Chapter 11

Møller Scattering

In the last chapter we apply our previous findings in order to calculate the cross section for the concrete example of an elastic scattering of two electrons:

$$e^- \, e^- \rightarrow e^- \, e^- \, . \tag{11.1}$$

This represents a paradigmatic scattering process in quantum field theory, which is named after the Danish physicist Christian Møller. The interaction between two electrons, that is idealized in the Møller scattering, forms the theoretical basis of many familiar physical phenomena such as, for instance, the repulsion between the two electrons of the helium atom. Furthermore, Møller scattering is a fundamental, purely pointlike process in quantum electrodynamics, which provides an important means to test the standard model of elementary particle physics. In addition, it is the dominant physical process in low-energy (< 100 MeV) electron scattering experiments. Thus, it is an important constraint in the design of electron scattering experiments that search for new physics beyond the standard model.

First we apply the perturbative technique worked out in Chapter 10 and determine the scattering matrix in the leading non-vanishing order, which turns out to be the quadratic one. Due to an intriguing cancellation of non-covariant terms the result is finally manifestly covariant and consists of two expressions. Taking into account the Feyman rules these two analytic expressions can be graphically represented in terms of Feynman diagrams. Secondly, we assume that the polarization is unknown for both the initial and the final electrons. This allows to average the square of the scattering matrix with respect to the polarizations of the involved electrons. The corresponding evaluation is quite technical and relies basically on the Clifford algebra of the Dirac matrices. Thirdly we analyze in detail the kinematics of such a two-particle scattering process by introducing the Lorentz-invariant Mandelstam variables. In particular, we specialize the relativistic scattering problem for two particles to the center of mass reference frame. This allows to express the Mandelstam variables just in terms of the scattering energy and the scattering angle. And, finally, we determine the scattering cross section for the Møller scattering and discuss both the ultra-relativistic and the non-relativistic limit. In the latter case we find that the Rutherford scattering formula is recovered for the forward peak.

11.1 Scattering Matrix

In the case of Møller scattering, one investigates a scattering process, where two electrons in the initial state

$$|\psi_i\rangle = |\mathbf{p}_{i_1}, s_{i_1}; \mathbf{p}_{i_2}, s_{i_2}\rangle \tag{11.2}$$

change into two electrons in the final state

$$|\psi_f\rangle = |\mathbf{p}_{f_1}, s_{f_1}; \mathbf{p}_{f_2}, s_{f_2}\rangle\,. \tag{11.3}$$

In the following we determine the matrix element of the scattering operator (10.186) up to the second order in the charge $q = -e$ with respect to the initial state (11.2) and the final state (11.3) according to (10.182). We observe that the zeroth order vanishes, since both states are orthogonal to each other for different momenta $\mathbf{p}_{i_1}, \mathbf{p}_{i_2} \neq \mathbf{p}_{f_1}, \mathbf{p}_{f_2}$:

$$\langle \mathbf{p}_{f_1}, s_{f_1}; \mathbf{p}_{f_2}, s_{f_2} | \mathbf{p}_{i_1}, s_{i_1}; \mathbf{p}_{i_2}, s_{i_2}\rangle = 0\,. \tag{11.4}$$

Furthermore, also the first order disappears, since both the initial and the final state (11.2) and (11.3) do not contain any photon and the first-order term in the scattering operator (10.186) involves the operator of the vector potential, whose plane wave decomposition (8.154) contains the annihilation and the creation of a photon. Therefore, the lowest non-vanishing perturbative order is the quadratic one, which turns out to consist of two contributions:

$$S_{fi}^{(2)} = S_{fi}^{(2,\text{inst})} + S_{fi}^{(2,\text{rad})}\,. \tag{11.5}$$

The first contribution stems from the instantaneous Coulomb self-interaction of the Dirac field

$$S_{fi}^{(2,\text{inst})} = \frac{-ie^2}{8\pi\hbar\epsilon_0 c^2} \int dt \int d^3x \int d^3x' \, \frac{\langle \psi_f | : \hat{j}^0(\mathbf{x},t)\hat{j}^0(\mathbf{x}',t) : |\psi_i\rangle}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{11.6}$$

while the second contribution represents an interaction between the Dirac and the Maxwell field:

$$S_{fi}^{(2,\text{rad})} = -\frac{e^2}{2\hbar^2 c^2} \int d^4x \int d^4x' \, \langle \psi_f | \hat{T}\left\{ : \hat{j}^k(x)\hat{A}_k(x) : : \hat{j}^l(x')\hat{A}_l(x') : \right\} |\psi_i\rangle\,. \tag{11.7}$$

Note that in (11.6) the time-like and in (11.7) the space-like components of the four-vector current density operator (10.91) occur, respectively:

$$\hat{j}^\mu(x) = c\hat{\overline{\psi}}(x)\gamma^\mu\hat{\psi}(x)\,. \tag{11.8}$$

Here we take into account the plane wave decompositions of the spinor field operators (9.431) and (9.432), which we rewrite according to

$$\hat{\overline{\psi}}(x) = \int d^3p_2 \sum_{s_2} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}} \left\{ e^{ip_2x/\hbar}\,\overline{u}(\mathbf{p}_2,s_2)\hat{b}^\dagger_{\mathbf{s}_2,s_2} + e^{-ip_2x/\hbar}\,\overline{v}(\mathbf{p}_2,s_2)\hat{d}_{\mathbf{p}_2,s_2}\right\}\,, \tag{11.9}$$

$$\hat{\psi}(x) = \int d^3p_1 \sum_{s_1} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}} \left\{ e^{-ip_1x/\hbar}\,u(\mathbf{p}_1,s_1)\hat{b}_{\mathbf{p}_1,s_1} + e^{ip_1x/\hbar}\,v(\mathbf{p}_1,s_1)\hat{d}^\dagger_{\mathbf{p}_1,s_1}\right\}\,, \tag{11.10}$$

where $s=\pm 1/2$ denotes the helicity. With this one obtains for the four-vector current density operator (11.8) the decomposition

$$\hat{j}^u(x) = c\int d^3p_1 \int d^3p_2 \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}\left\{e^{i(p_2-p_1)x/\hbar}\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1}\right.$$
$$+e^{i(p_2+p_1)x/\hbar}\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu v(\mathbf{p}_1,s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{d}^\dagger_{\mathbf{p}_1,s_1} + e^{-i(p_1+p_2)x/\hbar}\,\overline{v}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\hat{d}_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1}$$
$$\left.+e^{-i(p_2-p_1)x/\hbar}\,\overline{v}(\mathbf{p}_2,s_2)\gamma^\mu v(\mathbf{p}_1,s_1)\hat{d}_{\mathbf{p}_2,s_2}\hat{d}^\dagger_{\mathbf{p}_1,s_1}\right\}\,. \tag{11.11}$$

Evaluating the matrix element of the normal ordered operator $:\hat{j}^0(\mathbf{x},t)\hat{j}^0(\mathbf{x}',t):$ with the states

$$\langle \mathbf{p}_{f_1}, s_{f_1}; \mathbf{p}_{f_2}, s_{f_2}| = \langle 0|\,\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\,, \tag{11.12}$$
$$|\mathbf{p}_{i_1}, s_{i_1}; \mathbf{p}_{i_2}, s_{i_2}\rangle = \hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\,|0\rangle\,, \tag{11.13}$$

then only the first summand in (11.11) leads to a non-vanishing contribution. For the instantaneous self-interaction of the Dirac field (11.6) this results in

$$S^{(2,\text{inst})}_{fi} = \frac{-ie^2}{8\pi\hbar\epsilon_0 c^2}\int dt\int d^3x\int d^3x'\frac{1}{|\mathbf{x}-\mathbf{x}'|}\int d^3p_1\int d^3p_2\int d^3p_3\int d^3p_4\sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4}$$
$$\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}}\,e^{i\left(E_{\mathbf{p}_2}-E_{\mathbf{p}_1}\right)t/\hbar}e^{-i(\mathbf{p}_2-\mathbf{p}_1)\mathbf{x}/\hbar}$$
$$\times\overline{u}(\mathbf{p}_2,s_2)\gamma^0 u(\mathbf{p}_1,s_1)\,e^{i\left(E_{\mathbf{p}_4}-E_{\mathbf{p}_3}\right)t/\hbar}e^{-i(\mathbf{p}_4-\mathbf{p}_3)\mathbf{x}'/\hbar}\,\overline{u}(\mathbf{p}_4,s_4)\gamma^0 u(\mathbf{p}_3,s_3)$$
$$\times C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \tag{11.14}$$

Here we have introduced a vacuum expectation value of creation and annihilation operators as an abbreviation:

$$C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4) = \langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\,:\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\hat{b}_{\mathbf{p}_3,s_3}:\,\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}|0\rangle$$
$$= -\,\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\hat{b}_{\mathbf{p}_1,s_1}\hat{b}_{\mathbf{p}_3,s_3}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},si_2}|0\rangle\,, \tag{11.15}$$

where the evaluation of the normal ordering led to a minus sign due to the anti-commutator algebra of the fermionic operators (9.405). Afterwards, we evaluate the interaction (11.7) between the Dirac and the Maxwell fields. Here we use the bosonic definition of the time-ordering operator (7.124) and note that the operators $\hat{j}^k(x)$ and $\hat{A}_k(x)$ interchange with each other. Furthermore, taking into account the initial and the final state defined according to (11.2), (11.3), (11.12), and (11.13) yields

$$S^{(2,\text{rad})}_{fi} = -\frac{e^2}{2\hbar^2c^2}\int d^4x\int d^4x'\left\{\Theta(x^0-x'^0)\,\langle\psi_f|:\hat{j}^k(x)\hat{A}_k(x)::\hat{j}^l(x')\hat{A}'_l(x')|\psi_i\rangle\right.$$
$$\left.+\Theta(x'^0-x^0)\,\langle\psi_f|:\hat{j}^l(x')\hat{A}'_l(x')::\hat{j}^k(x)\hat{A}_k(x):|\psi_i\rangle\right\} = -\frac{e^2}{2\hbar^2c^2}\int d^4x\int d^4x'$$
$$\times\left\{\Theta(x^0-x'^0)\,\langle 0|\hat{A}_\mu(x)\hat{A}_\nu(x')|0\rangle\,\langle\mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\mu(x)::\hat{j}^\nu(x'):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle\right.$$
$$\left.+\Theta(x'^0-x^0)\,\langle 0|\hat{A}_\nu(x)\hat{A}_\mu(x')|0\rangle\,\langle\mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\nu(x)::\hat{j}^\mu(x'):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle\right\}\,. \tag{11.16}$$

In the last step, we replaced the summations over the spatial indices k, l by summations over the spatio-temporal indices μ, ν, since we have $\hat{A}_0(x) = 0$ in the radiation gauge. The normal ordering of the four-current density operator (11.11) leads to

$$
\begin{aligned}
:\hat{j}^\mu(x) := c\int d^3p_1 \int d^3p_2 \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}} \qquad (11.17)\\
\times\Big\{ e^{i(p_2-p_1)x/\hbar}\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1} + e^{i(p_2+p_1)x/\hbar}\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu v(\mathbf{p}_1,s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{d}^\dagger_{\mathbf{p}_1,s_1}\\
+e^{-i(p_1+p_2)x/\hbar}\,\overline{v}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\hat{d}_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1} - e^{-i(p_2-p_1)x/\hbar}\,\overline{v}(\mathbf{p}_2,s_2)\gamma^\mu v(\mathbf{p}_1,s_1)\hat{d}^\dagger_{\mathbf{p}_1,s_1}\hat{d}_{\mathbf{p}_2,s_2}\Big\}\,.
\end{aligned}
$$

Note that the normal ordering affected only the last term by changing its sign. Evaluating the matrix element for the product of two normally ordered four-vector current density operators $:\hat{j}^\mu(x)::\hat{j}^\nu(x):$ with the states (11.12) and (11.13), then only the first summand in (11.17) leads in both cases to a non-vanishing contribution:

$$
\begin{aligned}
\langle \mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\mu(x)::\hat{j}^\nu(x'):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle = c^2\int d^3p_1\int d^3p_2\int d^3p_3\int d^3p_4\\
\times\sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_2}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_4}}}\,e^{i(p_2-p_1)x/\hbar}\\
\times\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\,e^{i(p_4-p_3)x'/\hbar}\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\,\tilde{C}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \qquad (11.18)
\end{aligned}
$$

The vacuum expectation value introduced here reads

$$
\begin{aligned}
&\tilde{C}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4) = \langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\hat{b}_{\mathbf{p}_3,s_3}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}|0\rangle\\
&= -\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\hat{b}_{\mathbf{p}_1,s_1}\hat{b}_{\mathbf{p}_3,s_3}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}|0\rangle\\
&+\delta(\mathbf{p}_1-\mathbf{p}_4)\delta_{s_1,s_4}\,\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_3,s_3}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}|0\rangle\,, \qquad (11.19)
\end{aligned}
$$

where we have applied the anti-commutator algebra of the fermionic operators (9.405). In (11.19) the second term disappears due to the different momenta of the initial and the final state (11.12), and (11.13). Indeed, as (11.19) contains two creation (annihilation) operators for the initial (final) states but only one annihilation (creation) operator for an intermediate state, there always remains one creation (annihilation) operator, which finally annihilates the bra (ket) vacuum. Thus, a comparison with (11.15) yields:

$$
\tilde{C}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4) = C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \qquad (11.20)
$$

We conclude from (11.14), (11.16), (11.18), and (11.20) that both contributions of the scattering matrix (11.5) depend on the same vacuum expectation value (11.15). We now evaluate the latter by iteratively applying the underlying anti-commutator relations (9.405):

$$
\begin{aligned}
&C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)=\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\left(\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\right)\left(\hat{b}_{\mathbf{p}_3,s_3}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\right)|0\rangle\\
&-\delta(\mathbf{p}_3-\mathbf{p}_{i_1})\delta_{s_3,s_{i_1}}\,\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\left(\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\right)|0\rangle=\langle 0|\,\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\\
&\times\left[\cancel{-\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}_{\mathbf{p}_1,s_1}}+\delta(\mathbf{p}_1-\mathbf{p}_{i_1})\delta_{s_1,s_{i_1}}\right]\left[\cancel{-\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}_{\mathbf{p}_3,s_3}}+\delta(\mathbf{p}_1-\mathbf{p}_{i_1})\delta_{s_1,s_{i_1}}\right]|0\rangle-\delta(\mathbf{p}_3-\mathbf{p}_{i_1})\delta_{s_3,si_1}\\
&\times\langle 0|\,\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}\left[\cancel{\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}_{\mathbf{p}_1,s_1}}+\delta(\mathbf{p}_1-\mathbf{p}_{i_2})\delta_{s_1,s_{i_2}}\right]|0\rangle=\Big\{\delta(\mathbf{p}_1-\mathbf{p}_{i_1})\delta_{s_1,s_{i_1}}\\
&\times\delta(\mathbf{p}_3-\mathbf{p}_{i_2})\delta_{s_3,s_{i_2}}-\delta(\mathbf{p}_1-\mathbf{p}_{i_2})\delta_{s_1,s_{i_2}}\delta(\mathbf{p}_3-\mathbf{p}_{i_1})\delta_{s_3,s_{i_1}}\Big\}\,\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}|0\rangle\;. \qquad (11.21)
\end{aligned}
$$

Here the terms crossed out terms do not contribute as the creation operator of an initial state annihilates the bra vacuum due to $\mathbf{p}_{i_1},\mathbf{p}_{i_2}\neq\mathbf{p}_{f_1},\mathbf{p}_{f_2}$. The remaining vacuum expectation value (11.21) results in

$$
\begin{aligned}
&\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_4,s_4}|0\rangle=-\,\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_{f_1},s_{f_1}}\hat{b}^\dagger_{\mathbf{p}_4,s_4}|0\rangle+\delta(\mathbf{p}_2-\mathbf{p}_{f_1})\delta_{s_2,s_{f_1}} \qquad (11.22)\\
&\times\langle 0|\hat{b}_{\mathbf{p}_{f_2},s_{f_2}}\hat{b}^\dagger_{\mathbf{p}_4,s_4}|0\rangle=\delta(\mathbf{p}_2-\mathbf{p}_{f_1})\delta_{s_2,s_{f_1}}\delta(\mathbf{p}_4-\mathbf{p}_{f_2})\delta_{s_4,s_{f_2}}-\delta(\mathbf{p}_2-\mathbf{p}_{f_2})\delta_{s_2,s_{f_2}}\delta(\mathbf{p}_4-\mathbf{p}_{f_1})\delta_{s_4,s_{f_1}}\;.
\end{aligned}
$$

Inserting (11.22) into (11.21) yields in total four terms:

$$
\begin{aligned}
&C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)=\delta(\mathbf{p}_{f_1}-\mathbf{p}_2)\delta_{s_{f_1},S_2}\delta(\mathbf{p}_{f_2}-\mathbf{p}_4)\delta_{s_{f_2},s_4}\delta(\mathbf{p}_{i_1}-\mathbf{p}_1)\delta_{s_{i_1},s_1}\\
&\times\delta(\mathbf{p}_{i_2}-\mathbf{p}_3)\delta_{s_{i_2},s_3}+\delta(\mathbf{p}_{f_1}-\mathbf{p}_4)\delta_{s_{f_1},s_4}\delta(\mathbf{p}_{f_2}-\mathbf{p}_2)\delta_{s_{f_2},s_2}\delta(\mathbf{p}_{i_1}-\mathbf{p}_3)\delta_{s_{i_1},s_3}\delta(\mathbf{p}_{i_2}-\mathbf{p}_1)\delta_{s_{i_2},s_1}\\
&-\delta(\mathbf{p}_{f_1}-\mathbf{p}_2)\delta_{s_{f_1},s_2}\delta(\mathbf{p}_{f_2}-\mathbf{p}_4)\delta_{s_{f_2},s_4}\delta(\mathbf{p}_{i_1}-\mathbf{p}_3)\delta_{s_{i_1},s_3}\delta(\mathbf{p}_{i_2}-\mathbf{p}_1)\delta_{s_{i_2},s_1}\\
&-\delta(\mathbf{p}_{f_1}-\mathbf{p}_4)\delta_{s_{f_1},s_4}\delta(\mathbf{p}_{f_2}-\mathbf{p}_4)\delta_{s_{f_2},s_4}\delta(\mathbf{p}_{i_1}-\mathbf{p}_1)\delta_{s_{i_1},s_1}\delta(\mathbf{p}_{i_2}-\mathbf{p}_3)\delta_{s_{i_2},s_3}\;. \qquad (11.23)
\end{aligned}
$$

We recognize that the vacuum expectation value (11.23) turns out to have the symmetry

$$
C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)=C(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1,\mathbf{p}_2;s_3,s_4,s_1,s_2)\,, \qquad (11.24)
$$

where both the initial and the final momenta as well as the helicities are exchanged with respect to each other. Therefore, the substitutions $\mathbf{p}_1,s_1\leftrightarrow\mathbf{p}_3,s_3$ and $\mathbf{p}_2,s_2\leftrightarrow\mathbf{p}_4,s_4$ in (11.18) lead with (11.20) to a corresponding symmetry of the matrix element

$$
\begin{aligned}
&\langle\mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\mu(x)::\hat{j}^\nu(x'):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle\\
&=\langle\mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\nu(x')::\hat{j}^\mu(x):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle\;. \qquad (11.25)
\end{aligned}
$$

Using (11.25) in (11.16), the latter reduces to

$$
\begin{aligned}
S_{fi}^{(2,\mathrm{rad})}&=-\frac{e^2}{2\hbar^2c^2}\int d^4x\int d^4x'\langle\mathbf{p}_{f_1},s_{f_1};\mathbf{p}_{f_2},s_{f_2}|:\hat{j}^\mu(x):\\
&\quad\times:\hat{j}^\nu(x'):|\mathbf{p}_{i_1},s_{i_1};\mathbf{p}_{i_2},s_{i_2}\rangle\,D_{\mu\nu}(x,x')\,, \qquad (11.26)
\end{aligned}
$$

where we have introduced as an abbreviation the Maxwell propagator

$$
D_{\mu\nu}(x,x')=\Theta(x^0-x'^0)\,\langle 0|\hat{A}_\mu(x)\hat{A}_\nu(x')|0\rangle+\Theta(x'^0-x^0)\,\langle 0|\hat{A}_\nu(x')\hat{A}_\mu(x)|0\rangle\;. \qquad (11.27)
$$

Substituting (11.18) and (11.20) into (11.26), we obtain for the interaction between the Dirac and the Maxwell field:

$$S_{fi}^{(2,\mathrm{rad})} = -\frac{e^2}{2\hbar^2} \int d^4x \int d^4x' \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \sum_{s_1} \sum_{s_2} \sum_{s_3} \sum_{s_4}$$
$$\times \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}} D_{\mu\nu}(x, x')\, e^{i(p_2-p_1)x/\hbar}$$
$$\times \overline{u}(\mathbf{p}_2, s_2)\gamma^\mu u(\mathbf{p}_1, s_1)\, e^{i(p_4-p_3)x'/\hbar}\, \overline{u}(\mathbf{p}_4, s_4)\gamma^\nu u(\mathbf{p}_3, s_3)\, C(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4; s_1, s_2, s_3, s_4)\,. \qquad (11.28)$$

Based on the previous results we now establish an intriguing connection between both contributions (11.14) and (11.28) of the scattering matrix (11.5). To this end we first use the Fourier expansion of the Coulomb potential in (11.14)

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{\mathbf{k}^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\,, \qquad (11.29)$$

so that the scattering matrix contribution from the instantaneous Coulomb self-interaction of the Dirac field (11.14) reduces to

$$S_{fi}^{(2,\mathrm{inst})} = \frac{-ie^2}{8\pi\hbar\epsilon_0} \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \sum_{s_1} \sum_{s_2} \sum_{s_3} \sum_{s_4} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}$$
$$\times \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}}\, \overline{u}(\mathbf{p}_2, s_2)\gamma^0 u(\mathbf{p}_1, s_1)\, \overline{u}(\mathbf{p}_4, s_4)\gamma^0 u(\mathbf{p}_3, s_3)\, C(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4; s_1, s_2, s_3, s_4)$$
$$\times \int d^3k\, \frac{1}{2\pi^2 \mathbf{k}^2} \int dt\, e^{i(E_{\mathbf{p}_2}+E_{\mathbf{p}_4}-E_{\mathbf{p}_1}-E_{\mathbf{p}_3})t/\hbar} \int d^3x\, e^{i(\hbar\mathbf{k}-\mathbf{p}_2+\mathbf{p}_1)\mathbf{x}/\hbar} \int d^3x'\, e^{i(-\hbar\mathbf{k}-\mathbf{p}_4+\mathbf{p}_3)\mathbf{x}'/\hbar}\,, \quad (11.30)$$

where the evaluation of the respective spatial and temporal integrals yields

$$\int dt\, e^{i(E_{\mathbf{p}_2}+E_{\mathbf{p}_4}-E_{\mathbf{p}_1}-E_{\mathbf{p}_3})t/\hbar} = \frac{2\pi\hbar}{c}\, \delta(p_2^0 + p_4^0 - p_1^0 - p_3^0)\,, \qquad (11.31)$$
$$\int d^3x\, e^{i(\hbar\mathbf{k}-\mathbf{p}_2+\mathbf{p}_1)\mathbf{x}/\hbar} = (2\pi\hbar)^3\, \delta(\hbar\mathbf{k} - \mathbf{p}_2 + \mathbf{p}_1)\,, \qquad (11.32)$$
$$\int d^3x'\, e^{i(-\hbar\mathbf{k}-\mathbf{p}_4+\mathbf{p}_3)\mathbf{x}'/\hbar} = (2\pi\hbar)^3\, \delta(\hbar\mathbf{k} - \mathbf{p}_4 + \mathbf{p}_3)\,. \qquad (11.33)$$

Substituting (11.31)–(11.33) into (11.30) and evaluating the **k**-integral finally leads to

$$S_{fi}^{(2,\mathrm{inst})} = \frac{-i\hbar e^2}{2\epsilon_0 c} \int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \sum_{s_1} \sum_{s_2} \sum_{s_3} \sum_{s_4} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}$$
$$\times \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}}\, (2\pi\hbar)^4 \delta(p_1 + p_4 - p_1 - p_3)\, \frac{1}{(\mathbf{p}_2 - \mathbf{p}_1)^2}\, \overline{u}(\mathbf{p}_2, s_2)\gamma^0 u(\mathbf{p}_1, s_1)$$
$$\times \overline{u}(\mathbf{p}_4, s_4)\gamma^0 u(\mathbf{p}_3, S_3)\, C(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4; s_1, s_2, s_3, s_4)\,. \qquad (11.34)$$

On the other hand, with the help of the four-dimensional Fourier representation of the Maxwell propagator (8.198)

$$D_{\mu\nu}(x, x') = \frac{i\hbar}{c\epsilon_0} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik(x-x')} P_{\mu\nu}(k) \qquad (11.35)$$

the scattering matrix contribution (11.28) stemming from the interaction between the Dirac and the Maxwell field yields

$$\begin{aligned} S_{fi}^{(2,\text{rad})} &= -\frac{ie^2}{2\pi\hbar c\epsilon_0}\int d^3p_1\int d^3p_2\int d^3p_3\int d^3p_4\sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_2}}} \\ &\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_4}}}\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}P_{\mu\nu}(k) \\ &\times C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\int d^4x\,e^{i(\hbar k+p_2-p_1)x/\hbar}\int d^4x'e^{i(-\hbar k+p_4-p_3)x'/\hbar}\,. \end{aligned} \tag{11.36}$$

The evaluation of the two spatio-temporal integrals results in

$$\begin{aligned} \int d^4x\,e^{i(\hbar k+p_2-p_1)x/\hbar} &= (2\pi\hbar)^4\,\delta(\hbar k+p_2-p_1) \\ \int d^4x'e^{i(-\hbar k+p_4-p_3)x'/\hbar} &= (2\pi\hbar)^4\,\delta(-\hbar k+p_4-p_3)\,, \end{aligned} \tag{11.37}$$

so that the k-integral in (11.36) can be evaluated as follows:

$$\begin{aligned} S_{fi}^{(2,\text{rad})} &= -\frac{ie^2\hbar}{2\epsilon_0 c}\int d^3p_1\int d^3p_2\int d^3p_3\int d^3p_4\sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{{}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_2}}} \\ &\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_4}}}\,(2\pi\hbar)^4\delta(p_2+p_4-p_1-p_3)\,\frac{P_{\mu\nu}(p_2-p_1)}{(p_2-p_1)^2} \\ &\times\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\,\overline{u}(_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\,C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \end{aligned} \tag{11.38}$$

Inserting the polarization sum from (8.203)

$$P_{\mu\nu}(k) = -g_{\mu\nu} - k^2\frac{\xi_\mu\xi_\nu}{(k\xi)^2-k^2} - \frac{k_\mu k_\nu+(k\xi)(k_\mu\xi_\nu+k_\nu\xi_\mu)}{(k\xi)^2-k^2} \tag{11.39}$$

into (11.38), it turns out that its last term does not contribute. Namely, due to the algebraic equations (9.303) and (9.305) determining the Dirac spinor $u(\mathbf{p},s)$ and the Dirac adjoint Dirac spinor $\overline{u}(\mathbf{p},s)$, we conclude

$$\begin{aligned} &\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)(p_{2\mu}-p_{1\mu}) = \left\{\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu p_{2\mu}\right\}u(\mathbf{p}_1,s_1) - \overline{u}(\mathbf{p}_2,s_2)\left\{\gamma^\mu p_{1\mu}u(\mathbf{p}_1,s_1)\right\} \\ &= -Mc\,\overline{u}(\mathbf{p}_2,s_2)u(\mathbf{p}_1,s_1) + Mc\,\overline{u}(\mathbf{p}_2,s_2)u(\mathbf{p}_1,s_1) = 0 \end{aligned} \tag{11.40}$$

and, analogously, we also obtain

$$\begin{aligned} &\delta(p_2+p_4-p_1-p_3)\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)(p_{2\nu}-p_{1\nu}) \\ &= \delta(p_2+p_4-p_1-p_3)\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)(p_{4\nu}-p_{3\nu}) = 0\,. \end{aligned} \tag{11.41}$$

Note that the identities (11.40) and (11.41) are a consequence of the charge conservation at a vertex and can be studied in more detail in the framework of the so-called Ward-Takahashi identities. From (11.38)–(11.41) we then conclude

$$S_{fi}^{(2,\text{rad})} = -\frac{i\hbar e^2}{2\epsilon_0 c}\int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}$$
$$\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}}\frac{(2\pi\hbar)^4\delta(p_2+p_4-p_1-p_3)}{(p_2-p_1)^2}\left\{-g_{\mu\nu}-\frac{(p_2-p_1)^2\xi_\mu\xi_\nu}{[(p_2-p_1)\xi]^2-(p_2-p_1)^2}\right\}$$
$$\times\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\,C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \tag{11.42}$$

Adding now both contributions (11.34) and (11.42) to the scattering matrix element (11.5) and taking into account the explicit form of the time-like vector ξ according to (8.200) yields a manifestly covariant result:

$$S_{fi}^{(2)} = \frac{i\hbar e^2}{2\epsilon_0 c}\int d^3p_1 \int d^3p_2 \int d^3p_3 \int d^3p_4 \sum_{s_1}\sum_{s_2}\sum_{s_3}\sum_{s_4} \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_1}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_2}}}$$
$$\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_3}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_4}}}\,(2\pi\hbar)^4\delta(p_2+p_4-p_1-p_3)\,\frac{g_{\mu\nu}}{(p_2-p_1)^2}$$
$$\times\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\,C(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4;s_1,s_2,s_3,s_4)\,. \tag{11.43}$$

Substituting the vacuum expectation value (11.23) into (11.43), the first two and the last two terms yield the same contribution, respectively, due to the obvious identity

$$\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1)\,\overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3) = \overline{u}(\mathbf{p}_4,s_4)\gamma^\nu u(\mathbf{p}_3,s_3)\,\overline{u}(\mathbf{p}_2,s_2)\gamma^\mu u(\mathbf{p}_1,s_1) \tag{11.44}$$

and the symmetry of the integrand with respect to the substitutions $\mathbf{p}_1, s_1 \leftrightarrow \mathbf{p}_3, s_3$ and $\mathbf{p}_2, s_2 \leftrightarrow \mathbf{p}_4, s_4$. This results in a factor of 2, which just compensates for the factor 1/2 stemming from the second order in the Taylor expansion of the exponential function:

$$S_{fi}^{(2)} = \frac{i\hbar e^2}{\epsilon_0 c}(2\pi\hbar)^4\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_{i_1}}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_{i_2}}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_{f_1}}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_{f_2}}}}$$
$$\times\left\{\frac{g_{\mu\nu}}{(p_{f_1}-p_{i_1})^2}\,\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_1},s_{i_1})\,\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma^\nu u(\mathbf{p}_{i_2},s_{i_2})\right.$$
$$\left.-\frac{g_{\mu\nu}}{(p_{f_1}-p_{i_2})^2}\,\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_2},s_{i_2})\,\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma^\nu u(\mathbf{p}_{i_1},s_{i_1})\right\}. \tag{11.45}$$

This perturbative result for the scattering matrix element of the Møller scattering can be represented in the form of two Feynman diagrams, which are depicted in Fig. 11.1. Note that no momentum integrals occur in (11.45), which would correspond to an internal loops in the Feynman diagrams. Therefore, one calles the graphs in Fig. 11.1 to be tree-level graphs. The corresponding manifestly covariant Feynman rules for converting the scattering matrix element (11.45) into the Feynman diagrams of Fig. 11.1 and vice versa read in momentum space as follows:

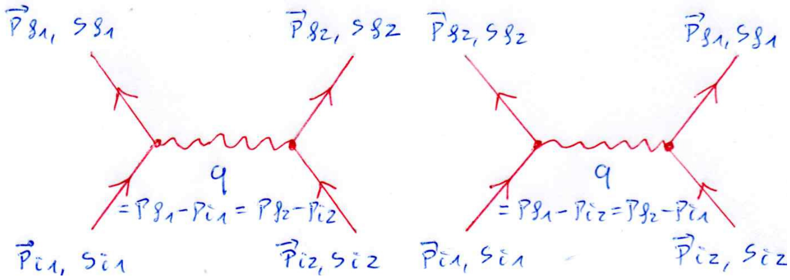

Figure 11.1: Direct (left) and exchange (right) Feynman diagram for the Møller scattering of two electrons.

(F1) The prefactor $(2\pi\hbar)^4\,\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})$ guarantees the conservation of energy and momentum in the scattering process.

(F2) An incoming electron corresponds to the factor $\sqrt{\dfrac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_i}}}\,u(\mathbf{p}_i, s_i)$.

(F3) An outgoing electron leads to the factor $\sqrt{\dfrac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}_f}}}\,\overline{u}(\mathbf{p}_f, s_f)$.

(F4) A vertex yields the factor $e\,\gamma^\mu$.

(F5) The Maxwell propagator corresponds to the covariant factor $\hbar g_{\mu\nu}/(\epsilon_0 c q^2)$, where q denotes the momentum transfer, see Fig. 11.1.

(F6) The phase of the scatter matrix element is calculated according to the following rule: $(-i)^{\text{number of vertices}}\,(-i)^{\text{number of inner lines}}$. Here, the minus sign for the number of vertices comes from the negative charge of the electron, while the minus sign for the inner line stems from the Maxwell propagator.

The phase rule (F6) leads directly to the correct phase of the direct graph: $(-i)^2(-i)^1 = +i$. Due to the indistinguishability of the two incoming and outgoing electrons, apart from the direct graph also the exchange graph contributes, where in the latter the two outgoing electrons are swapped. Due to the Fermi-Dirac statistics of the electrons, the exchange graph has an additional minus sign. Consequently, the entire scattering matrix is anti-symmetric with respect to the exchange of the two incoming or outgoing electrons. If we had calculated the scattering of identical bosons, the exchange graph would have the same sign as the direct graph and the total scattering amplitude would be symmetrical with respect to the exchange of the two incoming and outgoing bosons. Note that the Feynman diagrams in quantum electrodynamics always have the multiplicities ± 1 in contrast to other field theories such as the ϕ^4-theory of critical phenomena, where the multiplicities are highly non-trivial as they follow from involved combinatorial reasons.

11.2 Polarization Averaging

The second perturbative order of the Møller scattering matrix element in (11.45) factorises according to

$$\begin{aligned} S_{fi}^{(2)} &= \frac{i\hbar e^2}{\epsilon_0 c}\,(2\pi\hbar)^4\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{i_1}}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{i_2}}}} \\ &\quad\times\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{f_1}}}}\sqrt{\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{f_2}}}}\,M_{fi}^{(2)}\,, \end{aligned} \tag{11.46}$$

where we introduced the matrix element

$$\begin{aligned} M_{fi}^{(2)} &= \frac{g_{\mu\nu}}{(p_{f_1}-p_{i_1})^2}\,\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_1},s_{i_1})\,\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma^\nu u(\mathbf{p}_{i_2},s_{i_2}) \\ &\quad-\frac{g_{\mu\nu}}{(p_{f_1}-p_{i_2})^2}\,\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_2},s_{i_2})\,\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma^\nu u(\mathbf{p}_{i_1},s_{i_1})\,. \end{aligned} \tag{11.47}$$

Provided that the polarizations of both the incoming and the outgoing electrons are not detected during the scattering process, we have to calculate the scattering cross section from averaging the squared matrix element over all these polarisations:

$$\overline{|M_{fi}^{(2)}|^2} = \frac{1}{4}\sum_{s_{i_1}}\sum_{s_{i_2}}\sum_{s_{f_1}}\sum_{s_{f_2}}|M_{fi}^{(2)}|^2\,. \tag{11.48}$$

Substituting (11.47) into (11.48) leads in total to four terms:

$$\begin{aligned} \overline{|M_{fi}^{(2)}|^2} &= \frac{1}{4}\sum_{s_{i_1}}\sum_{s_{i_2}}\sum_{s_{f_1}}\sum_{s_{f_2}}\Bigg\{\frac{1}{(p_{f_1}-p_{i_1})^4}\left[\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_1},s_{i_1})\right]^*\left[\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma_\mu u(_{i_2},s_{i_2})\right]^* \\ &\quad\times\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\nu u(\mathbf{p}_{i_1},s_{i_1})\,\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma_\nu u(\mathbf{p}_{i_2},s_{i_2}) \\ &\quad-\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\left[\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma^\mu u(\mathbf{p}_{i_1},s_{i_1})\right]^*\left[\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma_\mu u(\mathbf{p}_{i_2},s_{i_2})\right]^* \\ &\quad\times\overline{u}(\mathbf{p}_{f_2},s_{f_2})\gamma^\nu u(\mathbf{p}_{i_1},s_{i_1})\overline{u}(\mathbf{p}_{f_1},s_{f_1})\gamma_\nu u(\mathbf{p}_{i_2},s_{i_2})+(p_{f_1}\leftrightarrow p_{f_2})\Bigg\}\,. \end{aligned} \tag{11.49}$$

Calculating the expression $[\overline{u}(\mathbf{p}_1,s_1)\gamma^\mu u(\mathbf{p}_2,s_2)]^*$, we note that $\overline{u}(\mathbf{p}_1,s_1)\gamma^\mu u(\mathbf{p}_2,s_2)$ coincides with its transpose as it is a scalar:

$$\begin{aligned} &[\overline{u}(\mathbf{p}_1,s_1)\gamma^\mu\,u(\mathbf{p}_2,s_2)]^* = [\overline{u}(\mathbf{p}_1,s_1)\gamma^\mu u(\mathbf{p}_2,s_2)]^\dagger \\ &= u^\dagger(\mathbf{p}_2,s_2)(\gamma^\mu)^\dagger\,\overline{u}^\dagger(\mathbf{p}_1,s_1) = \overline{u}(\mathbf{p}_2,s_2)\gamma^0(\gamma^\mu)^\dagger\gamma^0 u(\mathbf{p}_1,s_1)\,. \end{aligned} \tag{11.50}$$

From the chiral representation of the Dirac matrices (9.94) follows due to the hermiticity of the four Pauli matrices σ^μ:

$$(\gamma^\mu)^\dagger = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{cases} (\gamma^0)^\dagger = \gamma^0 \\ (\gamma^i)^\dagger = -\gamma^i \end{cases}\,. \tag{11.51}$$

With this we conclude

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu \qquad \Longrightarrow \qquad \begin{cases} \gamma^0(\gamma^0)^\dagger\gamma^0 = \gamma^0\gamma^0\gamma^0 = \gamma^0 \\ \gamma^0(\gamma^i)^\dagger\gamma^0 = -\gamma^0\gamma^i\gamma^0 = \gamma^i\gamma^0\gamma^0 = \gamma^i \end{cases} . \tag{11.52}$$

Substituting (11.52) into (11.50) then leads to the result

$$[\overline{u}(\mathbf{p}_1, s_1)\gamma^\mu u(\mathbf{p}_2, s_2)]^* = \overline{u}(\mathbf{p}_2, s_2)\gamma^\mu \, u(\mathbf{p}_1, s_1) \,. \tag{11.53}$$

Using (11.53) in (11.49) yields

$$\begin{aligned} \overline{|M_{fi}^{(2)}|^2} \;=\; & \frac{1}{4}\sum_{s_{i_1}}\sum_{s_{i_2}}\sum_{s_{f_1}}\sum_{s_{f_2}}\left\{\frac{1}{(p_{f_1}-p_{i_1})^4}\,\overline{u}(\mathbf{p}_{i_1}, s_{i_1})\gamma^\mu u(\mathbf{p}_{f_1}, s_{f_1})\,\overline{u}(\mathbf{p}_{i_2}, s_{i_2})\gamma_\mu u(\mathbf{p}_{f_2}, s_{f_2})\right. \\ & \times\overline{u}(\mathbf{p}_{f_1}, s_{f_1})\gamma^\nu u(\mathbf{p}_{i_1}, s_{i_1})\,\overline{u}(\mathbf{p}_{f_2}, s_{f_2})\gamma_\nu u(\mathbf{p}_{i_2}, s_{i_2}) \\ & -\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\overline{u}(\mathbf{p}_{i_1}, s_{i_1})\gamma^\mu u(\mathbf{p}_{f_1}, s_{f_1})\,\overline{u}(\mathbf{p}_{i_2}, s_{i_2})\gamma_\mu u(\mathbf{p}_{f_2}, s_{f_2}) \\ & \left.\times\overline{u}(\mathbf{p}_{f_2}, s_{f_2})\gamma^\nu u(\mathbf{p}_{i_1}, s_{i_1})\,\overline{u}(\mathbf{p}_{f_1}, s_{f_1})\gamma_\nu u(\mathbf{p}_{i_2}, s_{i_2}) + (p_{f_1} \leftrightarrow p_{f_2})\right\}. \end{aligned} \tag{11.54}$$

As the factors $\overline{u}(\mathbf{p}_1, s_1)\gamma^\mu u(\mathbf{p}_2, s_2)$ are scalars, their order can be changed:

$$\begin{aligned} \overline{|M_{fi}^{(2)}|^2} \;=\; & \frac{1}{4}\sum_{s_{f_1}}\sum_{s_{f_2}}\left\{\frac{1}{(p_{f_1}-p_{i_1})^4}\,\overline{u}(\mathbf{p}_{f_1}, s_{f_1})\gamma^\nu\left[\sum_{s_{i_1}} u(\mathbf{p}_{i_1}, s_{i_1})\overline{u}(\mathbf{p}_{i_1}, s_{i_1})\right]\gamma^\mu u(\mathbf{p}_{f_1}, s_{f_1})\right. \\ & \times\overline{u}(\mathbf{p}_{f_2}, s_{f_2})\gamma_\nu\left[\sum_{s_{i_2}} u(\mathbf{p}_{i_2}, s_{i_2})\overline{u}(\mathbf{p}_{i_2}, s_{i_2})\right]\gamma_\mu u(\mathbf{p}_{f_2}, s_{f_2}) \\ & -\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\overline{u}(\mathbf{p}_{f_2}, s_{f_2})\gamma^\nu\left[\sum_{s_{i_1}} u(\mathbf{p}_{i_1}, s_{i_1})\overline{u}(\mathbf{p}_{i_1}, s_{i_1})\right]\gamma^\mu u(\mathbf{p}_{f_1}, s_{f_1}) \\ & \left.\times\overline{u}(\mathbf{p}_{f_1}, s_{f_1})\gamma_\nu\left[\sum_{s_{i_2}} u(\mathbf{p}_{i_2}, s_{i_2})\overline{u}(\mathbf{p}_{i_2}, s_{i_2})\right]\gamma_\mu u(\mathbf{p}_{f_2}, s_{f_2}) + (p_{f_1} \leftrightarrow p_{f_2})\right\}. \end{aligned} \tag{11.55}$$

The polarisation sums occurring here with respect to s_{i_1}, s_{i_2} were already calculated according to (9.436) and (9.445). We implement now this result by introducing for the sake of clarity spinorial indices and by using for notational brevity the Einstein summation convention that implies summation over identical spinorial indices:

$$\begin{aligned} \overline{|M_{fi}^{(2)}|^2} \;=\; & \frac{1}{4}\sum_{s_{f_1}}\sum_{s_{f_2}}\left\{\frac{1}{(p_{f_1}-p_{i_1})^4}\,\overline{u}_\alpha(\mathbf{p}_{f_1}, s_{f_1})\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\,u_\delta(\mathbf{p}_{f_1}, s_{f_1})\right. \\ & \times\overline{u}_{\alpha'}(\mathbf{p}_{f_2}, s_{f_2})\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}\gamma_{\mu\gamma'\delta'}\,u_{\delta'}(\mathbf{p}_{f_2}, s_{f_2}) \\ & -\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\overline{u}_\alpha(\mathbf{p}_{f_2}, s_{f_2})\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\,u_\delta(\mathbf{p}_{f_1}, s_{f_1}) \\ & \left.\times\overline{u}_{\alpha'}(\mathbf{p}_{f_1}, s_{f_1})\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}\gamma_{\mu\gamma'\delta'}\,u_{\delta'}(\mathbf{p}_{f_2}, s_{f_2}) + (p_{f_1} \leftrightarrow p_{f_2})\right\}. \end{aligned} \tag{11.56}$$

Paying attention to the respective spinorial indices, the individual terms can be rearranged as follows

$$\overline{|M_{fi}^{(2)}|^2} = \frac{1}{4}\Bigg\{ \frac{1}{(p_{f_1}-p_{i_1})^4}\,\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\left[\sum_{s_{f_1}} u_\delta(\mathbf{p}_{f_1},s_{f_1})\overline{u}_\alpha(\mathbf{p}_{f_1},s_{f_1})\right]$$
$$\times\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}\gamma_{\mu\gamma'\delta'}\left[\sum_{s_{f_2}} u_{\delta'}(\mathbf{p}_{f_2},s_{f_2})\overline{u}_{\alpha'}(\mathbf{p}_{f_2},s_{f_2})\right]$$
$$-\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\left[\sum_{s_{f_1}} u_\delta(\mathbf{p}_{f_1},s_{f_1})u_{\delta'}(\mathbf{p}_{f_1},s_{f_1})\right]$$
$$\times\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}\gamma_{\mu\gamma'\delta'}\left[\sum_{s_{f_2}} u_{\delta'}(\mathbf{p}_{f_2},s_{f_2})\overline{u}_{\alpha}(\mathbf{p}_{f_2},s_{f_2})\right] + (p_{f_1}\leftrightarrow p_{f_2})\Bigg\}\,. \quad (11.57)$$

Here we take into account that also the polarisation sums with respect to s_{f_1}, s_{f_2} were already calculated according to (9.436) and (9.445), yielding:

$$\overline{|M_{fi}^{(2)}|^2} = \frac{1}{4}\Bigg\{ \frac{1}{(p_{f_1}-p_{i_1})^4}\,\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\delta\alpha}\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}$$
$$\times\gamma_{\mu\gamma'\delta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\delta'\alpha'} - \frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\gamma^\nu_{\alpha\beta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\beta\gamma}\gamma^\mu_{\gamma\delta}\left(\frac{p_{i_1}+Mc}{2Mc}\right)_{\delta\alpha'}$$
$$\times\gamma_{\nu\alpha'\beta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\beta'\gamma'}\gamma_{\mu\gamma'\delta'}\left(\frac{p_{i_2}+Mc}{2Mc}\right)_{\delta'\alpha} + (p_{f_1}\leftrightarrow p_{f_2})\Bigg\}\,. \quad (11.58)$$

The sums with respect to the spinorial indices can be interpreted as traces:

$$\overline{|M_{fi}^{(2)}|^2} = \frac{1}{4}\Bigg\{ \frac{1}{(p_{f_1}-p_{i_1})^4}\,\mathrm{Tr}\left[\gamma^\nu\frac{p_{i_1}+Mc}{2Mc}\gamma^\mu\frac{p_{f_1}+Mc}{2Mc}\right]\,\mathrm{Tr}\left[\gamma_\nu\frac{p_{i_2}+Mc}{2Mc}\gamma_\mu\frac{p_{f_2}+Mc}{2Mc}\right] \quad (11.59)$$
$$-\frac{1}{(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\,\mathrm{Tr}\left[\gamma^\nu\frac{p_{i_1}+Mc}{2Mc}\gamma^\mu\frac{p_{f_1}+Mc}{2Mc}\gamma_\nu\frac{p_{i_2}+Mc}{2Mc}\gamma_\mu\frac{p_{f_2}+Mc}{2Mc}\right] + (p_{f_1}\leftrightarrow p_{f_2})\Bigg\}$$

The first contribution in (11.59) is called the direct term

$$\overline{|M_{fi}^{(2)}|^2_{\mathrm{d}}} = \frac{\mathrm{Tr}\left[\gamma^\nu(p_{i_1}+Mc)\gamma^\mu(p_{f_1}+Mc)\right]\,\mathrm{Tr}\left[\gamma_\nu(p_{i_1}+Mc)\gamma_\mu(p_{f_2}+Mc)\right]}{64M^4c^4(p_{f_1}-p_{i_1})^4}\,. \quad (11.60)$$

It consists of the product of two traces of the same design type

$$\mathrm{Tr}\left[\gamma^\mu(p_{i_1}+Mc)\gamma^\nu(p_{f_1}+Mc)\right] = \mathrm{Tr}\left[\gamma^\mu p_{i_1}\gamma^\nu p_{f_1} + Mc\gamma^\mu p_{i_1}\gamma^\nu + Mc\gamma^\mu\gamma^\nu p_{f_1} + M^2c^2\gamma^\mu\gamma^\nu\right]$$
$$= p_{i_1\kappa}p_{f_1\lambda}\mathrm{Tr}\left[\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\right] + Mc\,p_{i_1\kappa}\mathrm{Tr}\left[\gamma^\mu\gamma^\kappa\gamma^\nu\right] + Mc\,p_{i_1\lambda}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\lambda\right] + M^2c^2\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right]\,. \quad (11.61)$$

11.3 Traces of Product of Dirac Matrices

Thus, according to (11.61), we have now to calculate traces over different products of γ-matrices. Due to the explicit form of the Dirac matrices (9.94), the trace over each individual γ-matrix disappears:

$$\mathrm{Tr}\left[\gamma^{\mu_1}\right] = 0\,. \tag{11.62}$$

The trace over the product of two γ-matrices can be calculated by using their property of representing a Clifford algebra (9.95):

$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\right] = \frac{1}{2}\,\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2} + \gamma^{\mu_2}\gamma^{\mu_1}\right] = g^{\mu_1\mu_2}\mathrm{Tr}[1] = 4g^{\nu_1\mu_2}\,. \tag{11.63}$$

We show now that the trace vanishes over a product of any odd number of γ-matrices. To this end we consider the γ_5-matrix defined in (9.149) that has the explicit form (9.152) and, thus, the property to be involutoric according to (9.153) as well as anti-commuting with any Dirac matrix according to (9.230). With this follows then for the trace of a product of γ-matrices:

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_n}\right] = \mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_n}\gamma_5\gamma_5\right] = \mathrm{Tr}\left[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_n}\gamma_5\right] \\
&= (-1)^n\,\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_n}\gamma_5\gamma_5\right] = (-1)^n\,\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_n}\right]\,,
\end{aligned} \tag{11.64}$$

so we obtain for n being odd:

$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\,\cdots\,\gamma^{\mu_{2n+1}}\right] = 0\,. \tag{11.65}$$

Thus, only the traces over a product of an even number of γ-matrices can be non-vanishing. Let us consider now the trace over a product of four γ-matrices. Successively applying the Clifford algebra property (9.95) together with (11.63) yields $4!! = 3$ terms:

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\right] = -\mathrm{Tr}\left[\gamma^{\mu_2}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\right] + 2g^{\mu_1\mu_2}\mathrm{Tr}\left[\gamma^{\mu_3}\gamma^{\mu_4}\right] = \mathrm{Tr}\left[\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_1}\gamma^{\mu_4}\right] + 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} \\
&-2g^{\mu_1\mu_3}\mathrm{Tr}\left[\gamma^{\mu_3}\gamma^{\mu_4}\right] = -\mathrm{Tr}[\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_1}] + 2g^{\mu_1\mu_4}\mathrm{Tr}[\gamma^{\mu_2}\gamma^{\mu_3}] + 8g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 8g^{\mu_1\mu_3}g^{\mu_2\mu_4} \\
&= \mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}] = 4\left(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right)\,.
\end{aligned} \tag{11.66}$$

With the help of the auxiliary calculations (11.62)–(11.66) we obtain for (11.61) the result

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu(\not{p}_{i_1} + Mc)\gamma^\nu(\not{p}_{f1} + Mc)] &= 4p_{i_1\kappa}p_{f_1\lambda}(g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\nu}g^{\kappa\lambda} + g^{\mu\lambda}g^{\kappa\nu}) + 4M^2c^2g^{\mu\nu} \\
&= 4(p_{i_1}^\mu p_{f_1}^\nu - p_{i_1}p_{f_1}g^{\mu\nu} + p_{i_1}^\nu p_{f_1}^\mu + M^2c^2g^{\mu\nu})\,.
\end{aligned} \tag{11.67}$$

Using (11.67) the direct term (11.60) yields

$$\overline{|M_{fi}^{(2)}|_{\mathrm{d}}^2} = \frac{[p_{i_1}^\mu p_{f_1}^\nu + p_{i_1}^\nu p_{f_1}^\mu + (M^2c^2 - p_{i_1}p_{f_1})g^{\mu\nu}][p_{i_2\mu}p_{f_2\nu} + p_{i_2\nu}p_{f_2\mu} + (M^2c^2 - p_{i_2}p_{f_2})g_{\mu\nu}]}{4M^4c^4(p_{f_1} - p_{i_1})^4}\,,$$

which finally reduces to

$$\overline{|M_{fi}^{(2)}|_{\mathrm{d}}^2} = \frac{(p_{i_1}p_{i_2})(p_{f_1}p_{f_2}) + (p_{f_1}p_{i_2})(p_{i_1}p_{f_2}) - M^2c^2p_{i_1}p_{f_1} - M^2c^2p_{i_2}p_{f_2} + 2M^4c^4}{2M^4c^4(p_{f_1} - p_{i_1})^4}\,. \tag{11.68}$$

The exchange term in (11.59) is formally obtained from the direct term (11.60) by interchanging the final momenta p_{f_1} and p_{f_2}:

$$\overline{|M_{fi}^{(2)}|_{\mathrm{ex}}^2} = \frac{\mathrm{Tr}\left[\gamma^\nu(p_{i_1} + Mc)\gamma^\mu(p_{f_2} + Mc)\right]\;\mathrm{Tr}\left[\gamma_\nu(p_{i_1} + Mc)\gamma_\mu(p_{f_1} + Mc)\right]}{64M^4c^4(p_{f_2} - p_{i_1})^4}\,. \tag{11.69}$$

Therefore, we obtain the result for evaluating the traces in (11.69) from (11.68) by interchanging the final momenta p_{f_1} and p_{f_2}:

$$\overline{|M_{fi}^{(2)}|_{\text{ex}}^2} = \frac{(p_{i_1}p_{i_2})(p_{f_1}p_{f_2}) + (p_{f_2}p_{i_2})(p_{i_1}p_{f_1}) - M^2c^2 p_{i_1}p_{f_2} - M^2c^2 p_{i_2}p_{f_1} + 2M^4c^4}{2M^4c^4(p_{f_2} - p_{i_1})^4} \,. \tag{11.70}$$

Thus, it only remains to consider the interference term between the direct and the exchange scattering in (11.59):

$$\overline{|M_{fi}^{(2)}|_{\text{i}}^2} = \frac{-\{\text{Tr}[\gamma^\mu(\not p_{i_2} + Mc)\gamma^\nu(\not p_{f_1} + Mc)\gamma_\mu(\not p_{i_2} + Mc)\gamma_\nu(\not p_{f_2} + Mc)] + (p_{f_1} \leftrightarrow p_{f_2})}{64M^4c^4(p_{f_1} - p_{i_1})^2(p_{f_2} - p_{i_1}\})^2} \,. \tag{11.71}$$

Let us restrict us for the time being to the evaluation of the first term in (11.71). The corresponding trace can be simplified due to (11.65) such that only the trace over products of an even number of γ-matrices occurs:

$$\begin{aligned}
&\text{Tr}[...] = \text{Tr}[(\gamma^\mu \not p_{i_1}\gamma^\nu \not p_{f_1} + Mc\gamma^\mu\gamma^\nu \not p_{f_1} + Mc\gamma^\mu \not p_{i_1}\gamma^\nu + M^2c^2\gamma^\mu\gamma^\nu) \qquad (11.72)\\
&\times(\gamma_\mu \not p_{i_1}\gamma_\nu \not p_{f_2} + Mc\gamma_\mu\gamma_\nu \not p_{f_2} + Mc\gamma_\mu \not p_{i_2}\gamma_\nu + M^2c^2\gamma_\mu\gamma_\nu)] = \text{Tr}[\gamma^\mu \not p_{i_1}\gamma^\nu \not p_{f_1}\gamma_\mu \not p_{i_2}\gamma_\nu \not p_{f_2}\\
&+M^2c^2\gamma^\mu \not p_{i_1}\gamma^\nu \not p_{f_1}\gamma_\mu\gamma_\nu + m^2c^2\gamma^\mu\gamma^\nu \not p_{f_1}\gamma^\nu\gamma_\mu\gamma_\nu \not p_{f_2} + M^2c^2\gamma^\mu\gamma^\nu \not p_{f_1}\gamma^\mu \not p_{i_2}\gamma_\nu\\
&+M^2c^2\gamma^\mu \not p_{i_1}\gamma^\nu\gamma_\mu\gamma_\nu \not p_{f_2} + M^2c^2\gamma^\mu \not p_{i_1}\gamma^\nu\gamma_\mu \not p_{i_2}\gamma_\nu + M^2c^2\gamma^\mu\gamma^\nu\gamma_\mu \not p_{i_2}\gamma_\nu \not p_{f_2} + M^4c^4\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu] \,.
\end{aligned}$$

These traces over products of an even number of γ-matrices should actually be calculated analogously to (11.63) and (11.66). However, the trace over a product of six (eight) γ-matrices, which appear here for the first time, leads in total to $6!! = 15$ $(8!! = 105)$ terms. Thus evaluating (11.72) with the previous calculational technique would be too involved. Instead we use the observation, that the contractions of γ-matrices occur in (11.72) within the trace, to our advantage. Namely it turns out that this circumstance drastically simplifies the trace calculation. With the help of the Clifford algebra (9.95) the contracted product of two γ-matrices can be calculated as follows:

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}\, g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g_{\mu\nu}g^{\mu\nu} = \delta^\mu{}_\mu = 4 \,. \tag{11.73}$$

In case of one γ-matrix between the two contracted γ-matrices we get by applying the Clifford algebra (9.95)

$$\gamma^\mu\gamma^\nu\gamma_\mu = (-\gamma^\nu\gamma^\mu\gamma_\mu + 2g^{\mu\nu})\gamma_\mu = -\gamma^\nu\gamma^\mu\gamma_\mu + 2g^{\mu\nu}\gamma_\mu = -2\gamma^\nu \,. \tag{11.74}$$

This result can be used to deal with two γ-matrices lying in between

$$\gamma^\mu\gamma^\nu\gamma^\kappa\gamma_\mu \;=\; (-\gamma^\nu\gamma^\mu + 2g^{\mu\nu})\gamma^\kappa\gamma_\mu = -\gamma^\nu(\gamma^\mu\gamma^\kappa\gamma_\mu) + 2g^{\mu\nu}\gamma^\kappa\gamma_\mu = 2[\gamma^\nu, \gamma^\kappa]_+ = 4g^{\nu\kappa} \,. \tag{11.75}$$

And, correspondingly, we yield for three γ-matrices:

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda\gamma_\mu &= (-\gamma^\nu\gamma^\mu + 2g^{\mu\nu})\gamma^\kappa\gamma\lambda\gamma_\mu = -\gamma^\nu(\gamma^\mu\gamma^\kappa\gamma^\lambda\gamma_\mu) + 2g^{\mu\nu}\gamma^\kappa\gamma^\lambda\gamma_\mu\\
&= -4\gamma^\nu g^{\kappa\lambda} + 2(\gamma^\kappa\gamma^\lambda)\gamma^\nu = -4\gamma^\nu g^{\kappa\lambda} + 2(-\gamma^\lambda\gamma^\kappa + 2g^{\kappa\lambda})\gamma^\nu = -2\gamma^\lambda\gamma^\kappa\gamma^\nu \,.
\end{aligned} \tag{11.76}$$

These contraction rules for γ-matrices can now be iteratively applied to the respective terms in the trace (11.72) of the interference term (11.71):

$$1)(\gamma^\mu\gamma^\nu\gamma_\mu)\gamma_\nu = -2\gamma^\nu\gamma_\nu = -8\,, \tag{11.77}$$

$$2)\gamma^\mu \not{p}_{i_1}\gamma^\nu \not{p}_{f_1}\gamma_\mu\gamma_\nu = p_{i_1\kappa}p_{f_1\lambda}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma_\mu)\gamma_\nu = p_{i_1\kappa}p_{f_1\lambda}(-2)\gamma^\lambda(\gamma^\nu\gamma^\kappa\gamma_\nu) = 4p_{i_1\kappa}p_{f_1\lambda}\gamma^\lambda\gamma^\kappa, \tag{11.78}$$

$$3)\gamma^\mu \not{p}_{i_1}\gamma^\nu\gamma_\mu \not{p}_{i_2}\gamma_\nu = p_{i_1\kappa}p_{i_2\lambda}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma_\mu)\gamma^\lambda\gamma_\nu = 4p_{i_1\kappa}p_{i_2\lambda}g^{\kappa\nu}\gamma^\lambda\gamma_\nu = 4p_{i_1\kappa}p_{i_2\lambda}\gamma^\lambda\gamma^\kappa\,, \tag{11.79}$$

$$4)\gamma^\mu\gamma^\nu \not{p}_{f_1}\gamma_\mu \not{p}_{i_2}\gamma_\nu = p_{i_1\kappa}p_{i_2\lambda}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma_\mu)\gamma^\lambda\gamma_\nu = 4p_{i_1\kappa}p_{i_2\lambda}\gamma^\lambda\gamma^\kappa\,, \tag{11.80}$$

$$5)\gamma^\mu \not{p}_{i_1}\gamma^\nu\gamma_\mu\gamma_\nu \not{p}_{f_2} = p_{i_1\kappa}p_{f_2\lambda}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma_\mu)\gamma_\nu\gamma^\lambda = 4p_{i_1\kappa}p_{f_2\lambda}g^{\kappa\nu}\gamma_\nu\gamma^\lambda = 4p_{i_1\kappa}p_{f_2\lambda}\gamma^\kappa\gamma^\lambda\,, \tag{11.81}$$

$$6)\gamma^\mu\gamma^\nu\gamma_\mu \not{p}_{i_2}\gamma_\nu \not{p}_{f_2} = p_{i_2\kappa}p_{f_2\lambda}(\gamma^\mu\gamma^\nu\gamma_\mu)\gamma^\kappa\gamma_\nu\gamma^\lambda = -2p_{i_2\kappa}p_{f_2\lambda}(\gamma^\nu\gamma^\kappa\gamma_\nu)\gamma^\lambda = 4p_{i_2\kappa}p_{f_2\lambda}\gamma^\kappa\gamma^\lambda\,, \tag{11.82}$$

$$7)\gamma^\mu\gamma^\nu \not{p}_{f_1}\gamma_\mu\gamma_\nu \not{p}_{f_2} = p_{f_1\kappa}p_{f_2\lambda}(\gamma^\mu\gamma^\nu\gamma^\kappa\gamma_\mu)\gamma_\nu\gamma^\lambda = 4p_{f_1\kappa}p_{f_2\lambda}\gamma^\kappa\gamma^\lambda\,, \tag{11.83}$$

$$\begin{aligned}8)\gamma^\mu \not{p}_{i_1}\gamma^\nu \not{p}_{f_1}\gamma_\mu \not{p}_{i_2}\gamma_\nu \not{p}_{f_2} &= p_{i_1\kappa}p_{f_1\lambda}p_{i_2\sigma}p_{f_2\tau}(\gamma^\mu\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma_\mu)\gamma^\tau\gamma_\nu\gamma^\tau \\ = -2p_{i_1\kappa}p_{f_1\lambda}p_{i_2\sigma}p_{f_2\tau}\gamma^\lambda(\gamma^\nu\gamma^\kappa\gamma^\sigma\gamma_\nu)\gamma^\tau &= -8p_{i_1\kappa}p_{f_1\lambda}p_{i_2\sigma}p_{f_2\tau}g^{\kappa\sigma}\gamma^\lambda\gamma^\tau = -8(p_{i_1}p_{i_2})p_{f_1\lambda}p_{f_2\tau}\gamma^\lambda\gamma^\tau\,.\end{aligned} \tag{11.84}$$

Using the auxiliary calculations (11.77)–(11.84) and taking into account (11.63) we obtain for (11.72) the following result

$$\begin{aligned}\mathrm{Tr}[...] &= -8(p_{i_1}p_{i_2})p_{f_1\lambda}p_{f_2\kappa}\mathrm{Tr}[\gamma^\lambda\gamma^\kappa] + 4M^2c^2\left\{p_{i_1\kappa}p_{f_1\lambda}\mathrm{Tr}[\gamma^\lambda\gamma^\kappa] + p_{i_1\kappa}p_{i_2\lambda}\mathrm{Tr}[\gamma^\lambda\gamma^\kappa]\right.\\ &\left.+p_{f_1\kappa}p_{i_2\lambda}\mathrm{Tr}[\gamma^\lambda\gamma^\kappa]\right\} + \left\{p_{i_1\kappa}p_{f_2\lambda}\mathrm{Tr}[\gamma^\kappa\gamma^\lambda] + p_{f_1\kappa}p_{f_2\lambda}\mathrm{Tr}[\gamma^\kappa\gamma^\lambda] + p_{i_2\kappa}p_{f_1\lambda}\mathrm{Tr}[\gamma^\kappa\gamma^\lambda]\right\} - 8M^4c^4\mathrm{Tr}[1] \\ &= -32(p_{i_1}p_{i_2})(p_{f_1}p_{f_2}) - 32M^4c^4 + 16M^2c^2(p_{i_1}p_{f_1} + p_{i_1}p_{i_2} + p_{f_1}p_{i_2} + p_{i_1}p_{f_2} + p_{f_1}p_{f_2} + p_{i_2}p_{f_1})\,.\end{aligned} \tag{11.85}$$

Substituting (11.84) into (11.71) leads to the final expression for the interference term between the direct and the exchange scattering:

$$\begin{aligned}\overline{\left|M_{fi}^{(2)}\right|_{\mathrm{i}}^2} &= \frac{1}{4M^4c^4(p_{f_1}-p_{i_1})^2(p_{f_2}-p_{i_1})^2}\left[2(p_{i_1}p_{i_2})(p_{f_1}p_{f_2}) + M^4c^4\right.\\ &\left.-M^2c^2(p_{i_1}p_{f_1} + p_{i_1}p_{i_2} + p_{f_1}p_{i_2} + p_{i_1}p_{f_2} + p_{f_1}p_{f_2} + p_{i_2}p_{f_1}) + (p_{f_1}\leftrightarrow p_{f_2})\right]\,.\end{aligned} \tag{11.86}$$

We conclude that the direct term (11.68), the exchange term (11.70), and the interference term (11.86) have the common property of having a manifestly covariant form as they only depend on the scalar product of momenta. Thus, it only remains to relate these scalar product of momenta to observable properties of the scattering process. This is achieved by introducing the Lorentz invariant Mandelstam variables.

11.4 Mandelstam Variables

Let us investigate now the kinematics of a general two-body scattering process

$$A + B \qquad \Longrightarrow \qquad C + D\,, \tag{11.87}$$

which is described by the four-vector momenta p_a, p_b, p_c, and p_d with a total of 16 components. The equivalence principle of special relativity requires that observable quantities, such as the scattering cross section, can be expressed by Lorentz invariants.

11.4.1 General Case

With the four-vector momenta p_i with $i = a, b, c, d$, one can form ten different scalar products $p_i p_j$ with $i \leq j$, four of which are fixed by the relativistic energy-momentum dispersion relations

$$p_i^2 = M_i^2 c^2 . \tag{11.88}$$

The remaining six degrees of freedom are still interdependent, as each scattering process must satisfy the energy-momentum conservation law:

$$p_a + p_b = p_c + p_d . \tag{11.89}$$

These four additional conditions lead to the fact that, ultimately, two kinematic variables are sufficient to describe the two-body scattering process (11.87), provided that one can perform an average over the polarisations of both the initial and the final particles. For historical reasons, one describes the two-body scattering process (11.87) by the following three Lorentz-invariant Mandelstam variables

$$s = (p_a + p_b)^2 = (p_c + p_d)^2 , \tag{11.90}$$
$$t = (p_c - p_a)^2 = (p_d - p_b)^2 , \tag{11.91}$$
$$u = (p_c - p_b)^2 = (p_d - p_a)^2 . \tag{11.92}$$

Due to (11.88) and (11.90)–(11.91) each of the six scalar products $p_i p_j$ with $i < j$ can be expressed by the three Mandelstam variables:

$$p_a p_b = \frac{1}{2}\left(s - M_a^2 c^2 - M_b^2 c^2\right) , \qquad p_c p_d = \frac{1}{2}\left(s - M_c^2 c^2 - M_d^2 c^2\right) \tag{11.93}$$
$$p_a p_c = -\frac{1}{2}\left(t - M_a^2 c^2 - M_c^2 c^2\right) , \qquad p_b p_d = -\frac{1}{2}\left(t - M_b^2 c^2 - M_d^2 c^2\right) , \tag{11.94}$$
$$p_b p_c = -\frac{1}{2}\left(u - M_b^2 c^2 - M_c^2 c^2\right) , \qquad p_a p_d = -\frac{1}{2}\left(u - M_a^2 c^2 - M_d^2 c^2\right) . \tag{11.95}$$

Furthermore, it is possible to derive a restriction for the three Mandelstam variables. At first we obtain from (11.90)–(11.92)

$$s + t + u = (p_a + p_b)^2 + (p_a - p_c)^2 + (p_a - p_d)^2 = 3p_a^2 + p_b^2 + p_c^2 + p_d^2 + 2p_a(p_b - p_c - p_d), \tag{11.96}$$

which reduces then with (11.88) and (11.89) to

$$s + t + u = p_a^2 + p_b^2 + p_c^2 + p_d^2 = (M_a^2 + M_b^2 + M_c^2 + M_d^2)c^2 . \tag{11.97}$$

Obviously, one of the three Mandelstam variables s, t, u can be eliminated with the help of (11.97). Nevertheless, all the three Mandelstam variables are often used, as the results for scattering cross sections turn out to acquire then a symmetrical form.

11.4.2 Equal Masses

Various simplifications occur for two-body scattering processes (11.87), where the involved particles have an equal mass:

$$M_a = M_b = M_c = M_d = M\,. \tag{11.98}$$

With the help of the identifications

$$p_a = p_{i_1}\,, \qquad p_b = p_{i_2}\,, \qquad p_c = p_{f_1}\,, \qquad p_d = p_{f_2} \tag{11.99}$$

the relativistic energy-momentum dispersion relations (11.88) go over to

$$p_{i_1}^2 = p_{i_2}^2 = p_{f_1}^2 = p_{f_2}^2 = M^2c^2\,. \tag{11.100}$$

Additionally the corresponding scalar products (11.93)–(11.95) read then as follows:

$$p_{i_1}p_{i_2} = p_{f_1}p_{f_2} = \frac{1}{2}\left(s - 2M^2c^2\right)\,, \tag{11.101}$$

$$p_{i_1}p_{f_1} = p_{i_2}p_{f_2} = -\frac{1}{2}\left(t - 2M^2c^2\right)\,, \tag{11.102}$$

$$p_{i_2}p_{f_1} = p_{i_1}p_{f_2} = -\frac{1}{2}\left(u - 2M^2c^2\right)\,. \tag{11.103}$$

And the definitions of the Mandelstam variables (11.90)–(11.92) take now the form

$$s = (p_{i_1} + p_{i_2})^2 = (p_{f_1} + p_{f_2})^2\,, \tag{11.104}$$

$$t = (p_{f_1} - p_{i_1})^2 = (p_{f_2} - p_{i_2})^2\,, \tag{11.105}$$

$$u = (p_{f_1} - p_{i_2})^2 = (p_{f_2} - p_{i_1})^2\,, \tag{11.106}$$

whereby the restriction (11.97) coverts into

$$s + t + u = 4M^2c^2\,. \tag{11.107}$$

11.4.3 Matrix Element

Now we return to the polarisation averaged matrix element of the Møller scattering and express the individual contributions with the help of (11.101)–(11.107) by the three Mandelstam variables s, t, u. For the direct term (11.68) we obtain

$$\overline{|M_{fi}^{(2)}|_{\mathrm{d}}^2} = \frac{(s-2M^2c^2)^2 + (u-2M^2c^2)^2 + 4M^2c^2t}{8M^4c^4t^2}\,. \tag{11.108}$$

The exchange term (11.70) follows from the direct term (11.68) by exchanging the final momenta p_{f_1} and p_{f_2}. At the level of the Mandelstam variables (11.101)–(11.107) this corresponds to an exchange of t and u:

$$\overline{|M_{fi}^{(2)}|_{\mathrm{ex}}^2} = \frac{(s-2M^2c^2)^2 + (t-2M^2c^2)^2 + 4M^2c^2u}{8M^4c^4u^2}\,. \tag{11.109}$$

The Feynman diagrams in Fig. 11.1, whose absolute square and a subsequent polarization average leads to the terms (11.108) and (11.109), are also called after the Mandelstam variable in the denominator to graphically represent the t- and the u-channel, respectively. Correspondingly, the interference term (11.85) yields

$$\overline{|M_{fi}^{(2)}|_{\mathrm{i}}^{2}} = \frac{1}{2M^4c^4tu}\Bigg\{\frac{1}{2}(s-2M^2c^2)^2 -M^2c^2\left[(s-2M^2c^2)-(t-2M^2c^2)-(u-2M^2c^2)\right]+2M^4c^4+(u\leftrightarrow t)\Bigg\}. \tag{11.110}$$

Both contributions in (11.110) are apparently identical and we obtain

$$\overline{|M_{fi}^{(2)}|_{\mathrm{i}}^{2}} = \frac{1}{2M^4c^4tu}\left[\frac{1}{2}(s-2M^2c^2)^2-M^2c^2(s-t-u)\right]. \tag{11.111}$$

Taking into account the restriction (11.107) this reduces to

$$\overline{|M_{fi}^{(2)}|_{\mathrm{i}}^{2}} = \frac{(s-2M^2c^2)(s-6M^2c^2)}{4M^4c^4tu}. \tag{11.112}$$

11.5 Center-of-Mass System

Now we specialize the kinematic analysis to the center of mass reference frame for two particles of equal mass.

11.5.1 Kinematics

Here the four-momentum vectors simplify even further:

$$p_{i_1} = \begin{pmatrix} E_{i_1}/c \\ \mathbf{p}_{i_1}\end{pmatrix}, \quad p_{i_2} = \begin{pmatrix} E_{i_2}/c \\ \mathbf{p}_{i_2}\end{pmatrix}, \quad p_{f_1} = \begin{pmatrix} E_{f_1}/c \\ \mathbf{p}_{f_1}\end{pmatrix}, \quad p_{f_2} = \begin{pmatrix} E_{f_2}/c \\ \mathbf{p}_{f_2}\end{pmatrix}. \tag{11.113}$$

The center of mass system is distinguished from other inertial systems by the fact that the total momentum of the two incoming particles disappears:

$$\mathbf{p}_{i_1}+\mathbf{p}_{i_2}=\mathbf{0} \qquad \Longrightarrow \qquad \mathbf{p}_{i_1}=\mathbf{p}, \quad \mathbf{p}_{i_2}=-\mathbf{p}. \tag{11.114}$$

From their respective energy-momentum dispersion relations (11.100)

$$E_{i_1}=\sqrt{\mathbf{p}_{i_1}^2c^2+M^2c^4}, \qquad E_{i_2}=\sqrt{\mathbf{p}_{i_2}^2c^2+M^2c^4} \tag{11.115}$$

then follows that the energies of the two incoming particles coincide:

$$E_{i_1}=E_{i_2}=E. \tag{11.116}$$

From the momentum conservation (11.89) in the center of mass reference frame follows with (11.99) and (11.114) for the momenta of the two outgoing particles

$$\mathbf{p}_{f_1} + \mathbf{p}_{f_2} = \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{p}_{f_1} = \mathbf{p}' , \quad \mathbf{p}_{f_2} = -\mathbf{p}' . \tag{11.117}$$

Thus, the corresponding energy-momentum dispersion relations (11.100)

$$E_{f_1} = \sqrt{\mathbf{p}_{f_1}^2 c^2 + M^2 c^4} , \qquad E_{f_2} = \sqrt{\mathbf{p}_{f_2}^2 c^2 + M^2 c^4} \tag{11.118}$$

imply that also the energies of the two outgoing particles are equal:

$$E_{f_1} = E_{f_2} = E' . \tag{11.119}$$

And from the energy conservation (11.89) in the center of mass reference frame

$$E_a + E_b = E_c + E_d \tag{11.120}$$

then follows with (11.99), (11.116), and (11.119) that the energy of the incoming and the outgoing particles E and E' coincide:

$$E_{i_1} + E_{i_2} = E_{f_1} + E_{f_2} \qquad \Longrightarrow \qquad E = E' . \tag{11.121}$$

We conclude from (11.114), (11.116), (11.117), (11.119), and (11.121) that the four-momentum vectors (11.113) are given in the center of mass reference frame as follows:

$$p_{i_1} = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix} , \quad p_{i_2} = \begin{pmatrix} E/c \\ -\mathbf{p} \end{pmatrix} , \quad p_{f_1} = \begin{pmatrix} E/c \\ \mathbf{p}' \end{pmatrix} , \quad p_{f_2} = \begin{pmatrix} E/c \\ -\mathbf{p}' \end{pmatrix} . \tag{11.122}$$

For the Mandelstam variables (11.104)–(11.106) this has due to (11.115), (11.118), (11.121), and (11.122) the consequence

$$s = (p_{i_1} + p_{i_2})^2 = \begin{pmatrix} 2E/c \\ \mathbf{0} \end{pmatrix}^2 = \frac{4E^2}{c^2} , \tag{11.123}$$

$$t = (p_{f_1} - p_{i_1})^2 = \begin{pmatrix} 0 \\ \mathbf{p}' - \mathbf{p} \end{pmatrix}^2 = -(\mathbf{p}' - \mathbf{p})^2 = -2\mathbf{p}^2(1 - \cos\theta) , \tag{11.124}$$

$$u = (p_{f_1} - p_{i_2})^2 = \begin{pmatrix} 0 \\ \mathbf{p}' + \mathbf{p} \end{pmatrix}^2 = -(\mathbf{p}' + \mathbf{p})^2 = -2\mathbf{p}^2(1 + \cos\theta) . \tag{11.125}$$

Here θ denotes the angle between the incoming and the outgoing electrons, which coincides with the angle between the momenta $\mathbf{p}$ and $\mathbf{p}'$ as illustrated in Fig. 11.2. Obviously, the Mandelstam variables s, t, u in the center of mass reference frame (11.123)–(11.125) satisfy the restriction (11.107) due to the relativistic energy-momentum dispersion relation (11.114)–(11.116):

$$s + t + u = \frac{4E^2}{c^2} - 2\mathbf{p}^2(1 - \cos\theta) - 2\mathbf{p}^2(1 + \cos\theta) = \frac{4}{c^2}\left(E^2 - \mathbf{p}^2 c^2\right) = 4M^2c^2 . \tag{11.126}$$

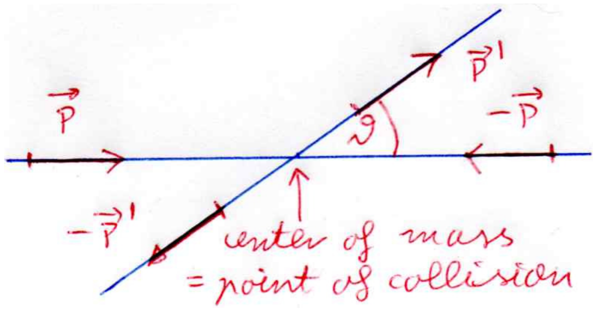

Figure 11.2: Geometry of the elastic Møller scattering in the center of mass reference frame with two incoming (outgoing) electrons of momenta $\pm\mathbf{p}$ ($\pm\mathbf{p}'$).

Furthermore, we read off from (11.114)–(11.116) that the two Mandelstam variables (11.124) and (11.125) can be rewritten as

$$t = -2\,\frac{E^2 - M^2c^4}{c^2}\,(1-\cos\theta)\,, \tag{11.127}$$

$$u = -2\,\frac{E^2 - M^2c^4}{c^2}\,(1+\cos\theta)\,. \tag{11.128}$$

Thus, for a scattering process of two particles with equal masses the Mandelstam variables (11.123), (11.127), and (11.128) in the center of mass reference frame depend on both the scattering energy E and the scattering angle ϑ.

11.5.2 Matrix Element

With the help of (11.123), (11.127), and (11.128) the individual contributions to the polarisation-averaged squared matrix element for the Møller scattering can be expressed as follows. The direct term (11.108) goes over into

$$\overline{|M_{fi}^{(2)}|_{\rm d}^2} = \frac{1}{8M^4c^4(E^2-M^2c^4)^2(1-\cos\theta)^2} \tag{11.129}$$
$$\times\left\{(2E^2-M^2c^4)^2+\left[(E^2-M^2c^4)(1+\cos\theta)+2M^2c^4\right]^2-2M^2c^4(E^2-M^2c^4)(1-\cos\theta)\right\}\,,$$

the exchange term (11.109) reads

$$\overline{|M_{fi}^{(2)}|_{\rm ex}^2} = \frac{1}{8M^4c^4(E^2-M^2c^4)^2(1+\cos\theta)^2} \tag{11.130}$$
$$\times\left\{(2E^2-M^2c^4)^2+\left[(E^2-M^2c^4)(1-\cos\theta)+M^2c^4\right]^2-2M^2c^4(E^2-M^2c^4)(1+\cos\theta)\right\}\,,$$

and the interference term (11.112) results in

$$\overline{|M_{fi}^{(2)}|_{\rm i}^2} = \frac{(2E^2-M^2c^4)(2E^2-3M^2c^4)}{4M^4c^4(E^2-M^2c^4)^2(1-\cos\theta)(1+\cos\theta)}\,. \tag{11.131}$$

These three contributions are now to added:

$$\overline{|M_{fi}^{(2)}|^2} = \overline{|M_{fi}^{(2)}|_{\rm d}^2}+\overline{|M_{fi}^{(2)}|_{\rm ex}^2}+\overline{|M_{fi}^{(2)}|_{\rm i}^2} = \frac{f(\theta)}{8M^4c^4(E^2-M^2c^4)^2(1-\cos\theta)^2(1+\cos\theta)^2}\,. \tag{11.132}$$

Due to straight-forward but lengthy manipulations the angle-dependent numerator results in

$$\begin{aligned} f(\theta) &= (1+2\cos\theta+\cos^2\theta)\left[(2E^2-M^2c^4)^2+E^4+2E^2(E^2-M^2c^4)\cos\theta\right. \\ &\left.+(E^2-M^2c^4)\cos^2\theta-2M^2c^4(E^2-M^2c^4)(1-\cos\theta)\right]+(1-2\cos\theta+\cos^2\theta) \\ &\times\left[(2E^2-M^2c^4)^2+E^4-2E^2(E^2-M^2c^4)\cos\theta+(E^2-M^2c^4)\cos^2\theta\right. \\ &\left.-2M^2c^4(E^2-M^2c^4)(1+\cos\theta)\right]+2(1-\cos^2\theta)(2E^2-M^2c^4)(2E^2-3M^2c^4)\,. \end{aligned} \tag{11.133}$$

It turns out to be useful to take into account the trigonometric Pythagoras

$$\sin^2\theta+\cos^2\theta=1 \tag{11.134}$$

in order to further simplify the expression (11.133), yielding after some algebraic manipulations the concise result:

$$f(\theta)=2\left[4(2E^2-M^2c^4)^2-(8E^4-4M^2c^4E^2-M^4c^8)\sin^2\theta+(E^2-M^2c^4)^2\sin^4\theta\right]\,. \tag{11.135}$$

Inserting (11.135) into (11.132) leads together with (11.134) the following angular dependence of the polarisation-averaged squared matrix element of the Møller scattering in the center of mass reference frame:

$$\overline{\left|M_{fi}^{(2)}\right|^2}=\frac{4(2E^2-M^2c^4)^2-(8E^4-4M^2c^4E^2-M^4c^8)\sin^2\theta+(E^2-M^2c^4)^2\sin^4\theta}{4M^4c^4(E^2-M^2c^4)^2\sin^4\theta}\,. \tag{11.136}$$

11.6 Transition Rate Per Volume

Now we return to the perturbative result for the scattering matrix of the Møller scattering (11.45) and evaluate its absolute square:

$$\begin{aligned} \left|S_{fi}^{(2)}\right|^2 &= \frac{\hbar^2e^4}{\epsilon_0^2c^2}\,(2\pi\hbar)^8\delta(0)\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2}) \\ &\times\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{i_1}}}\,\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{i_2}}}\,\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{f_1}}}\,\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}_{f_2}}}\left|M_{fi}^{(2)}\right|^2. \end{aligned} \tag{11.137}$$

The transition probability (11.137) is formally infinite due to the appearance of the singular factor $\delta(0)$. In order to deal with this singularity we reconsider the decomposition of the field operator $\hat{\psi}(x)$ into plane waves according to (11.10). However, instead we now assume, as is usual in solid-state physics, that an electron is located in a finite box with volume V. Then we have instead of (11.10) the following plane wave decomposition:

$$\hat{\psi}(x)=\sum_{\mathbf{p}}\sum_s\sqrt{\frac{Mc^2}{VE_{\mathbf{p}}}}\left\{e^{-ipx/\hbar}\,u(\mathbf{p},s)\,\hat{b}_{\mathbf{p},s}+e^{ipx/\hbar}\,v(\mathbf{p},s)\,\hat{d}^\dagger_{\mathbf{p},s}\right\}\,. \tag{11.138}$$

While the orthonormality relation of the plane waves in the continuum reads

$$\int d^4x\,e^{i(p-p')x/\hbar}=(2\pi\hbar)^4\,\delta(p-p')\,, \tag{11.139}$$

it reads in a finite box V and within a finite observation time T

$$\int_V d^3x \int_{-Tc/2}^{Tc/2} dx_0\, e^{i(p-p')x/\hbar} = VTc\,\delta_{p,p'}\,. \tag{11.140}$$

Note that the delta function in (11.139) is substituted by the Kronecker symbol in (11.140). Therefore, comparing (11.139) and (11.140) yields on formal grounds the following substitution rule

$$(2\pi\hbar)^4\,\delta(0) = VTc\,, \tag{11.141}$$

which suggests an appropriate regularisation for the singular term $\delta(0)$. We now follow the calculation strategy that both the initial and the final states of the scattering process are still considered to be continuous, while the intermediate states are treated as discrete ones as in (11.138). Thus, we would then have to repeat the whole perturbative calculation for the Møller scattering and calculate how the scattering matrix element (11.45) and its absolute square (11.137) change from this modified point of view. This would yield the result (11.137) with a regularization by the formal substitution rule (11.141) together with the identification $(2\pi\hbar)^3 \to V$. With this we obtain for the transition rate per volume from (11.137) and (11.141):

$$\frac{\left|S_{fi}^{(2)}\right|^2}{VT} = \frac{\hbar^2 e^4}{\epsilon_0^2 c}\,(2\pi\hbar)^4\,\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})\,\frac{(Mc^2)^4}{V^4 E_{\mathbf{p}_{i_1}} E_{\mathbf{p}_{i_2}} E_{\mathbf{p}_{f_1}} E_{\mathbf{p}_{f_2}}}\left|M_{fi}^{(2)}\right|^2. \tag{11.142}$$

This transition rate per volume is then to be integrated or summed up over all final states:

$$\frac{V}{(2\pi\hbar)^3}\int d^3p_{f_1}\,\frac{V}{(2\pi\hbar)^3}\int d^3p_{f_2}\sum_{s_{f_1}}\sum_{s_{f_2}} \tag{11.143}$$

and it is to be averaged over the polarizations of the initial states:

$$\frac{1}{4}\sum_{s_{i_1}}\sum_{s_{i_2}} \tag{11.144}$$

This yields the averaged transition rate per volume:

$$W = \frac{1}{4}\sum_{s_{i_1}}\sum_{s_{i_2}}\sum_{s_{f_1}}\sum_{s_{f_2}}\frac{V}{(2\pi\hbar)^3}\int d^3p_{f_1}\,\frac{V}{(2\pi\hbar)^3}\int d^3p_{f_2}\,\frac{|S_{fi}^{(2)}|^2}{VT}\,. \tag{11.145}$$

Inserting (11.142) into (11.145) as well as taking into account (11.48) then leads to

$$W = \frac{e^4 M^2 c^4}{4\pi^2\epsilon_0^2 c V^2 E_{\mathbf{p}_{i_1}} E_{\mathbf{p}_{i_2}}}\int d^3p_{f_1}\int d^3p_{f_2}\,\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})\,\frac{M^2c^4}{E_{\mathbf{p}_{f_1}} E_{\mathbf{p}_{f_2}}}\,\overline{\left|M_{fi}^{(2)}\right|^2}\,, \tag{11.146}$$

where the polarisation average of the squared matrix element (11.48) was already been calculated in (11.136). The two integrals over the outgoing momenta are of the following form:

$$I = \int \frac{d^3p_{f_1}}{2E_{\mathbf{p}_{f_1}}}\frac{d^3p_{f_2}}{2E_{\mathbf{p}_{f_2}}}\,\delta(p_{f_1}+p_{f_2}-p_{i_1}-p_{i_2})\,f(\mathbf{p}_{f_1},\mathbf{p}_{f_2})\,. \tag{11.147}$$

In order to evaluate (11.147) we perform at first the following auxiliary calculation

$$\int_0^\infty dp^0 \delta(p^2 - M^2c^2) = \int_0^\infty dp^0 \delta((p^0)^2 - \mathbf{p}^2 - M^2c^2)$$
$$= \int_0^\infty dp^0 \left[\frac{c}{2E_\mathbf{p}} \, \delta\left(p^0 - \frac{E_\mathbf{p}}{c}\right) + \frac{c}{2E_\mathbf{p}} \, \delta\left(p^0 + \frac{E_\mathbf{p}}{c}\right) \right] = \frac{c}{2E_\mathbf{p}} \,. \tag{11.148}$$

Note that we used here the distributional rule

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \, \delta(x - x_i) \,, \qquad g(x_i) = 0 \tag{11.149}$$

for the function

$$g(p^0) = (p^0)^2 - \frac{E_\mathbf{p}^2}{c^2} = \left(p^0 - \frac{E_\mathbf{p}}{c}\right)\left(p^0 + \frac{E_\mathbf{p}}{c}\right) \,, \qquad g'\left(p^0 = \pm \frac{E_\mathbf{p}}{c}\right) = \pm\, 2\, \frac{E_\mathbf{p}}{c} \,. \tag{11.150}$$

Inserting (11.148) into (11.147) leads to

$$I = \frac{1}{c} \int \frac{d^3p_{f_1}}{2E_{\mathbf{p}_{f_1}}} \int d^3p_{f_2} \int_0^\infty dp^0_{f_2} \, \delta(p^2_{f_2} - M^2c^2) \delta(p_{f_1} + p_{f_2} - p_{i_1} - p_{i_2}) f(\mathbf{p}_{f_1}, \mathbf{p}_{f_2})$$
$$= \frac{1}{c} \int \frac{d^3p_{f_1}}{2E_{\mathbf{p}_{f_1}}} \int d^4p_{f_2} \Theta(p^0_{f_2}) \delta(p^2_{f_2} - M^2c^2) \delta(p_{f_1} + p_{f_2} - p_{i_1} - p_{i_2}) f(\mathbf{p}_{f_1}, \mathbf{p}_{f_2}) \,. \tag{11.151}$$

Now the four-dimensional p_{f_2}-integral can formally be evaluated and we obtain the intermediate result:

$$I = \frac{1}{c} \int \frac{d^3p_{f_1}}{2E_{\mathbf{p}_{f_1}}} \, \Theta(p^0_{i_1} + p^0_{i_2} - p^0_{f_1}) \delta((p_{i_1} + p_{i_2} - p_{f_1})^2 - M^2c^2) f(\mathbf{p}_{f_1}, \mathbf{p}_{i_1} + \mathbf{p}_{i_2} - \mathbf{p}_{f_1}) \,. \tag{11.152}$$

In view of evaluating also the $\mathbf{p}_{f_1}$-integral we specialise for the center of mass reference frame, so that we can apply the considerations from the previous section. However, in contrast to (11.122), we cannot use the conservation of energy, as this is only established due the delta function in (11.152). Therefore, based on (11.114) and (11.117), we have to generalise the four-momentum vectors (11.122) accordingly:

$$p_{i_1} = \begin{pmatrix} E_\mathbf{p}/c \\ \mathbf{p} \end{pmatrix} \,, \quad p_{i_2} = \begin{pmatrix} E_\mathbf{p}/c \\ -\mathbf{p} \end{pmatrix} \,, \quad p_{f_1} = \begin{pmatrix} E_{\mathbf{p}'}/c \\ \mathbf{p}' \end{pmatrix} \,, \quad p_{f_2} = \begin{pmatrix} E_{\mathbf{p}'}/c \\ -\mathbf{p}' \end{pmatrix} \,. \tag{11.153}$$

From this we read off

$$p^0_{i_1} + p^0_{i_2} - p^0_{f_1} = \frac{2E_\mathbf{p} - E_{\mathbf{p}'}}{c} \,, \tag{11.154}$$
$$\mathbf{p}_{i_1} + \mathbf{p}_{i_2} - \mathbf{p}_{f_1} = -\mathbf{p}' \,, \tag{11.155}$$

as well as

$$(p_{i_1} + p_{i_2} - p_{f_1})^2 = (p_{i_1} + p_{i_2})^2 - 2(p_{i_1} + p_{i_2})p_{f_1} + p^2_{f_1} \tag{11.156}$$
$$= \left(\frac{2E_\mathbf{p}}{c}\right)^2 - 2\frac{2E_\mathbf{p}}{c}\frac{E_{\mathbf{p}'}}{c} + \frac{E^2_{\mathbf{p}'}}{c^2} - \mathbf{p}'^2 = \frac{4E_\mathbf{p}}{c^2}(E_\mathbf{p} - E_{\mathbf{p}'}) + M^2c^2 \,,$$

where in the last step the relativistic energy-momentum dispersion

$$E_{\mathbf{p}'}^2 = \mathbf{p}'^2 c^2 + M^2 c^4 \tag{11.157}$$

was used. With this (11.152) reads using (11.149) in the center of mass reference frame

$$I = \frac{1}{c} \int \frac{d^3 p'}{2E_{\mathbf{p}'}} \, \Theta(2E_{\mathbf{p}} - E_{\mathbf{p}'}) \, \frac{c^2}{4E_{\mathbf{p}}} \, \delta(E_{\mathbf{p}} - E_{\mathbf{p}'}) f(\mathbf{p}', -\mathbf{p}') \, . \tag{11.158}$$

In view of a further evaluation of the $\mathbf{p}'$-integral, we introduce spherical coordinates for which we get

$$d^3 p' = |\mathbf{p}'|^2 \, d|\mathbf{p}'| d\Omega \, , \qquad d\Omega = \sin\theta d\theta d\phi \, . \tag{11.159}$$

Furthermore, due to a comparison of (11.136), (11.146), and (11.147), we identify $f(\mathbf{p}', -\mathbf{p}')$ with $F(|\mathbf{p}'|, \theta)$:

$$I = \frac{c}{8E_{\mathbf{p}}} \int_0^\infty d|\mathbf{p}'| \, \frac{|\mathbf{p}'|^2}{E_{\mathbf{p}'}} \int d\Omega \, \Theta(2E_{\mathbf{p}} - E_{\mathbf{p}'}) \delta(E_{\mathbf{p}} - E_{\mathbf{p}'}) F(|\mathbf{p}'|, \theta) \, . \tag{11.160}$$

Due to the relativistic energy-momentum dispersion (11.157) we obtain the following substitution:

$$2E_{\mathbf{p}'} \, dE_{\mathbf{p}'} = 2|\mathbf{p}'| \, d|\mathbf{p}'| c^2 \qquad \Longrightarrow \qquad d|\mathbf{p}'| = \frac{E_{\mathbf{p}'}}{|\mathbf{p}'| c^2} \, dE_{\mathbf{p}'} \, , \tag{11.161}$$

so that (11.160) goes over into

$$\begin{aligned} I &= \frac{c}{8E_{\mathbf{p}}} \int_0^\infty dE_{\mathbf{p}'} \, \frac{E_{\mathbf{p}'}}{|\mathbf{p}'| c^2} \, \frac{|\mathbf{p}'|^2}{E_{\mathbf{p}'}} \int d\Omega \, \Theta(2E_{\mathbf{p}} - E_{\mathbf{p}'}) \delta(E_{\mathbf{p}} - E_{\mathbf{p}'}) F(|\mathbf{p}'|, \theta) \qquad (11.162) \\ &= \frac{1}{8c^2 E_{\mathbf{p}}} \int d\Omega \int_0^\infty dE_{\mathbf{p}'} \sqrt{E_{\mathbf{p}'}^2 - M^2 c^4} \, \Theta(2E_{\mathbf{p}} - E_{\mathbf{p}'}) \delta(E_{\mathbf{p}} - E_{\mathbf{p}'}) F\left(\sqrt{E_{\mathbf{p}'}^2/c^2 - M^2 c^2}, \theta\right) \, . \end{aligned}$$

Now the $E_{\mathbf{p}'}$ integral can be performed due to the delta function, yielding, finally, the conservation of energy $E_{\mathbf{p}'} = E_{\mathbf{p}}$:

$$I = \frac{\sqrt{E_{\mathbf{p}}^2 - M^2 c^4}}{8c^2 E_{\mathbf{p}}} \int d\Omega \, F\left(\sqrt{E_{\mathbf{p}}^2/c^2 - M^2 c^2}, \theta\right) \, . \tag{11.163}$$

Based on the result (11.163) for the two integrals (11.147) in the center of mass reference frame, we now obtain for the averaged transition rate per volume (11.146) with identifying $E = E_{\mathbf{p}}$:

$$W = \frac{e^4}{\pi^2 \epsilon_0^2 c} \, \frac{M^4 c^8}{V^2 E^2} \, \frac{\sqrt{E^2 - M^2 c^4}}{8c^2 E} \int d\Omega \, \overline{\left| M_{fi}^{(2)} \right|^2} \, . \tag{11.164}$$

Checking the physical units of (11.164) by taking into account (11.136) yields, indeed, as expected: $[W] = 1/(\mathrm{s}\,\mathrm{m}^3)$.

11.7 Cross Section

In order to calculate the cross section we still need the number of incoming electrons per time unit and area. For this purpose, we consider again the normal order of the four current density operator (11.17), but this time for electrons being confined in a finite volume V. To this end we apply (11.138) and its Dirac adjoint, yielding instead of (11.17):

$$
\begin{aligned}
&: \hat{j}^\mu(x) := c \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{s_1} \sum_{s_2} \sqrt{\frac{Mc^2}{VE_{\mathbf{p}_1}}} \sqrt{\frac{Mc^2}{VE_{\mathbf{p}_2}}} \qquad (11.165) \\
&\Big\{ e^{i(p_2-p_1)x/\hbar}\, \overline{u}(\mathbf{p}_2, s_2)\gamma^\mu u(\mathbf{p}_1, s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1} + e^{i(p_2+p_1)x/\hbar}\, \overline{u}(\mathbf{p}_2, s_2)\gamma^\mu v(\mathbf{p}_1, s_1)\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}^\dagger_{\mathbf{p}_1,s_1} \\
&+ e^{-i(p_2+p_1)x/\hbar}\, \overline{v}(\mathbf{p}_2, s_2)\gamma^\mu u(\mathbf{p}_1, s_1)\hat{d}_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_1,s_1} - e^{-i(p_2+p_1)x/\hbar}\, \overline{v}(\mathbf{p}_2, s_2)\gamma^\mu v(\mathbf{p}_1, s_1)\hat{d}^+_{\mathbf{p}_2,s_2}\hat{d}_{\mathbf{p}_1,s_1} \Big\} \,.
\end{aligned}
$$

Evaluating the matrix element of (11.165) with respect to the initial state (11.13) leads to

$$
\begin{aligned}
\langle \psi_i | : \hat{j}^\mu(x) : |\psi_i\rangle \;=\; & c \sum_{\mathbf{p}_1} \sum_{\mathbf{p}_2} \sum_{s_1} \sum_{s_2} \sqrt{\frac{Mc^2}{VE_{\mathbf{p}_1}}} \sqrt{\frac{Mc^2}{VE_{\mathbf{p}_2}}} \\
& \times e^{i(p_2-p_1)x/\hbar}\, \overline{u}(\mathbf{p}_2, s_2)\gamma^\mu u(\mathbf{p}_1, s_1)\, C(\mathbf{p}_1, \mathbf{p}_2; s_1, s_2)\,, \qquad (11.166)
\end{aligned}
$$

where we have introduced as an abbreviation the vaccum expectation value

$$
C(\mathbf{p}_1, \mathbf{p}_2; s_1, s_2) \;=\; \langle 0|\, \hat{b}_{\mathbf{p}_{i_2},s_{i_2}} \left(\hat{b}_{\mathbf{p}_{i_1},s_{i_1}} \hat{b}^\dagger_{2,s_2}\right) \left(\hat{b}_{\mathbf{p}_1,s_1} \hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\right) \hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle \,. \qquad (11.167)
$$

Applying the anti-commutator algebra (9.405) we obtain from (11.167)

$$
\begin{aligned}
\cdots &= \langle 0|\, \hat{b}_{\mathbf{p}_{i_2},s_{i_2}} \left(-\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_{i_1},s_{i_1}} + \delta_{\mathbf{p}_{i_1},\mathbf{p}_2}\delta_{s_{i_1},s_2}\right) \left(-\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}_{\mathbf{p}_1,s_1} + \delta_{\mathbf{p}_{i_1},\mathbf{p}_1}\delta_{s_{i_1},s_1}\right) \hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle \qquad (11.168) \\
&= \langle 0|\, (\hat{b}_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}^\dagger_{\mathbf{p}_2,s_2})\hat{b}_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}(\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}) |0\rangle - \delta_{\mathbf{p}_{i_1},\mathbf{p}_2}\delta_{s_{i_1},s_2} \langle 0|\, \hat{b}_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}_{\mathbf{p}_1,s_1}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle \\
&- \delta_{\mathbf{p}_{i_1},\mathbf{p}_1}\delta_{s_{i_1},s_1} \langle 0|\, \hat{b}_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle + \delta_{\mathbf{p}_{i_1},\mathbf{p}_1}\delta_{\mathbf{p}_{i_1},\mathbf{p}_2}\delta_{s_{i_1},s_2}\delta_{s_{i_1},s_2} \langle 0|\, \hat{b}_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle \,.
\end{aligned}
$$

Since it is assumed that the initial momenta $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}$ differ from each other, the respective fermionic operators $\hat{b}_{\mathbf{p}_{i_2},s_{i_2}}$, $\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}}$ and $\hat{b}_{\mathbf{p}_{i_1},s_{i_1}}$, $\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}$ anticommute, respectively. Therefore, the second and the third matrix element in (11.168) disappear, so we obtain

$$
\begin{aligned}
C(\mathbf{p}_1, \mathbf{p}_2; s_1, s_2) &= \langle 0| \Big(-\hat{b}^\dagger_{\mathbf{p}_2,s_2}\hat{b}_{\mathbf{p}_{i_2},s_{i_2}} + \delta_{\mathbf{p}_{i_2},\mathbf{p}_2}\delta_{s_{i_2},s_2}\Big) \hat{b}_{\mathbf{p}_{i_1},s_{i_1}}\hat{b}^\dagger_{\mathbf{p}_{i_1},s_{i_1}} \Big(-\hat{b}^\dagger_{\mathbf{p}_{i_2},s_{i_2}}\hat{b}_{\mathbf{p}_1,s_1} + \delta_{\mathbf{p}_{i_2},\mathbf{p}_1}\delta_{s_{i_2},s_1}\Big) |0\rangle \\
&+ \delta_{\mathbf{p}_{i_1},\mathbf{p}_1}\delta_{\mathbf{p}_{i_2},\mathbf{p}_2}\delta_{s_{i_1},s_1}\delta_{s_{i_1},s_2} \langle 0|\, 1 - \hat{b}^\dagger_{i_2,s_{i_2}}\hat{b}_{\mathbf{p}_{i_2},s_{i_2}} |0\rangle = \delta_{\mathbf{p}_1,\mathbf{p}_2}\delta_{s_1,s_2}\Big(\delta_{\mathbf{p}_1,\mathbf{p}_{i_1}}\delta_{s_1,s_{i_1}} + \delta_{\mathbf{p}_1,\mathbf{p}_{i_2}}\delta_{s_1,s_{i_2}}\Big) \,. \quad (11.169)
\end{aligned}
$$

Inserting the vaccum expectation value (11.169) into (11.166) leads to the matrix element

$$
\langle \psi_i | : \hat{j}^\mu(x) : |\psi_i\rangle = c\, \frac{Mc^2}{VE_{\mathbf{p}_{i_1}}}\, \overline{u}(\mathbf{p}_{i_1}, s_{i_1})\gamma^\mu u(\mathbf{p}_{i_1}, s_{i_1}) + c\, \frac{Mc^2}{VE_{\mathbf{p}_{i_2}}}\, \overline{u}(\mathbf{p}_{i_2}, s_{i_2})\gamma^\mu u(\mathbf{p}_{i_2}, s_{i_2})\,. \qquad (11.170)
$$

Afterwards, we average this current density with respect to the polarizations of both incoming electrons:

$$
J^\mu = \frac{1}{4} \sum_{s_{i_1}} \sum_{s_{i_2}} \langle \psi_i | : \hat{j}^\mu(x) : |\psi_i\rangle \,. \qquad (11.171)
$$

Substituting (11.170) into (11.171) we obtain

$$J^\mu = \frac{Mc^3}{2VE_{\mathbf{p}_{i_1}}} \sum_{s_{i_1}} \overline{u}_\alpha(\mathbf{p}_{i_1}, s_{i_1}) \gamma^\mu_{\alpha\beta} u_\beta(\mathbf{p}_{i_1}, s_{i_1}) + \frac{Mc^3}{2VE_{\mathbf{p}_{i_2}}} \sum_{s_{i_2}} \overline{u}_\alpha(\mathbf{p}_{i_2}, s_{i_2}) \gamma^\mu_{\alpha\beta} u_\beta(\mathbf{p}_{i_2}, s_{i_2}) \, . \tag{11.172}$$

The polarisation sums with respect to s_{i_1}, s_{i_2} were already calculated according to (9.436) and (9.445), yielding

$$J^\mu = \frac{Mc^3}{2VE_{\mathbf{p}_{i_1}}} \gamma^\mu_{\alpha\beta} \left(\frac{\mathbf{p}_{i_1\nu}\gamma^\nu + Mc}{2Mc} \right)_{\alpha\beta} + \frac{Mc^3}{2VE_{\mathbf{p}_{i_2}}} \gamma^\mu_{\alpha\beta} \left(\frac{\mathbf{p}_{i_2\nu}\gamma^\nu + Mc}{2Mc} \right)_{\alpha\beta} . \tag{11.173}$$

The sums with respect to the spinorial indices can be interpreted as traces:

$$J^\mu = \frac{c^2}{4VE_{\mathbf{p}_{i_1}}} \left\{ p_{i_1\nu} \operatorname{Tr}[\gamma^\mu\gamma^\nu] + Mc \operatorname{Tr}[\gamma^\mu] \right\} + \frac{c^2}{4VE_{\mathbf{p}_{i_2}}} \left\{ p_{i_2\nu} \operatorname{Tr}[\gamma^\mu\gamma^\nu] + Mc \operatorname{Tr}[\gamma^\mu] \right\} . \tag{11.174}$$

Due to the trace rules (11.62) and (11.63) the polarization averaged current density (11.174) reduces to

$$J^\mu = \frac{p^\mu_{i_1} c^2}{VE_{\mathbf{p}_{i_1}}} + \frac{p^\mu_{i_2} c^2}{VE_{\mathbf{p}_{i_2}}} \, . \tag{11.175}$$

In the center of mass reference frame (11.122) applies, so that the polarization averaged current density (11.175) vanishes:

$$J^\mu = 0 \, . \tag{11.176}$$

The relative current density, however, turns out to be

$$\Delta J = \frac{2|\mathbf{p}|c^2}{VE_{\mathbf{p}}} \tag{11.177}$$

and has, indeed, the correct physics unit $[\Delta J] = 1/(\mathrm{s\,m}^2)$. The cross section follows now from the quotient of the averaged transition rate per volume W and the averaged current density ΔJ per volume:

$$\sigma = \frac{W}{\Delta J / V} \, . \tag{11.178}$$

Substituting (11.164) and (11.177) into (11.178) yields the total cross section in the form of

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega} \, , \tag{11.179}$$

so that the differential cross section is definied by

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16\pi^2\epsilon_0^2} \frac{M^4c^4}{E^2} \overline{\left| M^{(2)}_{f_i} \right|^2} \, . \tag{11.180}$$

Inserting the polarisation-averaged matrix element (11.136) therein then yields

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2\hbar^2c^2}{4E^2} \left[1 - \frac{8E^4 - 4M^2c^4E^2 - M^4c^8}{(E^2 - M^2c^4)^2} \frac{1}{\sin^2\theta} + \frac{4(2E^2 - M^2c^4)^2}{(E^2 - M^2c^4)^2} \frac{1}{\sin^4\theta} \right] . \tag{11.181}$$

Here we have introduced the Sommerfeld fine-structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}\,, \tag{11.182}$$

which quantifies the strength of the electromagnetic interaction between elementary charged particles. It is a dimensionless quantity related to the elementary charge e, which denotes the strength of the coupling of an elementary charged particle with the electromagnetic field. As a dimensionless quantity, its numerical value is approximately given by

$$\alpha \approx \frac{1}{137}\,. \tag{11.183}$$

The result (11.181) predicts the differential cross section for the elastic scattering of two unpolarized electrons on the basis of quantum electrodynamics. It was first calculated in the ultra-relativistic regime by Christian Møller in 1932 based on some guesses and consistency requirements, not using quantunm electrodynamics. The full quantum electrodynamical calculation was provided only a few years later by Bethe and Fermi. Note that the indistinguishability of the two electrons involved in the scattering is represented by the forward-backward symmetry, i.e. the differential cross section is invariant with respect to the substitution $\theta \to \pi - \theta$. Within a classic experiment at the Laboratory of Nuclear Studies (Cornell University, Ithaca, New York) the Møller scattering formula (11.181) was checked in detail [Phys. Rev. **94**, 357 (1954)]. To this end the absolute differential electron-electron scattering cross section was measured for the incident electron energy in the laboratory frame varying in the interval from 0.6 to 1.2 Mev, which has to be compared with the rest energy of the electron of 0.513 MeV. The technique of measurement combined good resolution with large energy transfers between the particles, so this experiment allowed a sensitive test of the Møller scattering formula (11.181) in the relativistic regime. The results verified the theoretical predictions within a 7% experimental error.

In the ultra-relativistic limit $E \gg Mc^2$ the differential cross section (11.181) reduces to:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm ur} = \frac{\alpha^2\hbar^2c^2}{4E^2}\left(1 - \frac{8}{\sin^2\theta} + \frac{16}{\sin^4\theta}\right)\,. \tag{11.184}$$

With the help of the trigonometric formulae

$$\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}\sin\theta\,,\;\; \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1-\cos\theta\right),\;\; \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1+\cos\theta\right) \tag{11.185}$$

follows the trigonometric side calculation

$$\frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{2}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} + \frac{1+\sin^4\left(\frac{\theta}{2}\right)}{\cos^4\left(\frac{\theta}{2}\right)} = 2\left(1 - \frac{8}{\sin^2\theta} + \frac{16}{\sin^4\theta}\right)\,. \tag{11.186}$$

Inserting (11.186) into (11.184) leads to

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm ur} = \frac{\alpha^2\hbar^2c^2}{8E^2}\left[\frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{2}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} + \frac{1-\sin^4\left(\frac{\theta}{2}\right)}{\cos^4\left(\frac{\theta}{2}\right)}\right]\,. \tag{11.187}$$

In the opposite non-relativistic limit $E = Mc^2 + \epsilon$ we obtain with $\epsilon \ll Mc^2$ from (11.181)

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm nr} = \frac{\alpha^2\hbar^2c^2}{16\epsilon^2}\left(\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right) . \tag{11.188}$$

With the trigonometric formulae (11.185) follows the trigonometric side calculation

$$\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} + \frac{1}{\cos^4\left(\frac{\theta}{2}\right)} = \frac{16}{\sin^4\theta}\left\{\frac{1}{4}(1+\cos\theta)^2 + \frac{1}{4}(1-\cos\theta)^2 \right.$$
$$\left. -\frac{1}{4}(1-\cos\theta)(1+\cos\theta)\right\} = \frac{16}{\sin^4\theta}\left(1 - \frac{3}{4}\sin^2\theta\right) = 4\left(\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta}\right) . \tag{11.189}$$

With this (11.188) goes over into

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm nr} = \frac{\alpha^2\hbar^2c^2}{64\epsilon^2}\left[\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{\cos^4\left(\frac{\theta}{2}\right)} - \frac{1}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}\right] . \tag{11.190}$$

With the non-relativistic dispersion relation $\epsilon = \mathbf{p}^2/(2M)$ it follows finally

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm nr} = \frac{\alpha^2\hbar^2c^2M^2}{16\mathbf{p}^4}\left[\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{1}{\cos^4\left(\frac{\theta}{2}\right)} - \frac{1}{\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)}\right] . \tag{11.191}$$

The first term in (11.191) just corresponds to the cross section of the Rutherford scattering

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm R} = \frac{\alpha^2\hbar^2c^2M^2Z^2}{4\mathbf{p}^4}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} . \tag{11.192}$$

with the nuclear charge number $Z = 1$ and the mass M being substituted by the reduced mass $M/2$. This means that the forward peak of the non-relativistic Møller scattering at $\theta \approx 0$ agrees with the prediction of Rutherford prediction. Beyond that, however, there is another significant backward peak at $\theta = \pi$ that stems from interferences. Note that the latter must occur due to above mentioned forward-backward symmetry following from the indistinguishability of the electrons.

While formerly many particle colliders were designed specifically for electron-electron collisions, recently electron-positron colliders have become more common. Here one uses the so-called crossing symmetry, one of the useful tricks often used in quantum field theory to evaluate Feynman diagrams. Namely, from the Feynman rules follows directly that the unpolarized scattering matrix for any process involving a particle with momentum p in the initial state can be converted into the unpolarized scattering matrix for an otherwise identical process but with an anti-particle of momentum $-p$ in the final state. This implies that the Møller scattering between two electrons (11.1) goes over into the corresponding unpolarized cross section of the Bhabha scattering, i.e. the electron-positron scattering:

$$e^- e^+ \to e^- e^+ . \tag{11.193}$$

Applying this crossing symmetry to the unpolarized Møller cross section turns out to have the consequence that the unpolarized Bhabha cross section follows by interchaging the Mandelstam parameter s and u in (11.108), (11.109), and (11.112):

$$s \leftrightarrow u \,. \tag{11.194}$$

We refrain here from discussing the respective energy and angle dependence of the Bhabha differential cross section. Instead we refer to the above mentioned classic experiment at the Laboratory of Nuclear Studies, where the absolute differential positron-electron scattering cross section was checked in the energy interval from 0.6 to 1.0 Mev, which verified the Bhabha formula within the 10% experimental error. Furthermore, the ratio of the Møller and the Bhabha cross sections was also measured with somewhat increased accuracy, yielding a verification within about 8% experimental error.

In the last three decades Bhabha scattering has been used as a luminosity monitor in a number of e^-e^+ collider physics experiments. The accurate measurement of luminosity is necessary for accurate measurements of cross sections. Small-angle Bhabha scattering was used to measure the luminosity of the 1993 run of the Stanford Large Detector (SLD), with a relative uncertainty of less than 0.5%. Electron-positron colliders operating in the region of the low-lying hadronic resonances (about 1 GeV to 10 GeV), such as the Beijing Electron Synchrotron (BES) and the Belle and BaBar "B-factory" experiments, use large-angle Bhabha scattering as a luminosity monitor. To achieve the desired precision at the 0.1% level, the experimental measurements must be compared to a theoretical calculation including next-to-leading-order radiative corrections. The high-precision measurement of the total hadronic cross section at these low energies is, for instance, a crucial input into the theoretical calculation of the anomalous magnetic dipole moment of the muon, which is used to constrain supersymmetry and other models of physics beyond the Standard Model.