Chapter 10

Relativistic Light-Matter Interaction

Quantum electrodynamics is the relativistic quantum field theory of electrodynamics. It describes how light and matter interact and represents historically the first successful many-body theory, which unites quantum mechanics and special relativity. It involves all phenomena, where electrically charged particles interact by means of an exchange of photons. This chapter focuses on working out the relativistic light-matter interaction consecutively at first on a classical, then on a first quantized, and, finally, on a second quantized description level. At all three stages the common guiding principle to introduce an interaction between the free theories of light and matter consists of a minimal coupling scheme, which is based on a local gauge theory. As the main result we derive the second quantized Hamilton operator underlying quantum electrodynamics. Apart from the free Maxwell and the free Dirac theory, which have already been discussed in the two previous chapters, we also obtain an interaction term, whose physical consequences have to be studied perturbatively. To this end we concisely review the Dirac interaction picture, which allows to treat the relativistic light-matter interaction systematically order by order. As a special case we outline how to treat a generic scattering problem with the help of a corresponding perturbative expansion of the scattering operator, whose matrix elements determine the cross section.

10.1 Relativistic Mechanics

After having summarized concisely the basic principles of relativistic mechanics, we discuss first a free particle and then we introduce the description of a charged particle.

10.1.1 Basics

The trajectory of a classical relativistic particle is described by specifying both the time coordinate t and the space coordinates $\mathbf{x}$ as a function of some parameter σ:

$$(x^\lambda(\sigma)) = (ct(\sigma), \mathbf{x}(\sigma)) . \tag{10.1}$$

Thus, the velocity with respect to this trajectory parameter σ reads

$$\left(\dot{x}^\lambda(\sigma)\right) = \left(\frac{dx^\lambda(\sigma)}{d\sigma}\right) = \left(c\,\frac{dt(\sigma)}{d\sigma}, \frac{d\mathbf{x}(\sigma)}{d\sigma}\right) . \tag{10.2}$$

The action represents a functional of the trajectory in the four-dimensional space-time

$$\mathcal{A} = \mathcal{A}\left[x^\lambda(\bullet)\right] \tag{10.3}$$

and is defined as the integral of the Lagrange function with respect to the chosen trajectory parameter σ:

$$\mathcal{A} = \int_{\sigma_i}^{\sigma_f} d\sigma L\left(x^\lambda(\sigma); \dot{x}^\lambda(\sigma)\right) . \tag{10.4}$$

Then the Hamilton principle leads to the underlying equations of motion in form of the Euler-Lagrange equations:

$$\frac{\delta\mathcal{A}}{\delta x^\mu(\sigma)} = \frac{\partial L}{\partial x^\mu(\sigma)} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu(\sigma)} = 0\,. \tag{10.5}$$

Note that the Hamilton principle exhibits the mechanical gauge invariance that a regauging of the Lagrange function according to

$$L'\left(x^\lambda; \dot{x}^\lambda\right) = L\left(x^\lambda; \dot{x}^\lambda\right) + \frac{d}{d\sigma}\,\chi\left(x^\lambda\right) = L\left(x^\lambda; \dot{x}^\lambda\right) + \partial_\nu\chi\left(x^\lambda\right)\dot{x}^\nu \tag{10.6}$$

only leads to additional surface terms of the action (10.3):

$$\mathcal{A}' = \mathcal{A} + \chi\left(x^\lambda(\sigma_f)\right) - \chi\left(x^\lambda(\sigma_i)\right) \tag{10.7}$$

and, therefore, does not change the equations of motion. In fact, for the transformed Lagrange function (10.6) one obtains the same Euler-Lagrange equations

$$\frac{\partial L'}{\partial x^\mu(\sigma)} - \frac{d}{d\sigma}\frac{\partial L'}{\partial \dot{x}^\mu(\sigma)} = \frac{\partial L}{\partial x^\mu(\sigma)} + \partial_\mu\partial_\nu\chi(x^\lambda(\sigma))\dot{x}^\nu(\sigma) - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\mu(\sigma)} - \partial_\nu\partial_\mu\chi(x^\lambda(\sigma))\dot{x}^\nu(\sigma) \tag{10.8}$$

since the gauge function $\chi\left(x^\lambda\right)$ is supposed to be twice continuously differentiable and therefore satisfies the Schwarz theorem:

$$\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)\chi\left(x^\lambda\right) = 0\,. \tag{10.9}$$

In addition to this mechanical gauge invariance, relativistic mechanics has even further symmetries that take into account the principles of special relativity. Since the laws of physics are supposed to have the same form in all inertial frames, the action must be invariant under Lorentz transformations. In addition, however, the description of the trajectory should also be independent of the choice of the parameter σ, so that the action must also be form invariant under any transformation of the trajectory parameter:

$$\sigma = \sigma\left(\sigma'\right) . \tag{10.10}$$

This reparametrisation invariance is guaranteed by the fact that the Lagrange function represents a homogeneous function of the velocities of first order:

$$L\left(x^\lambda;\alpha\dot{x}^\lambda\right) = \alpha L\left(x^\lambda;\alpha\dot{x}^\lambda\right)\,. \tag{10.11}$$

Then applying (10.10) and (10.11) to the action (10.4) yields

$$\begin{aligned}\mathcal{A} &= \int_{\sigma_i}^{\sigma_f} d\sigma L\left(x^\lambda(\sigma);\dot{x}^\lambda(\sigma)\right) = \int_{\sigma_i'}^{\sigma_f'} d\sigma'\,\frac{d\sigma}{d\sigma'}\,L\left(x^\lambda(\sigma');\dot{x}^\lambda(\sigma')\frac{d\sigma'}{d\sigma}\right)\\ &= \int_{\sigma_i'}^{\sigma_f'} d\sigma' L(x^\lambda\left(\sigma'\right);\dot{x}^\lambda(\sigma'))\,.\end{aligned} \tag{10.12}$$

Differentiating the condition (10.11) with respect to α and evaluating it then at the point $\alpha = 1$ yields the corresponding Euler theorem. It states that the Hamilton function of relativistic mechanics vanishes:

$$H = \frac{\partial L}{\partial \dot{x}^\mu}\,\dot{x}^\mu - L = 0\,. \tag{10.13}$$

This result is at first glance puzzling in view of the question how a relativistic mechanical systems is supposed to be quantized. The generic operator approach to determine from the underlying Hamilton function a corresponding quantum mechanical operator seems not to be possible due to (10.13). This can be considered as a motivation of Richard Feynman to work out an alternative formulation of quantum mechanics, which does not rely on Hamilton mechanics but is based instead on Lagrange mechanics.

10.1.2 Free Particle

Let us consider at first a free relativistic particle of mass M, whose action is motivated as follows. With the help of the Minkowski metric $g_{\mu\nu}$ one can determine the distance between two inifinitesimally adjacent space-time points x^μ and $x^\mu + dx^\mu$ according to

$$ds = \sqrt{g_{\mu\nu}\,dx^\mu dx^\nu}\,. \tag{10.14}$$

Decomposing this Lorentz invariant length element ds into the temporal and spatial contributions

$$ds = \sqrt{c^2dt^2 - d\mathbf{x}^2}\,, \tag{10.15}$$

its physical meaning becomes apparent. Considering two infinitesimally adjacent space-time points in the rest frame of the particle, where we have $d\mathbf{x}_R = \mathbf{0}$, then ds becomes the proper length and, correspondingly, $\tau = ds/c$ denotes the proper time. The length of a trajectory between two different space-time points follows then from integrating (10.14) with respect to the chosen trajectory parameter σ:

$$\int_{s_i}^{s_f} ds = \int_{\sigma_i}^{\sigma_f} d\sigma\,\frac{ds}{d\sigma} = \int_{\sigma_i}^{\sigma_f} d\sigma\,\sqrt{g_{\mu\nu}\,\dot{x}^\mu(\sigma)\dot{x}^\nu(\sigma)}\,. \tag{10.16}$$

We remark that the trajectory length (10.16) is a Lorentz invariant quantity, which is also reparametrization invariant as the integrand is homogeneous in the velocities of first order in the sense of (10.11). This suggests that (10.16) is a viable candidate for an action in relativistic mechanics. Therefore, we argue now that

$$\mathcal{A}^{(0)} = -Mc \int_{s_i}^{s_f} ds = -Mc \int_{\sigma_i}^{\sigma_f} d\sigma \sqrt{g_{\mu\nu}\, \dot{x}^\mu(\sigma) \dot{x}^\nu(\sigma)} \tag{10.17}$$

represents the action of a free relativistic particle of mass M. We justify our choice by proving that it has the correct non-relativistic limit. Using the time t as the trajectory parameter σ, (10.17) namely leads to

$$\mathcal{A}^{(0)} = -Mc^2 \int_{t_i}^{t_f} dt \sqrt{1 - \frac{\dot{\mathbf{x}}(t)^2}{c^2}}\,, \tag{10.18}$$

so that the limes $c \to \infty$ yields the leading contribution

$$\mathcal{A}^{(0)} = \int_{t_i}^{t_f} dt \left\{ \frac{M}{2}\, \dot{x}(t)^2 - Mc^2 \right\} + \dots\,. \tag{10.19}$$

This is the action of a free non-relativistic particle of mass M for which the energy scale is just shifted by the rest energy Mc^2.

10.1.3 Charged Particle

If a non-relativistic particle has a charge q, its interaction with a scalar potential $\varphi(\mathbf{x}, t)$ reads

$$\mathcal{A}^{(\mathrm{int})} = -q \int_{t_i}^{t_f} dt\, \varphi(\mathbf{x}(t), t)\,. \tag{10.20}$$

Taking into account (8.34) and (10.1), this can also be written as

$$\mathcal{A}^{(\mathrm{int})} = -q \int_{\sigma_i}^{\sigma_f} d\sigma\, \dot{x}^0(\sigma) A_0(x^\lambda(\sigma))\,. \tag{10.21}$$

Generalising (10.21) in a relativistic covariant way yields the interaction of a relativistic particle with the entire electromagnetic field, which is described by the four-vector potential $A_\mu(x^\lambda)$:

$$\mathcal{A}^{(\mathrm{int})} = -q \int_{\sigma_i}^{\sigma_f} d\sigma\, \dot{x}^\mu(\sigma) A_\mu(x^\lambda(\sigma))\,. \tag{10.22}$$

Thus, one can consider the charge q as a formal coupling constant, which measures the strength of interaction between the particle four-velocity and the four-vector potential. Note that also the interaction (10.22) is reparametrisation invariant as its integrand is homogeneous in the velocities of first order in the sense of (10.11) like the free action (10.17). Adding the free action (10.17) and the interaction (10.22) leads to a resulting action (10.3) with the Lagrange function

$$L\left(x^\mu; \dot{x}^\mu\right) = -Mc \sqrt{g_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu} - q\, A_\mu(x^\lambda) \dot{x}^\mu\,. \tag{10.23}$$

An electromagnetic gauge transformation

$$A'_{\mu}(x^{\lambda}) = A_{\mu}(x^{\lambda}) + \partial_{\mu}\Lambda(x^{\lambda}) \tag{10.24}$$

see (8.42), leads according to (10.4) and (10.23) to a mechanical gauge transformation (10.6), where the mechanical gauge function χ and the electromagnetic gauge function Λ turn out to be proportional to each other:

$$\chi\left(x^{\lambda}\right) = -q\Lambda(x^{\lambda})\,. \tag{10.25}$$

Let us form the partial derivatives of the Lagrange function (10.23)

$$\frac{\partial L}{\partial x^{\mu}} = -q\,\partial_{\mu}A_{\nu}(x^{\lambda})\dot{x}^{\nu}\,, \tag{10.26}$$

$$\frac{\partial L}{\partial \dot{x}^{\mu}} = -Mc\,\frac{g_{\mu\nu}\,\dot{x}^{\nu}}{\sqrt{g_{\kappa\lambda}\,\dot{x}^{\kappa}\dot{x}^{\lambda}}} - qA_{\mu}(x^{\lambda})\,. \tag{10.27}$$

Furthermore, we introduce for the derivative of the proper length s with respect to the trajectory parameter σ according to (10.14) the shortcut notation

$$\dot{s}(\sigma) = \sqrt{g_{\mu\nu}\,\dot{x}^{\mu}(\sigma)\dot{x}^{\nu}(\sigma)}\,. \tag{10.28}$$

Then the Euler-Lagrange equations (10.5) following from (10.26) and (10.27) read as follows

$$M\ddot{x}^{\mu} = M\,\frac{\ddot{s}}{\dot{s}}\,\dot{x}^{\mu} + \frac{q\dot{s}}{c}\,g^{\mu\kappa}\left\{\partial_{\kappa}A_{\nu}(x^{\lambda}) - \partial_{\nu}A_{\kappa}(x^{\lambda})\right\}\dot{x}^{\nu}\,. \tag{10.29}$$

Due to the reparametrization invariance of relativistic mechanics we are free to make a physically reasonable choice for the trajectory parameter. To this end we choose the trajectory parameter σ to be the proper time $\tau = s/c$. On the one hand this corresponds to the time which passes in the rest frame of the moving particle. On the other hand this simplifies the equations of motion (10.29) due to $\dot{s} = c$ and $\ddot{s} = 0$:

$$M\ddot{x}^{\mu} = q\,F^{\mu}{}_{\nu}\left(x^{\lambda}\right)\dot{x}^{\nu}\,. \tag{10.30}$$

Here the electrodynamic field strength sensor

$$F^{\mu}{}_{\nu}\left(x^{\lambda}\right) = g^{\mu\kappa}\,F_{\kappa\nu}\left(x^{\lambda}\right) \tag{10.31}$$

was introduced as an abbreviation. Note with (8.20) and (10.31) we recognize at the right-hand side of (10.30) the relativistic generalization of the Lorentz force.

10.1.4 Minimal Coupling

In order to investigate in more detail the description of a charged particle in relativistic mechanics we choose the trajectory parameter σ to be the time t in the laboratory frame. Then the action (10.3), (10.4) reduces to

$$\mathcal{A} = \mathcal{A}[\mathbf{x}(\bullet)] = \int_{t_i}^{t_f} dt\,L\left(\mathbf{x}(t);\dot{\mathbf{x}}(t);t\right) \tag{10.32}$$

and the Lagrange function (10.23) specializes due to (8.34) and (10.1) to

$$L = -Mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} - q\varphi(\mathbf{x},t) + q\,\dot{\mathbf{x}}\,\mathbf{A}(\mathbf{x},t)\,. \tag{10.33}$$

Thus, the canonical momentum reads

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{M\dot{\mathbf{x}}}{\sqrt{1-\dot{\mathbf{x}}^2/c^2}} + q\mathbf{A}(\mathbf{x},t)\,. \tag{10.34}$$

Here the first term represents the kinetic momentum and the second term the corresponding contribution of the vector potential, so (10.34) corresponds to

$$\mathbf{p} = \mathbf{p}_{\rm kin} + q\mathbf{A}(\mathbf{x},t)\,. \tag{10.35}$$

In view of performing a Legendre transformation from the Lagrange function to the Hamilton function, we have to invert the relation (10.34) between the momentum $\mathbf{p}$ and the velocity $\dot{\mathbf{x}}$. A straight-forward algebraic calculation yields

$$\dot{\mathbf{x}} = \frac{c\,[\mathbf{p}-q\mathbf{A}(\mathbf{x},t)]}{\sqrt{[\mathbf{p}-q\mathbf{A}(\mathbf{x},t)]^2 + M^2c^2}}\,. \tag{10.36}$$

Thus, evaluating the Legendre transformation

$$H = \dot{\mathbf{x}}\,\frac{\partial L}{\partial \dot{\mathbf{x}}} - L \tag{10.37}$$

by using (10.33) and (10.36) we obtain for the Hamilton function of a relativistic charged particle in the electromagnetic field

$$H = c\sqrt{[\mathbf{p}-q\mathbf{A}(\mathbf{x},t)]^2 + M^2c^2} + q\varphi(\mathbf{x},t)\,. \tag{10.38}$$

We remark that this result reduces in the non-relativistic limit $c \to \infty$ to the leading contribution

$$H = Mc^2 + \frac{[\mathbf{p}-q\mathbf{A}(\mathbf{x},t)]^2}{2M} + q\varphi(\mathbf{x},t) + \ldots\,. \tag{10.39}$$

Furthermore, analogous to (10.35), we interpret (10.38) such that the first term describes the kinetic energy and the second term the potential energy:

$$H = H_{\rm kin} + q\varphi(\mathbf{x},t)\,. \tag{10.40}$$

Conversely, we read off from (10.35) and (10.40) that a free theory with $q = 0$ can formally be transferred into the corresponding interacting one with $q \neq 0$ by substituting the momentum (energy) via

$$\mathbf{p} \quad \to \quad \mathbf{p} - q\mathbf{A}(\mathbf{x},t)\,, \qquad H \quad \to \quad H - q\varphi(\mathbf{x},t)\,. \tag{10.41}$$

This so-called minimal coupling of charged particle to the electromagnetic field can now be covariantly formulated in terms of a covariant momentum four-vector (6.16) by taking into account (8.34) according to

$$p_\mu \quad \to \quad p_\mu - qA_\mu(x^\lambda)\,. \tag{10.42}$$

In the following the minimal coupling rule (10.42) is applied to the realm of relativistic quantum field theory by combining it with the Jordan rule.

10.2 QED Actions

Quantum electrodynamics describes the interaction between charged massive particles and the electromagnetic field. One distinguishes, in principle, between scalar quantum electrodynamics for charged spin 0-particles as, for instance, pions $\pi^{\pm}$, and spinor quantum electrodynamics for charged spin 1/2-particles as, for instance, electrons e^- or positrons e^+. As a consequence, the underlying equations of motion for massive particles and for the electromagnetic field are coupled by additional interaction terms. In the Lagrangian density this leads to an additional interaction term in addition to the free Lagrangian density, whose strength depends on the coupling constant of electrodynamics, i.e. the charge q. In the following, we examine at first scalar quantum electrodynamics.

10.2.1 Scalar QED

We start with the relativistic covariant action of the free Klein-Gordon field

$$\mathcal{A}\left[\Psi(\bullet);\Psi^*(\bullet)\right] = \frac{1}{c}\int d^4x\,\mathcal{L}\left(\Psi(x^\lambda),\partial_\mu\Psi(x^\lambda);\Psi^*(x^\lambda),\partial_\mu\Psi^*(x^\lambda)\right)\,, \tag{10.43}$$

where the Lagrange density reads according to Section 7.1

$$\mathcal{L} = \frac{\hbar^2}{2M}\,g^{\mu\nu}\partial_\mu\Psi^*(x^\lambda)\partial_\nu\Psi(x^\lambda) - \frac{Mc^2}{2}\,\Psi^*(x^\lambda)\Psi(x^\lambda)\,. \tag{10.44}$$

A minimal coupling of the Klein-Gordon field to the electromagnetic field is now implemented by combining the substitution rule (10.42) with the Jordan rule, see (6.96):

$$p_\mu \quad\rightarrow\quad i\hbar\partial_\mu\,. \tag{10.45}$$

This leads to the catchy substitution rule

$$\partial_\mu\Psi(x^\lambda) \quad\rightarrow\quad D_\mu\Psi(x^\lambda)\,, \qquad \partial_\mu\Psi^*(x^\lambda) \quad\rightarrow\quad D^*_\mu\Psi^*(x^\lambda)\,, \tag{10.46}$$

where D_μ denotes the so-called gauge covariant derivative:

$$D_\mu = \partial_\mu + \frac{iq}{\hbar}\,A_\mu(x^\lambda)\,. \tag{10.47}$$

Applying (10.46) and (10.47) to the Lagrangian density (10.44) we get

$$\mathcal{L} = \frac{\hbar}{2M}\,g^{\mu\nu}\left\{\partial_\mu - \frac{iq}{\hbar}\,A_\mu(x^\lambda)\right\}\Psi^*(x^\lambda)\left\{\partial_\mu + \frac{iq}{\hbar}\,A_\mu(x^\lambda)\right\}\Psi(x^\lambda) - \frac{Mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda)\,, \tag{10.48}$$

which can be rewritten in a form which resembles that of a free Lagrangian density of the Klein-Gordon field:

$$\mathcal{L} = \frac{\hbar}{2M}\,g^{\mu\nu}D^*_\mu\Psi^*(x^\lambda)D_\nu\Psi(x^\lambda) - \frac{Mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda)\,. \tag{10.49}$$

We now examine the consequences of an electrodynamic gauge transformation (10.24). As the fields in quantum mechanics are only uniquely determined up to a phase factor, an electrodynamic gauge transformation can only change the phase. Supplementing an electrodynamic gauge transformation (10.24) accordingly with a quantum mechanical gauge transformation

$$\Psi'(x^\lambda) = \exp\left\{-\frac{iq}{\hbar}\,\Lambda(x^\lambda)\right\}\Psi(x^\lambda)\,, \tag{10.50}$$

the gauge covariant derivative (10.47) turns out to transform like the Klein-Gordon field $\Psi(x^\lambda)$:

$$\begin{aligned} D'_\mu\Psi'(x^\lambda) &= \left\{\partial_\mu + \frac{iq}{\hbar}\,A_\mu(x^\lambda) + \frac{iq}{\hbar}\,\partial_\mu\,\Lambda(x^\lambda)\right\}\exp\left\{-\frac{iq}{\hbar}\,\Lambda(x^\lambda)\right\}\Psi(x^\lambda) \\ &= \exp\left\{-\frac{iq}{\hbar}\,\Lambda(x^\lambda)\right\}D_\mu\Psi(x^\lambda)\,. \end{aligned} \tag{10.51}$$

Analogously, one obtains for the adjoint field

$$\Psi'^*(x^\lambda) = \exp\left\{\frac{iq}{\hbar}\,\Lambda(x^\lambda)\right\}\Psi^*(x^\lambda)\,, \tag{10.52}$$

$$D'^*_\mu\Psi'^*(x^\lambda) = \exp\left\{\frac{iq}{\hbar}\,\Lambda(x^\lambda)\right\}D^*_\mu\Psi^*(x^\lambda)\,. \tag{10.53}$$

Then it follows straight-forwardly from (10.50)–(10.52) that the Lagrangian density (10.49) is invariant under an electrodynamic gauge transformation:

$$\begin{aligned} \mathcal{L}' &= \frac{\hbar}{2M}\,g^{\mu\nu}D'^*_\mu\Psi'^*(x^\lambda)D'_\nu\Psi'(x^\lambda) - \frac{Mc^2}{2}\,\Psi'^*(x^\lambda)\Psi'(x^\lambda) \\ &= \frac{\hbar}{2M}\,g^{\mu\nu}D^*_\mu\Psi(x^\lambda)D_\mu\Psi(x^\lambda) - \frac{Mc^2}{2}\Psi^*(x^\lambda)\,\Psi(x^\lambda) = \mathcal{L}\,. \end{aligned} \tag{10.54}$$

If we consider the four-vector potential $A_\mu(x^\lambda)$ not as a given quantity but as a dynamic field, we must add to the Lagrangian density (10.48) the Lagrangian density of the free Maxwell field from Subsections 8.5 and 8.6, which is also invariant under the local gauge transformation (10.24). Accordingly, scalar quantum electrodynamics has the gauge invariant action

$$\mathcal{A}\left[\Psi(\bullet);\Psi^*(\bullet);A_\nu(\bullet)\right] = \frac{1}{c}\int d^4x\,\mathcal{L}\,, \tag{10.55}$$

where the Lagrange density is of the form

$$\mathcal{L} = \mathcal{L}\left(\Psi(x^\lambda),\partial_\mu\Psi(x^\lambda);\Psi^*(x^\lambda),\partial_\mu\Psi^*(x^\lambda);A_\nu(x^\lambda),\partial_\mu A_\nu(x^\lambda)\right) \tag{10.56}$$

and reads explicitly

$$\mathcal{L} = \frac{\hbar}{2M}\,g^{\mu\nu}\left(\partial_\mu - \frac{iq}{\hbar}\,A_\mu\right)\Psi^*\left(\partial_\mu + \frac{iq}{\hbar}\,A_\mu\right)\Psi - \frac{Mc^2}{2}\Psi^*\Psi - \frac{1}{4\mu_0}\,F_{\mu\nu}F^{\mu\nu}\,. \tag{10.57}$$

The Lagrange density thus decomposes according to

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(\mathrm{int})}\,. \tag{10.58}$$

Here $\mathcal{L}^{(0)}$ includes the free Lagrange densities of the Klein-Gordon field and the Maxwell field

$$\mathcal{L}^{(0)} = \frac{\hbar}{2M}\, g^{\mu\nu}\partial_\mu\Psi^*(x^\lambda)\partial_\nu\Psi(x^\lambda) - \frac{Mc^2}{2}\Psi^*(x^\lambda)\Psi(x^\lambda) - \frac{1}{4\mu_0}\, F_{\mu\nu}(x^\lambda)F^{\mu\nu}(x^\lambda) \tag{10.59}$$

and the interaction term turns out to have the structure

$$\mathcal{L}^{(\text{int})} = -j^\mu(x^\lambda)A_\mu(x^\lambda)\,. \tag{10.60}$$

The four-vector potential thus couples to the four-vector current density, which follows from applying to the free four-vector current density

$$j^\mu(x^\lambda) = \frac{i\hbar q}{2M}\, g^{\mu\nu}\Big\{\Psi^*(x^\lambda)\partial_\nu\Psi(x^\lambda) - \Psi(x^\lambda)\partial_\nu\Psi^*(x^\lambda)\Big\}\,, \tag{10.61}$$

see Section 7.2, the catchy substitution rule (10.46), (10.47), yielding

$$j^\mu(x^\lambda) = \frac{i\hbar q}{2M}\, g^{\mu\nu}\Big\{\Psi^*(x^\lambda)\partial_\nu\Psi(x^\lambda) - \Psi(x^\lambda)\partial_\nu\Psi^*(x^\lambda)\Big\} - \frac{q^2}{M}\, g^{\mu\nu}\, A_\nu(x^\lambda)\Psi^*(x^\lambda)\Psi(x^\lambda)\,. \tag{10.62}$$

The respective partial derivatives of the Lagrange density (10.57) read as follows

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = \frac{i\hbar q}{2M}\, g^{\nu\kappa}\,\Psi\left(\partial_\kappa - \frac{iq}{\hbar}\,A_\kappa\right)\Psi^* - \frac{i\hbar q}{2M}\, g^{\nu\kappa}\,\Psi^*\left(\partial_\kappa + \frac{iq}{\hbar}\,A_\kappa\right)\Psi\,, \tag{10.63}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{\mu_0}\,F^{\mu\nu}\,, \tag{10.64}$$

$$\frac{\partial\mathcal{L}}{\partial\Psi^*} = -\frac{i\hbar q}{2M} g^{\mu\nu}\,A_\mu\left(\partial_\nu + \frac{iq}{\hbar}\,A_\nu\right)\Psi - \frac{Mc^2}{2}\Psi\,, \tag{10.65}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)} = \frac{\hbar^2}{2M}\, g^{\mu\nu}\left(\partial_\nu + \frac{iq}{\hbar}\,A_\nu\right)\Psi\,, \tag{10.66}$$

$$\frac{\partial\mathcal{L}}{\partial\Psi} = \frac{i\hbar q}{2M}\, g^{\mu\nu}\,A_\mu\left(\partial_\nu - \frac{iq}{\hbar}\,A_\nu\right)\Psi - \frac{Mc^2}{2}\Psi^*\,, \tag{10.67}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = \frac{\hbar^2}{2M}\, g^{\mu\nu}\left(\partial_\nu - \frac{iq}{\hbar}\,A_\nu\right)\Psi^*\,. \tag{10.68}$$

With this we obtain the Euler-Lagrange equations of scalar quantum electrodynamics. For the Maxwell field the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \tag{10.69}$$

are specified as follows:

$$\partial_\mu F^{\mu\nu} = \mu_0\, g^{\nu\kappa}\,\frac{i\hbar q}{2M}\left\{\Psi^*\left(\partial_\kappa + \frac{iq}{\hbar}\,A_\kappa\right)\Psi - \Psi\left(\partial_\kappa - \frac{iq}{\hbar}\,A_\kappa\right)\Psi^*\right\}\,, \tag{10.70}$$

and for the Klein-Gordon field we get

$$\frac{\partial\mathcal{L}}{\partial\Psi^*} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)} = 0 \quad\Rightarrow\quad g^{\mu\nu}\left(\partial_\mu + \frac{iq}{\hbar}\,A_\mu\right)\left(\partial_\nu + \frac{iq}{\hbar}\,A_\nu\right)\Psi + \frac{M^2c^2}{\hbar^2}\,\Psi = 0\,, \tag{10.71}$$

$$\frac{\partial\mathcal{L}}{\partial\Psi} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = 0 \quad\Rightarrow\quad g^{\mu\nu}\left(\partial_\mu - \frac{iq}{\hbar}\,A_\mu\right)\left(\partial_\nu - \frac{iq}{\hbar}\,A_\nu\right)\Psi^* + \frac{M^2c^2}{\hbar^2}\,\Psi^* = 0\,. \tag{10.72}$$

The equations of motion (10.70) represent inhomogeneous Maxwell equations (8.29) with the current density (10.62). Furthermore, the equations of motion (10.71) and (10.72) arise from the free Klein-Gordon equations by applying the catchy substitution rule (10.46), (10.47).

10.2.2 Spinor QED

Now we construct the corresponding Lagrange density of spinor quantum electrodynamics by applying the principle of local gauge invariance. The starting point is the Lagrange density of the free Dirac field, see Sections 9.6 and 9.13:

$$\mathcal{L} = \overline{\psi}(x) \left(i\hbar c\gamma^\mu \partial_\mu - Mc^2\right) \psi(x) . \tag{10.73}$$

Obviously, this Lagrange density is invariant with respect to a global phase transformation of the form

$$\psi'(x) = e^{-iq\Lambda/\hbar}\, \psi(x) , \qquad \overline{\psi}'(x) = e^{iq\Lambda/\hbar}\, \overline{\psi}(x) , \tag{10.74}$$

where q denotes the charge of the massive spin 1/2-particle. This global $U(1)$ invariance is used in the Appendix to derive the continuity equation of charge conservation for the free Dirac theory by applying the Noether theorem. We now try to achieve that one can choose any phase at any space-time point, so that the above global phase Λ becomes a space- and time-dependent quantity $\Lambda(x)$. Accordingly, we consider the local $U(1)$ phase transformation

$$\psi'(x) = e^{-iq\Lambda(x)/\hbar}\, \psi(x) , \qquad \overline{\psi}'(x) = e^{iq\Lambda(x)/\hbar}\, \overline{\psi}(x) . \tag{10.75}$$

The Lagrange density of the free Dirac field (10.73) is then no longer invariant under such a local phase transformation, since an additional term appears due to the partial derivative of the Dirac spinor:

$$\partial_\mu \psi'(x) \quad = \quad e^{-iq\Lambda(x)/\hbar} \left\{ \partial_\mu \psi(x) - \frac{iq}{\hbar}\, \partial_\mu \Lambda(x) \psi(x) \right\} \tag{10.76}$$

and we get

$$\mathcal{L}' = \overline{\psi}'(x) \left(i\hbar c\gamma^\mu \partial_\mu - Mc^2\right) \psi'(x) = \mathcal{L} + qc\overline{\psi}(x)\gamma^\mu \partial_\mu \Lambda(x) \psi(x) . \tag{10.77}$$

In order to establish a local gauge invariance, additional fields must be introduced and the Lagrange density (10.73) must be extended correspondingly. Since the additional term in (10.77) depends on the gradient of the phase $\partial_\mu \Lambda(x)$ and, therefore, represents a Lorentz vector, we introduce a gauge field $A_\mu(x)$, which couples to the spinor with the coupling constant q. To this end we replace the partial derivative of the spinor by

$$\partial_\mu \psi(x) \quad \to \quad \mathcal{D}_\mu \psi(x) , \tag{10.78}$$

where the gauge covariant derivative of the spinor is defined by

$$\mathcal{D}_\mu = \partial_\mu + \frac{iq}{\hbar} A_\mu(x) . \tag{10.79}$$

Then we determine the transformation behaviour of the gauge field by requiring that the gauge covariant derivative of the spinor transforms like the spinor itself:

$$\mathcal{D}'_\mu \psi'(x) = e^{-iq\Lambda(x)/\hbar}\, \mathcal{D}_\mu \psi(x) . \tag{10.80}$$

Substituting (10.79) into (10.80) then leads to the condition

$$\partial_\mu \psi'(x) + \frac{iq}{\hbar} A'_\mu(x)\psi'(x) = e^{iq\Lambda(x)/\hbar} \left\{ \partial_\mu \psi(x) + \frac{iq}{\hbar} A_\mu(x)\psi(x) \right\} . \tag{10.81}$$

With the help of (10.75) and (10.76) this reduces, finally, to the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) . \tag{10.82}$$

Since the gauge field $A_\mu(x)$ transforms just like the four-vector potential of electrodynamics in (10.24), it is identified with the latter in the following. The substitution rule (10.80), (10.81) then corresponds to the minimal coupling of the Dirac field to the Maxwell field. For the sake of completeness we note that the substitution rule for the Dirac adjoint spinor is given analogous to (10.78) by

$$\partial_\mu \overline{\psi}(x) \quad \rightarrow \quad \mathcal{D}^*_\mu \overline{\psi}(x) . \tag{10.83}$$

The gauge-covariant derivative of the Dirac-adjoint spinor transforms then via

$$\begin{aligned} \mathcal{D}'^*_\mu \overline{\psi'}(x) &= \left\{ \partial_\mu - \frac{iq}{\hbar} A_\mu(x) - \frac{iq}{\hbar} \partial_\mu \Lambda(x) \right\} e^{iq\Lambda(x)/\hbar} \overline{\psi}(x) \\ &= e^{iq\Lambda(x)/\hbar} \mathcal{D}^*_\mu \overline{\psi}(x) . \end{aligned} \tag{10.84}$$

Performing the substitution (10.78) in the Lagrange density of the free Dirac field (10.73), we obtain

$$\mathcal{L} = \overline{\psi}(x) \left(i\hbar c \gamma^\mu \mathcal{D}_\mu - Mc^2 \right) \Psi(x) . \tag{10.85}$$

Decomposing the gauge covariant derivative D_μ according to (10.81), then in addition to the original free Lagrangian density of the Dirac field (10.73) also an interaction term arises:

$$\mathcal{L} = \overline{\psi}(x) \left\{ i\hbar c \gamma^\mu \left[\partial_\mu + \frac{iq}{\hbar} A_\mu(x) \right] - Mc^2 \right\} \Psi(x) . \tag{10.86}$$

If we also consider the vector potential $A_\mu(x)$ as a dynamic field, we must add to the Lagrangian density (10.86) the Lagrangian density of the free Maxwell field. The resulting Lagrangian density turns out to be then manifestly local gauge invariant due to (10.75), (10.80), and (10.82). It represents the Lagrangian density of spinor quantum electrodynamics:

$$\mathcal{L} = \overline{\psi}(x) \left\{ i\hbar c \gamma^\mu \left[\partial_\mu + \frac{iq}{\hbar} A_\mu(x) \right] - Mc^2 \right\} \psi(x) - \frac{1}{4\mu_0} F_{\mu\nu}(x) F^{\mu\nu}(x) . \tag{10.87}$$

This Lagrange density decomposes according to

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(\text{int})} , \tag{10.88}$$

where $\mathcal{L}^{(0)}$ representing the free Lagrangian density including both the Dirac field and the Maxwell field

$$\mathcal{L}^{(0)} = \overline{\psi}(x) \left(i\hbar c \gamma^\mu \partial_\mu - Mc^2 \right) \psi(x) - \frac{1}{4\mu_0} F_{\mu\nu}(x) F^{\mu\nu}(x) \tag{10.89}$$

and the interaction term turns out to have the structure

$$\mathcal{L}^{(\mathrm{int})} = -j^\mu(x) A_\mu(x) \,. \tag{10.90}$$

The four-vector potential thus couples to the four-current density of the free Dirac field, see Section 9.10 and 9.13:

$$j^\mu(x) = qc\,\overline{\psi}(x)\gamma^\mu\psi(x) \,. \tag{10.91}$$

The respective partial derivatives of the Lagrange density (10.87) lead to

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = -qc\overline{\psi}\gamma^\nu\psi \,, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{\mu_0} F^{\mu\nu} \,, \tag{10.92}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = \left(i\hbar c\gamma^\mu\partial_\mu - Mc^2\right)\psi - qc\gamma^\mu\Psi A_\mu \,, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0 \,, \tag{10.93}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -Mc^2\,\overline{\psi} - qc\overline{\psi}\,\gamma^\mu A_\mu \,, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)} = i\hbar c\overline{\psi}(x)\gamma^\mu \,. \tag{10.94}$$

The Euler-Lagrange equations of spinor quantum electrodynamics thus result in

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \qquad \Rightarrow \qquad \partial_\mu F^{\mu\nu} = \mu_0 qc\overline{\psi}\,\gamma^\mu\psi \,, \tag{10.95}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\overline{\psi})} = 0 \qquad \Rightarrow \qquad i\gamma^\mu\left(\partial_\mu + \frac{iq}{\hbar}A_\mu\right)\psi - \frac{Mc}{\hbar}\psi = 0 \,, \tag{10.96}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)} = 0 \qquad \Rightarrow \qquad i\left(\partial_\mu - \frac{iq}{\hbar}A_\mu\right)\overline{\psi}\,\gamma^\mu + \frac{Mc}{\hbar}\overline{\psi} = 0 \,. \tag{10.97}$$

The equations of motion (10.95) agree with the inhomogeneous Maxwell equations (8.29) with the current density (10.91). Furthermore, the equations of motion (10.96) and (10.97) emerge from the free Dirac equations by applying the minimal couplings (10.78) and (10.83), which involve the gauge-covariant derivative (10.79).

10.3 QED Hamilton Function

Starting from the Lagrange density of spinor quantum electrodynamics in (10.87), we now calculate the corresponding Hamilton density. At first, we express the contribution of the free Maxwell field in terms of the electric field strength **E** and the magnetic field strength **B**, see Section 8.6:

$$\mathcal{L} = \overline{\psi}\left(i\hbar c\gamma^\mu\partial_\mu - Mc^2\right)\psi + \frac{\epsilon_0}{2}\,\mathbf{E}^2 - \frac{1}{2\mu_0}\,\mathbf{B}^2 - qc\overline{\psi}\,\gamma^\mu\psi A_\mu \,. \tag{10.98}$$

Then we express the electric field strength **E** and the magnetic field strength **B** by the scalar potential φ and the vector potential **A** due to (8.7) and (8.8), yielding

$$\begin{aligned}\mathcal{L} = {} & \overline{\psi}(\mathbf{x},t)\left(i\hbar c\gamma^\mu\partial_\mu - Mc^2\right)\psi(\mathbf{x},t) + \frac{\epsilon_0}{2}\left[\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\right]^2 + \epsilon_0\,\frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t}\,\boldsymbol{\nabla}\varphi(\mathbf{x},t) \\ & + \frac{\epsilon_0}{2}\left[\boldsymbol{\nabla}\varphi(\mathbf{x},t)\right]^2 - \frac{1}{2\mu_0}\left[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\right]^2 - qc\overline{\psi}(\mathbf{x},t)\,\gamma^\mu\psi(\mathbf{x},t)A_\mu(\mathbf{x},t) \,. \end{aligned} \tag{10.99}$$

Note that the Coulomb gauge (8.13) leads to the fact that the scalar potential φ no longer represents a dynamic field but is determined by (8.16) and (9.278):

$$\varphi(\mathbf{x},t) = \int d^3x' \, \frac{q\,\psi^\dagger(\mathbf{x}',t)\psi(\mathbf{x}',t)}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|} \,. \tag{10.100}$$

The canonical conjugated momentum fields then follow from (10.99) to be

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi(\mathbf{x},t)}{\partial t}\right)} = i\hbar\overline{\psi}(\mathbf{x},t)\gamma^0 = i\hbar\psi^\dagger(\mathbf{x},t)\,, \tag{10.101}$$

$$\overline{\pi}(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\overline{\psi}(\mathbf{x},t)}{\partial t}\right)} = 0\,, \tag{10.102}$$

$$\boldsymbol{\pi}(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}\right)} = \epsilon_0\left\{\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} + \boldsymbol{\nabla}\varphi(\mathbf{x},t)\right\}. \tag{10.103}$$

Note that the last term in (10.103) did not appear in Section 8.6, as there we considered the free Maxwell field in vacuum. A subsequent Legendre transformation leads then to the corresponding Hamilton density:

$$\mathcal{H} = \pi(\mathbf{x},t)\,\frac{\partial\psi(\mathbf{x},t)}{\partial t} + \frac{\partial\overline{\psi}(\mathbf{x},t)}{\partial t}\,\overline{\pi}(\mathbf{x},t) + \boldsymbol{\pi}(\mathbf{x},t)\,\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} - \mathcal{L}\,. \tag{10.104}$$

Thus, using (10.99) and (10.101)–(10.104) as well as (9.249) and (9.250) we obtain:

$$\begin{aligned}\mathcal{H} &= \psi^\dagger(\mathbf{x},t)\left(-i\hbar c\,\boldsymbol{\alpha}\boldsymbol{\nabla} + Mc^2\beta\right)\psi(\mathbf{x},t) + \frac{\epsilon_0}{2}\left[\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}\right]^2 + \frac{1}{2\mu_0}\left[\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)\right]^2 \\ &\quad -\frac{\epsilon_0}{2}\left[\boldsymbol{\nabla}\varphi(\mathbf{x},t)\right]^2 + q\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\varphi(\mathbf{x},t) - qc\,\psi^\dagger(\mathbf{x},t)\,\boldsymbol{\alpha}\psi(\mathbf{x},t)\mathbf{A}(\mathbf{x},t)\,.\end{aligned} \tag{10.105}$$

Going over to the Hamiltonian function, we yield by partial integration and by taking into account the Coulomb gauge (8.13), see Section 8.6:

$$\begin{aligned}H = \int d^3x \Big\{ &\psi^\dagger(\mathbf{x},t)\left(-i\hbar c\boldsymbol{\alpha}\boldsymbol{\nabla} + Mc^2\beta\right)\psi(\mathbf{x},t) + \frac{\epsilon_0}{2}\left[\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t}\right]^2 + \frac{1}{2\mu_0}\partial_k A_l(\mathbf{x},t)\partial_k A_l(\mathbf{x},t) \\ &+\frac{\epsilon_0}{2}\,\varphi(\mathbf{x},t)\Delta\varphi(\mathbf{x},t) + q\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\varphi(\mathbf{x},t) - qc\psi^\dagger(\mathbf{x},t)\boldsymbol{\alpha}\psi(\mathbf{x},t)\mathbf{A}(\mathbf{x},t)\}\Big\}\,.\end{aligned} \tag{10.106}$$

At this stage we use the Poisson equation for a point charge for determining that the Green function of the Poisson equation is given by the Coulomb potential:

$$\Delta\,\frac{q}{4\pi\epsilon_0\,|\mathbf{x}-\mathbf{x}'|} = -\frac{q}{\epsilon_0}\,\delta(\mathbf{x}-\mathbf{x}') \qquad \Rightarrow \qquad \Delta\,\frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x}-\mathbf{x}')\,. \tag{10.107}$$

Thus, taking into account (10.100) and (10.107) we yield the auxiliary calculation

$$\begin{aligned}&\frac{\epsilon_0}{2}\int d^3x\,\varphi(\mathbf{x},t)\Delta\varphi(\mathbf{x},t) = \frac{\epsilon_0}{2}\int d^3x\int d^3x'\,\varphi(\mathbf{x},t)\,\frac{q\psi^\dagger(\mathbf{x}',t)\psi(\mathbf{x}',t)}{4\pi\epsilon_0}\,\Delta\,\frac{1}{|\mathbf{x}-\mathbf{x}'|} \\ &-\frac{\epsilon_0}{2}\int d^3x\,\varphi(\mathbf{x},t)\,\frac{q\psi^\dagger(\mathbf{x}',t)\psi(\mathbf{x}',t)}{4\pi\epsilon_0}\,4\pi\delta(\mathbf{x}-\mathbf{x}') = -\frac{q}{2}\int d^3x\,\varphi(\mathbf{x},t)\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,.\end{aligned} \tag{10.108}$$

Substituting (10.100) and (10.108) into (10.106), the Hamilton function of spinor quantum electrodynamics decomposes according to

$$H = H^{(0)} + H^{(\mathrm{int})} . \tag{10.109}$$

where $H^{(0)}$ represents the free contributions of both the Dirac field and the Maxwell field:

$$\begin{aligned} H^{(0)} &= \int d^3x \Big\{ \psi^\dagger(\mathbf{x},t) \left(-i\hbar c \boldsymbol{\alpha} \boldsymbol{\nabla} + Mc^2\beta\right) \psi(\mathbf{x},t) + \frac{\epsilon_0}{2} \frac{\partial A_k(\mathbf{x},t)}{\partial t} \frac{\partial A_k(\mathbf{x},t)}{\partial t} \\ &\quad + \frac{1}{2\mu_0} \partial_k A_l(\mathbf{x},t) \partial_k A_l(\mathbf{x},t) \Big\} . \end{aligned} \tag{10.110}$$

The term $H^{(\mathrm{int})}$ represents the interaction between the Dirac and the Maxwell field:

$$\begin{aligned} H^{(\mathrm{int})} &= -qc\, \overline{\psi}(\mathbf{x},t)\, \boldsymbol{\gamma}\, \psi(\mathbf{x},t) \mathbf{A}(\mathbf{x},t) \\ &\quad + \frac{q^2}{8\pi\epsilon_0} \int d^3x \int d^3x' \, \frac{\overline{\psi}(\mathbf{x},t)\gamma^0\psi(\mathbf{x},t)\overline{\psi}(\mathbf{x}',t)\gamma^0\psi(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} . \end{aligned} \tag{10.111}$$

The first term in (10.111) arises from the free Hamilton function of the Dirac field in (10.110) by performing a minimal coupling to the vector potential in accordance with (10.79):

$$\boldsymbol{\nabla} \quad \rightarrow \quad \boldsymbol{\nabla} - \frac{iq}{\hbar}\, \mathbf{A}(\mathbf{x},t) . \tag{10.112}$$

The second term in (10.111) represents an instantaneous Coulomb self-interaction of the Dirac field. It is non-trivial to prove that such an instantaneous self-interaction does not contradict the principles of special relativity. Later we show by a concrete example of a scattering process that the instantaneous Coulomb self-interaction in (10.111) turns out to compensate an unwanted contribution (8.206) of the Maxwell propagator (8.204), which comes from the Coulomb gauge, thus yielding at the end manifestly covariant physical results.

10.4 Dirac Picture

In quantum field theory the quantisation of free fields is basically trivial, since the Hamilton function and, thus, the second quantized Hamilton operator is quadratic in the fields and the field operators, respectively. This has the consequence that the Fourier operators occurring in plane wave expansions of the field operators represent physically the creation and the annihilation of individual particles with well-defined properties. But the quantisation of interacting fields is non-trivial as it leads to interesting physical processes due to the involved nonlinearities. The Hamilton function contains higher powers of the same field in the case of a self-interaction or products of different fields as in quantum electrodynamics. The resulting dynamics of the field operators is, thus, complicated because, at each instant, the Fourier operators cause the creation and annihilation of particles with different properties. For instance, preparing an annihilation operator at in initial time t_0, it may happen that at a later time instant $t > t_0$ it evolves into a certain superposition of creation and annihilation operators.

Basically, it is not possible to solve exactly an interacting quantum field theory. However, if the interaction is sufficiently weak, reliable approximations can be obtained with the help of perturbation theory. Then neither the Schrödinger picture, in which the state vectors are time-dependent and the operators time-independent, nor the Heisenberg picture, in which conversely the state vectors are time-independent and the operators time-dependent, is suitable, see Section 3.4. Instead, in perturbation theory the Dirac or interaction picture turns out to be more favorable, since the time dependencies are then distributed appropriately between both the state vectors and the operators.

10.4.1 Derivation

The starting point of perturbation theory is the assumption that the Hamilton operator of the system under consideration can be split into two parts in the Schrödinger picture:

$$\hat{H}_S = \hat{H}_S^{(0)} + \hat{H}_S^{(\text{int})} \,. \tag{10.113}$$

Here $\hat{H}_S^{(0)}$ represents the Hamilton operator of a system of free fields and $\hat{H}_S^{(\text{int})}$ denotes the interacting part of the Hamilton operator. In the Schrödinger picture the time-dependent state vector $|\psi_S(t)\rangle$ fulfills the Schrödinger equation (3.53), which has the formal solution (3.55). Thus, the time dependence of $|\psi_S(t)\rangle$ is determined by the mutual influence of both the unperturbed and the perturbed Hamilton operator $\hat{H}_S^{(0)}$ and $\hat{H}_S^{(\text{int})}$. The idea for introducing the Dirac picture is now to redo the temporal evolution with the free Hamilton operator $\hat{H}_S^{(0)}$ according to

$$|\psi_D(t)\rangle = e^{i\hat{H}_S^{(0)}t/\hbar}\,|\psi_S(t)\rangle \qquad \Longleftrightarrow \qquad |\psi_S(t)\rangle = e^{-i\hat{H}_S^{(0)}t/\hbar}\,|\psi_D(t)\rangle \,. \tag{10.114}$$

In order to determine the operator $\hat{O}_D(t)$ in the Dirac picture, we require that the expectation values do not change during the transition from the Schrödinger picture to the Dirac picture:

$$\langle\psi_D(t)|\hat{O}_D(t)|\psi_D(t)\rangle = \langle\psi_S(t)|\hat{O}_S|\psi_S(t)\rangle \,. \tag{10.115}$$

Inserting (10.114) into (10.115) then actually leads to determine the operator $\hat{O}_D(t)$ in the Dirac picture

$$\begin{aligned} \langle\psi_D(t)|\, e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{O}_S\, e^{-i\hat{H}_S^{(0)}t/\hbar}\,|\psi_D(t)\rangle &= \langle\psi_D(t)|\hat{O}_D(t)|\psi_D(t)\rangle \\ \Rightarrow \qquad \hat{O}_D(t) &= e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{O}_S\, e^{-i\hat{H}_S^{(0)}t/\hbar} \,. \end{aligned} \tag{10.116}$$

For example, for the free Hamilton operator $\hat{O}_S = \hat{H}_S^{(0)}$ follows that it does not change its shape during the transition from the Schrödinger picture to the Dirac picture:

$$\hat{H}_D^{(0)}(t) = e^{i\hat{H}_S^{(0)}t}\,\hat{H}_S^{(0)}\,e^{-i\hat{H}_S^{(0)}t} = \hat{H}_S^{(0)} \,. \tag{10.117}$$

With (10.114) and (10.116) we have, thus, defined the Dirac picture both for the state vectors and the operators. It remains to investigate their respective equations of motion. Based on the equation of motion of a state vector in the Schrödinger picture (3.53) together with (10.113)

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat{H}_S|\psi_S(t)\rangle = \left(\hat{H}_S^{(0)} + \hat{H}_S^{(\mathrm{int})}\right)|\psi_S(t)\rangle \tag{10.118}$$

and taking into account (10.114) we then obtain the equation of motion of the corresponding state vector in the Dirac picture, which is called the Tomonaga-Schwinger equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}|\psi_D(t)\rangle &= e^{i\hat{H}_S^{(0)}t/\hbar}\left[i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle - \hat{H}_S^{(0)}|\psi_S(t)\rangle\right] = e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{H}_S^{(\mathrm{int})}|\psi_S(t)\rangle \\ \Longrightarrow\ & i\hbar\frac{\partial}{\partial t}|\psi_D(t)\rangle = \hat{H}_D^{(\mathrm{int})}(t)\,|\psi_D(t)\rangle\,. \end{aligned} \tag{10.119}$$

Here the interacting part of the Hamilton operator is transferred from the Schrödinger picture to the Dirac picture according to (10.116):

$$\hat{H}_D^{(\mathrm{int})}(t) = e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{H}_S^{(\mathrm{int})}\,e^{-i\hat{H}_S^{(0)}t/\hbar}\,. \tag{10.120}$$

Furthermore, starting from the equation of motion of an operator in the Schrödinger picture

$$i\hbar\frac{\partial}{\partial t}\hat{O}_S = 0\,, \tag{10.121}$$

we use (10.116) in order to derive the equation of motion of the corresponding operator in the Dirac picture as follows:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\hat{O}_D(t) &= e^{i\hat{H}_S^{(0)}t/\hbar}\left[\hat{O}_S\hat{H}_S^{(0)} - \hat{H}_S^{(0)}\hat{O}_S\right]e^{-i\hat{H}_S^{(0)}t/\hbar} \\ &= e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{O}_S\,e^{-i\hat{H}_S^{(0)}t/\hbar}\hat{H}_S^{(0)} - \hat{H}_S^{(0)}e^{i\hat{H}_S^{(0)}t/\hbar}\hat{O}_S\,e^{-i\hat{H}_S^{(0)}t/\hbar} = [\hat{O}_D(t),\hat{H}_S^{(0)}]_-\,. \end{aligned} \tag{10.122}$$

While in the Dirac picture the dynamics of the state vectors is determined by the interacting part of the Hamilton operator according to (10.119), only the free Hamilton operator enters the dynamics of the operators according to (10.122). The latter result has the consequence that the field operators in the Dirac picture still retain their respective properties of a free theory to create and annihilate particles.

10.4.2 Example

In order to illustrate the latter point we consider the quantum field-theoretic description of non-relativistic bosons, see Chapter 4. In the Schrödinger picture, the field operators $\hat{\psi}_S(\mathbf{x})$ and $\hat{\psi}_S^\dagger(\mathbf{x})$ satisfy the canonical commutator relations

$$\left[\hat{\psi}_S(\mathbf{x}),\hat{\psi}_S(\mathbf{x}')\right]_- = \left[\hat{\psi}_S^\dagger(\mathbf{x}),\hat{\psi}_S^\dagger(\mathbf{x}')\right]_- = 0\,,\qquad \left[\hat{\psi}_S(\mathbf{x}),\hat{\psi}_S^\dagger(\mathbf{x}')\right]_- = \delta(\mathbf{x}-\mathbf{x}')\,. \tag{10.123}$$

Thus, $\hat{\psi}_S(\mathbf{x})$ and $\hat{\psi}_S^\dagger(\mathbf{x})$ describe the annihilation and creation of a bosonic particle at space point $\mathbf{x}$, respectively. With the help of basis functions $u_{\mathbf{p}}(\mathbf{x})$, which fulfill the orthonormality relation

$$\int d^3x\, u_{\mathbf{p}}^*(\mathbf{x})\,u_{\mathbf{p}'}(\mathbf{x}) = \delta(\mathbf{p}-\mathbf{p}') \tag{10.124}$$

and the completeness relation

$$\int d^3p\, u^*_{\mathbf{p}}(\mathbf{x})\, u_{\mathbf{p}}(\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,, \tag{10.125}$$

the field operators $\hat{\psi}_S(\mathbf{x})$ and $\hat{\psi}^\dagger_S(\mathbf{x})$ can be expanded as follows:

$$\hat{\psi}_S(\mathbf{x}) = \int d^3p\, u_{\mathbf{p}}(\mathbf{x})\hat{a}_S(\mathbf{p})\,, \tag{10.126}$$

$$\hat{\psi}^\dagger_S(\mathbf{x}) = \int d^3p\, u^*_{\mathbf{p}}(\mathbf{x})\hat{a}^\dagger_S(\mathbf{p})\,. \tag{10.127}$$

Using (10.124), the expansions (10.126) and (10.127) are then inverted according to

$$\int d^3x\, u^*_{\mathbf{p}}(\mathbf{x})\hat{\psi}_S(\mathbf{x}) = \hat{a}_S(\mathbf{p})\,, \tag{10.128}$$

$$\int d^3x\, u_{\mathbf{p}}(\mathbf{x})\hat{\psi}^\dagger_S(\mathbf{x}) = \hat{a}^\dagger_S(\mathbf{p})\,. \tag{10.129}$$

With this the commutator relations (10.123) of the expansion operators $\hat{a}_S(\mathbf{p})$ and $\hat{a}^\dagger_S(\mathbf{p})$ result in

$$\left[\hat{a}_S(\mathbf{p}), \hat{a}_S(\mathbf{p}')\right]_- = \left[\hat{a}^\dagger_S(\mathbf{p}), \hat{a}^\dagger_S(\mathbf{p}')\right]_- = 0\,, \qquad \left[\hat{a}_S(\mathbf{p}), \hat{a}^\dagger_S(\mathbf{p}')\right]_- = \delta(\mathbf{p}-\mathbf{p}')\,. \tag{10.130}$$

Accordingly, the operator $\hat{a}_S(\mathbf{p})$ ($\hat{a}^\dagger_S(\mathbf{p})$) describes the annihilation (creation) of a particle of momentum $\mathbf{p}$. Let us assume for the sake of simplicity that the free Hamiltonian operator has already a diagonal form with an energy-momentum dispersion $E_{\mathbf{p}}$ in the Schrödinger picture:

$$\hat{H}^{(0)}_S = \int d^3p\, E_{\mathbf{p}}\, \hat{a}^\dagger_S(\mathbf{p})\hat{a}_S(\mathbf{p})\,. \tag{10.131}$$

The Heisenberg equation for the evolution of the annihilation operator in the Dirac picture (10.122) results then in

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\,\hat{a}_D(\mathbf{p},t) &= \left[\hat{a}_D(\mathbf{p},t), \hat{H}^{(0)}_S\right]_- = e^{i\hat{H}^{(0)}_S t/\hbar}\left[\hat{a}_S(\mathbf{p}), \hat{H}^{(0)}_S\right]_- e^{-i\hat{H}^{(0)}_S t/\hbar} \\ &= \int d^3p'\, E_{\mathbf{p}'}\, e^{i\hat{H}^{(0)}_S t/\hbar}\left[\hat{a}_S(\mathbf{p}), \hat{a}^\dagger_S(\mathbf{p}')\hat{a}_S(\mathbf{p}')\right]_- e^{-i\hat{H}^{(0)}_S t/\hbar} = E_{\mathbf{p}}\,\hat{a}_D(\mathbf{p},t)\,. \end{aligned} \tag{10.132}$$

The solution of this first-order differential equation with the initial condition

$$\hat{a}_D(\mathbf{p},0) = \hat{a}_S(\mathbf{p}) \tag{10.133}$$

is given by

$$\hat{a}_D(\mathbf{p},t) = e^{-iE_{\mathbf{p}}t/\hbar}\,\hat{a}_S(\mathbf{p})\,. \tag{10.134}$$

Thus, the time evolution of the creation operator yields

$$\hat{a}^\dagger_D(\mathbf{p},t) = e^{iE_{\mathbf{p}}t/\hbar}\,\hat{a}^\dagger_S(\mathbf{p})\,. \tag{10.135}$$

Due to (10.116), (10.131), (10.134), and (10.135) we then prove (10.117) as expected:

$$\hat{H}_D^{(0)}(t) = e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{H}_S^{(0)}\,e^{-i\hat{H}_S^{(0)}t/\hbar} = \int d^3p\,E_{\mathbf{p}}\,e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{a}_S^\dagger(\mathbf{p})\,e^{-i\hat{H}_S^{(0)}t/\hbar}\,e^{i\hat{H}_S^{(0)}t/\hbar}\,\hat{a}_S(\mathbf{p})\,e^{-i\hat{H}_S^{(0)}t/\hbar}$$
$$= \int d^3p\,E_{\mathbf{p}}\,\hat{a}_D^\dagger(\mathbf{p},t)\hat{a}_D(\mathbf{p},t) = \int d^3p\,E_{\mathbf{p}}\,\hat{a}_S^\dagger(\mathbf{p})\hat{a}_S(\mathbf{p}) = \hat{H}_S^{(0)}\,. \quad (10.136)$$

From (10.134) and (10.135) we read off that the creation and annihilation operators in the Dirac picture differ only by one additional phase factor from their counterparts in the Schrödinger picture. This means that the creation and annihilation operators in the Dirac picture do not change their character as single-particle operators during the time evolution. In particular, it follows directly from (10.130), (10.134), and (10.135) that the equal-time commutator relations in the Dirac picture coincide with those in the Schrödinger picture:

$$\left[\hat{a}_D(\mathbf{p},t),\hat{a}_D(\mathbf{p}',t)\right]_- = \left[\hat{a}_D^\dagger(\mathbf{p},t),\hat{a}_D^\dagger(\mathbf{p}',t)\right]_- = 0\,,\quad \left[\hat{a}_D(\mathbf{p},t),\hat{a}_D^\dagger(\mathbf{p}',t)\right]_- = \delta(\mathbf{p}-\mathbf{p}')\,. \quad (10.137)$$

This means that $\hat{a}_D(\mathbf{p},t)$ and $\hat{a}_D^\dagger(\mathbf{p},t)$ annihilate and create a particle with momentum $\mathbf{p}$ at time t. Furthermore, the field operators (10.126) and (10.127) in the Schrödinger picture change in the Dirac picture into

$$\hat{\psi}_D(\mathbf{x},t) = e^{i\hat{H}_S^{(0)}t/\hbar}\hat{\psi}_S(\mathbf{x})e^{-i\hat{H}_S^{(0)}t/\hbar} = \int d^3p\,u_{\mathbf{p}}(\mathbf{x})\,\hat{a}_D(\mathbf{p},t)\,, \quad (10.138)$$
$$\hat{\psi}_D^\dagger(\mathbf{x},t) = e^{i\hat{H}_S^{(0)}t/\hbar}\hat{\psi}_S^\dagger(\mathbf{p})e^{-i\hat{H}_S^{(0)}t/\hbar} = \int d^3p\,u_{\mathbf{p}}^*(\mathbf{x})\,\hat{a}_D^\dagger(\mathbf{p},t)\,. \quad (10.139)$$

Thus, according to (10.138) and (10.139), the field operators in the Dirac picture can be expanded with respect to creation and annihilation operators in exactly the same way as in the Heisenberg picture, see Section 3.4. Moreover, we obtain for the equal-time commutator relations of the field operators in the Dirac picture:

$$\left[\hat{\psi}_D(\mathbf{x},t),\hat{\psi}_D(\mathbf{x}',t)\right]_- = \left[\hat{\psi}_D^\dagger(\mathbf{x},t),\hat{\psi}_D^\dagger(\mathbf{x}',t)\right]_- = 0\,,\quad \left[\hat{\psi}_D(\mathbf{x},t),\hat{\psi}_D^\dagger(\mathbf{x}',t)\right]_- = \delta(\mathbf{x}-\mathbf{x}')\,. \quad (10.140)$$

Thus, we have in the Dirac picture the same equal-time commutator relations for the field operators as in the Heisenberg picture for free particles. This means that $\hat{\psi}_D(\mathbf{x},t)$ and $\hat{\psi}_D^\dagger(\mathbf{x},t)$ annihilate and create a particle at space point $\mathbf{x}$ at time t.

10.5 Canonical Field Quantisation

We now perform a canonical field quantisation of spinor quantum electrodynamics in the Dirac picture. According to the previous section, this means that we demand the same equal-time commutator or anti-commutator relations for the interacting theory in the Dirac picture as for the free theory in the Heisenberg picture. As we work from now on only in the Dirac picture we simplify our notation by omitting the index D, which indicates the Dirac picture. Concerning the Dirac field, equal-time anti-commutator relations are required for the independent field operators $\hat{\psi}_\alpha(\mathbf{x},t)$ and $\hat{\pi}_\beta(\mathbf{x},t)$:

$$\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}_\beta(\mathbf{x}',t)\right]_+ = \left[\hat{\pi}_\alpha(\mathbf{x},t),\hat{\pi}_\beta(\mathbf{x}',t)\right]_+ = 0\,,\ \left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\pi}_\beta(\mathbf{x}',t)\right]_+ = i\hbar\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \tag{10.141}$$

Concerning the Maxwell field, equal-time commutator relations are used for the independent field operators $\hat{A}_k(\mathbf{x},t)$ and $\hat{\pi}_l(\mathbf{x},t)$:

$$\left[\hat{A}_k(\mathbf{x},t),\hat{A}_l(\mathbf{x}',t)\right]_+ = \left[\hat{\pi}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_+ = 0\,,\ \left[\hat{A}_k(\mathbf{x},t),\hat{\pi}_l(\mathbf{x}',t)\right]_+ = i\hbar\delta^T_{kl}(\mathbf{x}-\mathbf{x}')\,, \tag{10.142}$$

where the transveral delta function (8.86) ensures analogous to Section 8.7 that the Coulomb gauge also holds for the field operators $\hat{A}_k(\mathbf{x},t)$ and $\hat{\pi}_l(\mathbf{x},t)$. And, due to their independence, equal-time commutator relations are required between the field operators of the Dirac and the Maxwell fields:

$$\begin{aligned}\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{A}_k(\mathbf{x}',t)\right]_- &= \left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\pi}_k(\mathbf{x}',t)\right]_- \\ = \left[\hat{\pi}_\alpha(\mathbf{x},t),\hat{A}_k(\mathbf{x}',t)\right]_- &= \left[\hat{\pi}_\alpha(\mathbf{x},t),\hat{\pi}_k(\mathbf{x}',t)\right]_- = 0\,.\end{aligned} \tag{10.143}$$

Applying the field quantization to the momentum fields (10.101) and (10.103) yields for the corresponding momentum operators:

$$\hat{\pi}(\mathbf{x},t) = i\hbar\hat{\overline{\psi}}(\mathbf{x},t)\gamma^0 = i\hbar\hat{\psi}^\dagger(\mathbf{x},t)\,, \tag{10.144}$$

$$\hat{\boldsymbol{\pi}}(\mathbf{x},t) = \epsilon_0\left\{\frac{\partial\hat{\mathbf{A}}(\mathbf{x},t)}{\partial t} + \boldsymbol{\nabla}\hat{\varphi}(\mathbf{x},t)\right\}\,, \tag{10.145}$$

where the scalar field operator follows from (10.100):

$$\hat{\varphi}(\mathbf{x},t) = \int d^3x'\,\frac{q\,\hat{\psi}^\dagger(\mathbf{x}',t)\hat{\psi}(\mathbf{x}',t)}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}'|}\,. \tag{10.146}$$

Thus, we can also use instead of the momentum field operators $\hat{\pi}_\alpha(\mathbf{x},t)$ and $\hat{\pi}_l(\mathbf{x},t)$ the field operators $\hat{\psi}^\dagger_\alpha(\mathbf{x},t)$ and $\partial\hat{A}_k(\mathbf{x},t)/\partial t$ in order to define the underlying equal-time (anti-)commutator relations of spinor QED. For instance, (10.141) can be directly rewritten as

$$\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}_\beta(\mathbf{x}',t)\right]_+ = \left[\hat{\psi}^\dagger_\alpha(\mathbf{x},t),\hat{\psi}^\dagger_\beta(\mathbf{x}',t)\right]_+ = 0\,,\ \left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}^\dagger_\beta(\mathbf{x}',t)\right]_+ = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}') \tag{10.147}$$

Accordingly, we obtain from (10.143) straight-forwardly

$$\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{A}_k(\mathbf{x}',t)\right]_- = \left[\hat{\psi}^\dagger_\alpha(\mathbf{x},t),\hat{A}_k(\mathbf{x}',t)\right]_- = 0\,. \tag{10.148}$$

Furthermore, taking into account (8.86), (10.142), (10.145), and (10.146) we get at first

$$\begin{aligned}\left[\hat{\psi}_\alpha(\mathbf{x},t),\frac{\partial\hat{A}_k(\mathbf{x}',t)}{\partial t}\right]_- &= \left[\hat{\psi}_\alpha(\mathbf{x},t),\frac{1}{\epsilon_0}\,\hat{\pi}_k(\mathbf{x}',t)\right]_- - \left[\hat{\psi}_\alpha(\mathbf{x},t),\partial'_k\hat{\varphi}(\mathbf{x}',t)\right]_- \\ &= -\frac{q}{4\pi\epsilon_0}\partial'_k\int d^3x''\,\frac{1}{|\mathbf{x}'-\mathbf{x}''|}\,\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}^\dagger_\beta(\mathbf{x}'',t)\hat{\psi}_\beta(\mathbf{x}'',t)\right]_-\,.\end{aligned} \tag{10.149}$$

Applying the operator identity (3.94) and (10.141) this reduces to

$$\left[\hat{\psi}_\alpha(\mathbf{x},t), \frac{\partial \hat{A}_k(\mathbf{x}',t)}{\partial t}\right]_- = \frac{q}{4\pi\epsilon_0}\left[\partial_k \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\hat{\psi}_\alpha(\mathbf{x},t) = -\frac{q}{4\pi\epsilon_0}\frac{(\mathbf{x}-\mathbf{x}')_k}{|\mathbf{x}-\mathbf{x}'|^3}\hat{\psi}_\alpha(\mathbf{x},t)\,. \quad (10.150)$$

Similarly we also yield

$$\left[\hat{\psi}^\dagger_\alpha(\mathbf{x},t), \frac{\partial \hat{A}_k(\mathbf{x}',t)}{\partial t}\right]_- = -\frac{q}{4\pi\epsilon_0}\frac{(\mathbf{x}'-\mathbf{x}')_k}{|\mathbf{x}'-\mathbf{x}'|^3}\hat{\psi}^\dagger_\alpha(\mathbf{x},t)\,. \quad (10.151)$$

Note that the non-locality of the commutator relations (10.149) and (10.151) is typical for the Coulomb gauge used here. Finally, we also convert (10.142) correspondingly. At first we get

$$\left[\hat{A}_k(\mathbf{x},t), \hat{A}_l(\mathbf{x}',t)\right]_- = 0 \quad (10.152)$$

and then we take into account (10.142), (10.145), and (10.146) in order to yield

$$\left[\hat{A}_k(\mathbf{x},t), \frac{\partial \hat{A}_l(\mathbf{x}',t)}{\partial t}\right]_- = \left[\hat{A}_k(\mathbf{x},t), \frac{1}{\epsilon_0}\hat{\pi}_l(\mathbf{x}',t)\right]_- - \left[\hat{A}_k(\mathbf{x},t), \partial'_l\hat{\varphi}_l(\mathbf{x}',t)\right]_-$$
$$= \frac{i\hbar}{\epsilon_0}\delta^T_{kl}(\mathbf{x}-\mathbf{x}') - \partial'_l\int d^3x''\,\frac{q}{4\pi\epsilon_0|\mathbf{x}'-\mathbf{x}''|}\left[\hat{A}_k(\mathbf{x},t), \hat{\psi}^\dagger_\alpha(\mathbf{x}'',t)\hat{\psi}_\alpha(\mathbf{x}'',t)\right]_- \,. \quad (10.153)$$

Applying (3.43) and (10.148) this reduces finally to

$$\left[\hat{A}_k(\mathbf{x},t), \frac{\partial \hat{A}_l(\mathbf{x}',t)}{\partial t}\right]_- = \frac{i\hbar}{\epsilon_0}\delta^T_{kl}(\mathbf{x}-\mathbf{x}')\,. \quad (10.154)$$

In the same way we also obtain

$$\left[\frac{\partial \hat{A}_k(\mathbf{x},t)}{\partial t}, \frac{\partial \hat{A}_l(\mathbf{x}',t)}{\partial t}\right]_- = \frac{1}{\epsilon_0^2}\left[\hat{\pi}_k(\mathbf{x},t), \hat{\pi}_l(\mathbf{x}',t)\right]_- + \partial_k\int d^3x''\,\frac{q}{4\pi\epsilon_0^2|\mathbf{x}-\mathbf{x}''|} \quad (10.155)$$
$$\times\left[\hat{\pi}_l(\mathbf{x}',t), \hat{\psi}^\dagger_\alpha(\mathbf{x}'',t)\hat{\psi}_\alpha(\mathbf{x}'',t)\right]_- - \partial'_l\int d^3x'''\frac{q}{4\pi\epsilon_0^2|\mathbf{x}'-\mathbf{x}'''|}\left[\hat{\pi}_k(\mathbf{x},t), \hat{\psi}^\dagger_\beta(\mathbf{x}''',t)\hat{\psi}_\beta(\mathbf{x}''',t)\right]_-$$
$$+\partial_k\partial'_l\int d^3x''\int d^3x'''\frac{q}{4\pi\epsilon_0^2|\mathbf{x}-\mathbf{x}''|}\,\frac{q}{4\pi\epsilon_0^2|\mathbf{x}'-\mathbf{x}'''|}\left[\hat{\psi}^\dagger_\alpha(\mathbf{x}'',t)\hat{\psi}_\alpha(\mathbf{x}'',t), \hat{\psi}^\dagger_\beta(\mathbf{x}''',t)\hat{\psi}_\beta(\mathbf{x}''',t)\right]_- \,.$$

Thus, finally, after applying (3.43), (10.142), (10.147), and (10.148) we end up with

$$\left[\frac{\partial \hat{A}_k(\mathbf{x},t)}{\partial t}, \frac{\partial \hat{A}_l(\mathbf{x}',t)}{\partial t}\right]_- = 0\,. \quad (10.156)$$

In the canonical field quantisation in the Dirac picture, the dynamics of the state vectors is determined according to (10.119) by the interacting part of the Hamilton operator. In spinor quantum electrodynamics it consists of two parts due to (10.111):

$$\hat{H}^{(\mathrm{int})}_D(t) = -qc\int d^3x : \hat{\overline{\psi}}(\mathbf{x},t)\boldsymbol{\gamma}\,\hat{\psi}(\mathbf{x},t) : \hat{\mathbf{A}}(\mathbf{x},t)$$
$$+\frac{q^2}{8\pi\epsilon_0}\int d^3x\int d^3x'\,\frac{:\hat{\overline{\psi}}(\mathbf{x},t)\gamma^0\hat{\psi}(\mathbf{x},t)\hat{\overline{\psi}}(\mathbf{x}',t)\gamma^0\hat{\psi}(\mathbf{x}',t):}{|\mathbf{x}-\mathbf{x}'|}\,. \quad (10.157)$$

Here, the normal ordering of the field operators was additionally used.

10.6 Time Evolution Operator

In the Dirac picture the interaction affects the dynamics of the state vectors according to (10.119). In order to investigate this in more detail we introduce the time evolution operator $\hat{U}(t_2, t_1)$, which connects the state vectors $|\psi_D(t_1)\rangle$ and $|\psi_D(t_2)\rangle$ at two consecutive times t_1 and t_2, respectively:

$$|\psi_D(t_2)\rangle = \hat{U}(t_2, t_1)|\psi_D(t_1)\rangle \,. \tag{10.158}$$

With the help of (10.114) and the formal solution of the Schrödinger equation (10.118)

$$|\psi_S(t_2)\rangle = e^{-i\hat{H}_S(t_2-t_1)/\hbar}|\psi_S(t_1)\rangle \tag{10.159}$$

we conclude

$$\begin{aligned} |\psi_D(t_2)\rangle &= e^{i\hat{H}_S^{(0)}t_2/\hbar}\,|\psi_S(t_2)\rangle = e^{i\hat{H}_S^{(0)}t_2/\hbar}\,e^{-i\hat{H}_S(t_2-t_1)/\hbar}\,|\psi_S(t_1)\rangle \\ &= e^{i\hat{H}_S^{(0)}t_2/\hbar}e^{-i\hat{H}_S(t_2-t_1)/\hbar}e^{-i\hat{H}_S^{(0)}t_1/\hbar}|\psi_D(t_1)\rangle \,. \end{aligned} \tag{10.160}$$

Thus, a comparison with (10.158) leads to a formal expression for the time evolution operator $\hat{U}(t_2, t_1)$:

$$\hat{U}(t_2, t_1) = e^{i\hat{H}_S^{(0)}t_2/\hbar}\,e^{-i\hat{H}_S(t_2-t_1)/\hbar}\,e^{i\hat{H}_S^{(0)}t_1/\hbar}\,. \tag{10.161}$$

Since the Hamilton operators $\hat{H}_S^{(0)}$ and $\hat{H}_S$ generally do not commute with each other, it is important to take into account the particular operator ordering in (10.161). With the help of the formal expression (10.161), various properties of the time evolution operator can be proved. It has the initial condition

$$\hat{U}(t_1, t_1) = 1 \tag{10.162}$$

and fulfills the group property

$$\hat{U}(t_3, t_2)\hat{U}(t_2, t_1) = \hat{U}(t_3, t_1)\,. \tag{10.163}$$

Indeed, we obtain from applying (10.161)

$$\begin{aligned} \hat{U}(t_3, t_2)\hat{U}(t_2, t_1) &= e^{i\hat{H}_S^{(0)}t_3/\hbar}\,e^{-i\hat{H}_S(t_3-t_2)/\hbar}\,e^{-i\hat{H}_S^{(0)}t_2/\hbar}\,e^{i\hat{H}_S^{(0)}t_2/\hbar}\,e^{-i\hat{H}_S(t_2-t_1)/\hbar}\,e^{-i\hat{H}_S^{(0)}t_1/\hbar} \\ &= e^{i\hat{H}_S^{(0)}t_3/\hbar}\,e^{-i\hat{H}_S(t_3-t_1)/\hbar}\,e^{-i\hat{H}_S^{(0)}t_1/\hbar} = \hat{U}(t_3, t_1)\,. \end{aligned} \tag{10.164}$$

Furthermore, we read off from evaluating (10.163) for $t_3 = t_1$ together (10.162) the inverse time evolution operator

$$\hat{U}^{-1}(t_2, t_1) = \hat{U}(t_1, t_2)\,. \tag{10.165}$$

And we deduce from (10.161) and (10.165) that the time evolution operator is unitary:

$$\hat{U}^{\dagger}(t_2, t_1) = e^{i\hat{H}_S^{(0)}t_1/\hbar}\,e^{-i\hat{H}_S(t_1-t_2)/\hbar}\,e^{-i\hat{H}_S^{(0)}t_2/\hbar} = \hat{U}(t_1, t_2) = \hat{U}^{-1}(t_2, t_1)\,. \tag{10.166}$$

Finally, we determine which differential equation the time evolution operator $\hat{U}(t_2, t_1)$ solves. Differentiating (10.161) with respect to t_2 and taking into account (10.113) yields

$$i\hbar \frac{\partial}{\partial t_2} \hat{U}^\dagger(t_2, t_1) = e^{i\hat{H}_S^{(0)} t_2/\hbar} \, \hat{H}_S^{(\text{int})} \, e^{-i\hat{H}_S^{(0)} t_2/\hbar} \, e^{i\hat{H}_S^{(0)} t_2/\hbar} \, e^{-i\hat{H}_S (t_2 - t_1)/\hbar} \, e^{-i\hat{H}_S^{(0)} t_1/\hbar} \,. \tag{10.167}$$

Thus, we conclude from (10.120), (10.161), and (10.167) that $\hat{U}(t_2, t_1)$ fulfills the differential equation

$$i\hbar \frac{\partial}{\partial t_2} \hat{U}(t_2, t_1) = \hat{H}_D^{(\text{int})}(t_2) \hat{U}(t_2, t_1) \,. \tag{10.168}$$

The initial value problem (10.162) and (10.168) can be formally rewritten in form of an integral equation:

$$\hat{U}(t_2, t_1) = 1 - \frac{i}{\hbar} \int_{t_1}^{t_2} dt_1' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{U}(t_1', t_1) \,. \tag{10.169}$$

Successively reinserting the left-hand side of (10.169) into the right-hand side, one obtains the von Neumann series

$$\begin{aligned} \hat{U}(t_2, t_1) = 1 - \frac{i}{\hbar} \int_{t_1}^{t_2} dt_1' \, \hat{H}_D^{(\text{int})}(t_1') + \left(\frac{-i}{\hbar} \right)^2 \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') + \ldots \\ + \left(\frac{-i}{\hbar} \right)^n \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \ldots \int_{t_1}^{t_{n-1}'} dt_n' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') \cdots \hat{H}_D^{(\text{int})}(t_n') + \ldots \,. \end{aligned} \tag{10.170}$$

It is noticeable in the nth summand of the von Neumann series that the time arguments of the multiple integrals are ordered in decreasing order: $t_1' > t_2' > \ldots > t_n'$. According to an idea of Freeman Dyson, all n integrals can be rewritten such that they are all performed over the same interval $[t_1, t_2]$ by using the time-ordered product of operators. Although the time ordering of operators has already been introduced previously for calculating the propagators of the Klein-Gordon field, the Maxwell field, and the Dirac field in the Chapters 7–9, its original motivation becomes apparent only now. To this end we consider exemplarily the second term in the von Neumann series (10.170) and reorganize it as follows:

$$\int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') = \int_{t_1}^{t_2} dt_2' \int_{t_2'}^{t_2} dt_1' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') \,. \tag{10.171}$$

Here we use the fact that the hatched triangle in Fig. 10.1 can be integrated in two ways. Either we first integrate over t_2' and then over t_1' or, conversely, first over t_1' and then over t_2'. Exchanging both integration variables at the right-hand side of (10.171) we conclude

$$\begin{aligned} 2 \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') = \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') \\ + \int_{t_1}^{t_2} dt_1' \int_{t_1'}^{t_2} dt_2' \, \hat{H}_D^{(\text{int})}(t_2') \, \hat{H}_D^{(\text{int})}(t_1') = \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_1' \, \Theta(t_1' - t_2') \, \hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') \\ + \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_1' \, \Theta(t_2' - t_1') \, \hat{H}_D^{(\text{int})}(t_2') \, \hat{H}_D^{(\text{int})}(t_1') = \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_1' \, \hat{T} \left(\hat{H}_D^{(\text{int})}(t_1') \, \hat{H}_D^{(\text{int})}(t_2') \right) \,. \end{aligned} \tag{10.172}$$

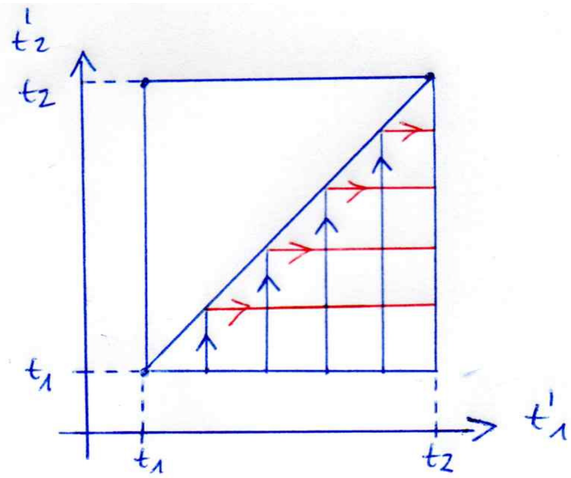

Figure 10.1: The hatched triangle can be integrated in two ways, which allows to rearrange the integral (10.171).

In the last step we assumed that the interacting Hamilton operator in the Dirac picture $\hat{H}_D^{(\mathrm{int})}(t)$ is bosonic, so the time ordering was used for two bosonic operators whose time order is not yet fixed:

$$\hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t_1')\hat{H}_D^{(\mathrm{int})}(t_2')\right) = \Theta(t_1'-t_2')\hat{H}_D^{(\mathrm{int})}(t_1')\hat{H}_D^{(\mathrm{int})}(t_2') + \Theta(t_2'-t_1')\hat{H}_D^{(\mathrm{int})}(t_2')\hat{H}_D^{(\mathrm{int})}(t_1') \,. \quad (10.173)$$

Analogous to (10.172), also all other terms in the von Neumann series (10.170) can be rewritten as multiple integrals over the entire interval $[t_1, t_2]$ with the help of the time-ordered product of operators. In the case of the nth-order term, one has to take into account in total $n!$ permutations of the time arguments. Therefore the generalisation of (10.172) reads

$$\begin{aligned} & n! \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} dt_2' \cdots \int_{t_1}^{t_{n-1}'} dt_n' \, \hat{H}_D^{(\mathrm{int})}(t_1')\, \hat{H}_D^{(\mathrm{int})}(t_2') \cdots \hat{H}_D^{(\mathrm{int})}(t_n') \\ & = \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_2' \cdots \int_{t_1}^{t_2} dt_n' \, \hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t_1')\, \hat{H}_D^{(\mathrm{int})}(t_2') \cdots \hat{H}_D^{(\mathrm{int})}(t_n')\right) . \end{aligned} \quad (10.174)$$

This obvious result can be proven by complete induction. With the help of (10.174) the von Neumann series (10.170) for the time evolution operator is finally given by

$$\hat{U}(t_2, t_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{t_1}^{t_2} dt_1' \cdots \int_{t_1}^{t_2} dt_n' \, \hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t_1') \cdots \hat{H}_D^{(\mathrm{int})}(t_n')\right) . \quad (10.175)$$

We can explicitly verify that the von Neumann series (10.175) solves the differential equation (10.168). Differentiating (10.175) with respect to t_2 we obtain due to the symmetry of the integrand with respect to the integration variables $t_1', t_2', ..., t_n'$:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t_2} \hat{U}(t_2, t_1) \;=\; & \sum_{n=1}^{\infty} \frac{i\hbar}{n!} \left(\frac{-i}{\hbar}\right)^n n \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_2} dt_2' \cdots \int_{t_1}^{t_2} dt_{n-1}' \\ & \times \hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t_1')\, \hat{H}_D^{(\mathrm{int})}(t_2') \cdots \hat{H}_D^{(\mathrm{int})}(t_{n-1}')\hat{H}_D^{(\mathrm{int})}(t_2)\right) . \end{aligned} \quad (10.176)$$

Due to the fact that the time t_2 is larger than all remaining integration variables t'_1, t'_2, ..., t'_{n-1} and using the definition (10.173) of the time-ordered product of operators, one can pull the operator $\hat{H}_D^{(\mathrm{int})}(t_2)$ out of the time ordering and obtain together with (10.175)

$$i\hbar\frac{\partial}{\partial t_2}\,\hat{U}(t_2,t_1) = \hat{H}_D^{(\mathrm{int})}(t'_2)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{-i}{\hbar}\right)^{n-1}\int_{t_1}^{t_2}dt'_1\int_{t_1}^{t_2}dt'_2\cdots\int_{t_1}^{t_2}dt'_{n-1}$$
$$\times\hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t'_1)\hat{H}_D^{(\mathrm{int})}(t'_2)\cdots\hat{H}_D^{(\mathrm{int})}(t'_{n-1})\right) = \hat{H}_D^{(\mathrm{int})}(t_2)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^{n}\int_{t_1}^{t_2}dt'_1\int_{t_1}^{t_2}dt'_2\cdots\int_{t_1}^{t_2}dt'_n$$
$$\times\hat{T}\left(\hat{H}_D^{(\mathrm{int})}(t'_1)\hat{H}_D^{(\mathrm{int})}(t'_2)\cdots\hat{H}_D^{(\mathrm{int})}(t'_n)\right) = \hat{H}_D^{(\mathrm{int})}(t_2)\hat{U}(t_2,t_1)\,. \tag{10.177}$$

Formally, the von Neumann series (10.175) can be summed up to a time-ordered exponential function:

$$\hat{U}(t_2,t_1) = \hat{T}\exp\left\{\frac{-i}{\hbar}\int_{t_1}^{t_2}dt\,\hat{H}_D^{(\mathrm{int})}(t)\right\}. \tag{10.178}$$

By taking into account that the time evolution operator (10.178) is defined by the von Neumann series (10.175) one can calculate perturbatively the cross sections of scattering processes.

10.7 Scattering Operator

We now consider a generic scenario for a scattering problem in the realm of relativistic quantum field theory. To this end we denote with $|\psi(t)\rangle$ a time-dependent state vector, which evolves starting from an initial state $|\psi_i\rangle$ in the limit $t\to-\infty$:

$$|\psi(-\infty)\rangle = |\psi_i\rangle\,. \tag{10.179}$$

The time evolution of the state vector $|\psi(t)\rangle$ under the influence of the interaction is determined in the Dirac picture by the time evolution operator $\hat{U}(t,-\infty)$ according to (10.158):

$$|\psi(t)\rangle = \hat{U}(t,-\infty)|\psi_i\rangle\,. \tag{10.180}$$

The scattering matrix S_{fi} denotes then the projection of the state vector $|\psi(t)\rangle$ in the limit $t\to+\infty$ onto the final state $|\psi_f\rangle$:

$$S_{fi} = \lim_{t\to+\infty}\langle\psi_f|\psi(t)\rangle\,. \tag{10.181}$$

From the knowledge of the scattering matrix (10.181) all observable quantities such as the scattering cross sections and decay rates can be calculated from the square of its absolute values and some kinetic considerations. According to (10.180) and (10.181), the probability amplitude S_{fi} for the transition from $|\psi_i\rangle$ to $|\psi_f\rangle$ can also be calculated as the matrix element

$$S_{fi} = \langle\psi_f|\hat{S}|\psi_i\rangle \tag{10.182}$$

of the scattering operator

$$\hat{S} = \hat{U}(+\infty, -\infty)\,. \tag{10.183}$$

According to (10.178) the scattering operator is explicitly given by

$$\hat{S} = \hat{T}\, \exp\left\{ \frac{-i}{\hbar} \int_{-\infty}^{+\infty} dt\, \hat{H}_D^{(\mathrm{int})}(t) \right\}\,. \tag{10.184}$$

In spinor quantum electrodynamics, the scattering operator reads according to (10.157) and (10.184):

$$\begin{aligned} \hat{S} \;=\; & \hat{T} \exp\left\{ \frac{iq}{\hbar} \int d^4x : \hat{\overline{\psi}}(x) \boldsymbol{\gamma} \hat{\psi}(x) : \hat{\mathbf{A}}(x) \right. \\ & \left. - \frac{iq^2}{8\pi\hbar\epsilon_0} \int dt \int d^3x \int d^3x' \, \frac{: \hat{\overline{\psi}}(\mathbf{x},t)\gamma^0 \hat{\psi}(\mathbf{x},t) \hat{\overline{\psi}}(\mathbf{x}',t)\gamma^0\hat{\psi}(\mathbf{x}',t) :}{|\mathbf{x}-\mathbf{x}'|} \right\}\,. \end{aligned} \tag{10.185}$$

Expanding the scattering operator up to the second order in the charge q, we obtain:

$$\begin{aligned} \hat{S} \;=\; & 1 + \frac{iq}{\hbar} \int d^4x : \hat{\overline{\psi}}(x) \boldsymbol{\gamma} \hat{\psi}(x) : \hat{\mathbf{A}}(x) \\ & - \frac{iq^2}{8\pi\hbar\epsilon_0} \int dt \int d^3x \int d^3x' \, \frac{: \hat{\overline{\psi}}(\mathbf{x},t)\gamma^0 \hat{\psi}(\mathbf{x},t) \hat{\overline{\psi}}(\mathbf{x}',t)\gamma^0\hat{\psi}(\mathbf{x}',t) :}{|\mathbf{x}-\mathbf{x}'|} \\ & + \frac{1}{2}\left(\frac{iq}{\hbar}\right)^2 \int d^4x \int d^4x'\, \hat{T} \left\{ : \left[\hat{\overline{\psi}}(x) \boldsymbol{\gamma} \overline{\psi}(x) : \hat{\mathbf{A}}(x)\right] \left[: \hat{\overline{\psi}}(x') \boldsymbol{\gamma} \hat{\psi}(x') \; : \hat{\mathbf{A}}(x')\right] \right\} + \ldots\,. \end{aligned} \tag{10.186}$$

We summarize that (10.182) and (10.186) represent the starting point for determining the cross sections of scattering processes.