finite. For instance, one introduces an ultraviolet cut-off Λ in momentum space or one follows the notion of Gerard 't Hooft and calculates the momentum integrals in $D = 4 - \varepsilon$ dimensions. By construction the infinities of the integrals then emerge in the limit $\Lambda \to \infty$ or $\varepsilon \to 0$. In a second step one shows then that the infinities can be absorbed by the few parameters of the theory as the mass, the coupling constant, and the fields. Depending on the available time we plan to perform this renormalization scheme in quantum electrodynamics explicitly in the lowest perturbative order. The general proof, that quantum electrodynamics is renormalizable to all orders of perturbation theory, is due to Freeman Dyson.