Chapter 2

Identical Particles

Here we deal with identical particles, thus they have exactly the same properties like mass, spin or charge. From all experiments performed so far in the realm of quantum mechanics one can deduce that such identical particles are indistinguishable. Nevertheless we start with describing a quantum many-particle system in Section 2.1 as if their identical particles would be distinguishable. Based on that we investigate then in Section 2.2 the consequences for postulating the indistinguishability of identical particles. Namely, it turns out in three spatial dimensions that identical particles are either bosons or fermions, which are characterized by a symmetric and anti-symmetric many-body wave function, respectively. We illustrate the corresponding complications in concrete calculations by the illustrative example of non-interacting identical particles in Section 2.3.

2.1 Distinguishable Particles

A many-body system of identical nonrelativistic particles of mass M is classically described by the Lagrange function

$$L(\mathbf{x}_1,\ldots,\mathbf{x}_n;\dot{\mathbf{x}}_1,\ldots,\dot{\mathbf{x}}_n) = \sum_{\nu=1}^{n} \frac{M}{2}\,\dot{\mathbf{x}}_\nu^2 - V(\mathbf{x}_1,\ldots,\mathbf{x}_n)\,. \tag{2.1}$$

The n-particle potential $V(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is usually additive in both the 1-particle potentials $V_1(\mathbf{x}_\nu)$ and the 2-particle potentials $V_2(\mathbf{x}_\nu - \mathbf{x}_\mu)$:

$$V(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\nu=1}^{n} V_1(\mathbf{x}_\nu) + \frac{1}{2}\sum_{\nu=1}^{n}\sum_{\mu=1}^{n} V_2(\mathbf{x}_\nu - \mathbf{x}_\mu)\,. \tag{2.2}$$

Note that the latter must obey the symmetry

$$V_2(\mathbf{x}_\nu - \mathbf{x}_\mu) = V_2(\mathbf{x}_\mu - \mathbf{x}_\nu) \tag{2.3}$$

due to the Newton axiom "action = - reactio". The Euler-Lagrange equations

$$\frac{\partial L}{\partial \mathbf{x}_\nu} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}_\nu} = 0 \tag{2.4}$$

corresponding to the Lagrange function (2.1), (2.2) lead to the Newton equations of motion:

$$M\ddot{\mathbf{x}}_\nu = -\frac{\partial V_1(\mathbf{x}_\nu)}{\partial \mathbf{x}_\nu} - \sum_{\mu=1}^{n} \frac{\partial V_2(\mathbf{x}_\nu - \mathbf{x}_\mu)}{\partial \mathbf{x}_\nu} \,. \tag{2.5}$$

The transition to the Hamilton formalism is implemented by introducing the canonically conjugated momenta

$$\mathbf{p}_\nu = \frac{\partial L}{\partial \dot{\mathbf{x}}_\nu} = M\dot{\mathbf{x}}_\nu \tag{2.6}$$

and by performing the Legendre transformation

$$H(\mathbf{p}_1,\ldots,\mathbf{p}_n;\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\nu=1}^{n} \mathbf{p}_\nu \dot{\mathbf{x}}_\nu - L(\mathbf{x}_1,\ldots,\mathbf{x}_n;\dot{\mathbf{x}}_1,\ldots,\dot{\mathbf{x}}_n) \tag{2.7}$$

which yields the Hamilton function

$$H(\mathbf{p}_1,\ldots,\mathbf{p}_n;\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\nu=1}^{n} \frac{\mathbf{p}_\nu^2}{2M} + \sum_{\nu=1}^{n} V_1(\mathbf{x}_\nu) + \frac{1}{2}\sum_{\nu=1}^{n}\sum_{\mu=1}^{n} V_2(\mathbf{x}_\nu - \mathbf{x}_\mu) \,. \tag{2.8}$$

The corresponding Hamilton equations

$$\dot{\mathbf{x}}_\nu = \frac{\partial H}{\partial \mathbf{p}_\nu} = \frac{\mathbf{p}_\nu}{M} \,, \tag{2.9}$$

$$\dot{\mathbf{p}}_\nu = -\frac{\partial H}{\partial \mathbf{x}_\nu} = -\frac{\partial V_1(\mathbf{x}_\nu)}{\partial \mathbf{x}_\nu} - \sum_{\mu=1}^{n} \frac{\partial V_2(\mathbf{x}_\nu - \mathbf{x}_\mu)}{\partial \mathbf{x}_\nu} \tag{2.10}$$

turn out to be equivalent to the Newton equations of motion (2.5).

The transition from classical mechanics to quantum mechanics is achieved by assigning operators to observables:

$$\mathbf{x}_\nu \to \hat{\mathbf{x}}_\nu \,, \quad \mathbf{p}_\nu \to \hat{\mathbf{p}}_\nu \,, \quad H(\mathbf{p}_1,\ldots,\mathbf{p}_n;\mathbf{x}_1,\ldots,\mathbf{x}_n) \to H(\hat{\mathbf{p}}_1,\ldots,\hat{\mathbf{p}}_n;\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_n) \,. \tag{2.11}$$

In order to obey the Heisenberg uncertainty relation, we postulate here the following canonical commutation relations

$$\left[\hat{x}_{j\nu}, \hat{x}_{k\mu}\right]_- = \left[\hat{p}_{j\nu}, \hat{p}_{k\mu}\right]_- = 0 \,, \qquad \left[\hat{p}_{j\nu}, \hat{x}_{k\mu}\right]_- = \frac{\hbar}{i}\,\delta_{jk}\,\delta_{\nu\mu} \,, \tag{2.12}$$

where the commutator between two quantum mechanical operators $\hat{A}$ and $\hat{B}$ is defined by

$$\left[\hat{A}, \hat{B}\right]_- = \hat{A}\hat{B} - \hat{B}\hat{A} \,. \tag{2.13}$$

The time evolution of a quantum mechanical state vector $|\psi(t)\rangle$ is described by the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}|\psi(t)\rangle \,. \tag{2.14}$$

In order to convert this representation independent formulation of quantum mechanics to the spatial representation, one chooses as a basis the eigenstates $|\mathbf{x}_1, \dots, \mathbf{x}_n\rangle$ of the coordinate operators $\hat{\mathbf{x}}_\nu$. They fulfill the eigenvalue problem

$$\hat{\mathbf{x}}_\nu |\mathbf{x}_1, \dots, \mathbf{x}_n\rangle = \mathbf{x}_\nu |\mathbf{x}_1, \dots, \mathbf{x}_n\rangle \tag{2.15}$$

as well as the orthonormality relation

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_n | \mathbf{x}'_1, \dots, \mathbf{x}'_n\rangle = \delta(\mathbf{x}_1 - \mathbf{x}'_1) \cdot \ldots \cdot \delta(\mathbf{x}_n - \mathbf{x}'_n) \tag{2.16}$$

and the completeness relation

$$\int d^3x_1 \cdot \ldots \cdot \int d^3x_n \, |\mathbf{x}_1, \dots, \mathbf{x}_n\rangle\langle \mathbf{x}_1, \dots, \mathbf{x}_n| = 1 \, . \tag{2.17}$$

The spatial representation of the momentum operators $\hat{\mathbf{p}}_\nu$ is given by the Jordan rule:

$$\langle \mathbf{x}_1, \dots, \mathbf{x}_n | \hat{\mathbf{p}}_\nu = \frac{\hbar}{i} \frac{\partial}{\partial \mathbf{x}_\nu} \langle \mathbf{x}_1, \dots, \mathbf{x}_n | \, . \tag{2.18}$$

Evolving the quantum mechanical state vector $|\psi(t)\rangle$ with respect to this basis yields due to the completeness relation (2.17)

$$|\psi(t)\rangle = \int d^3x_1 \cdot \ldots \cdot \int d^3x_n \, \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \, |\mathbf{x}_1, \dots, \mathbf{x}_n\rangle \, , \tag{2.19}$$

where the expansion coefficients represent the n-particle wave function

$$\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) = \langle \mathbf{x}_1, \dots, \mathbf{x}_n | \psi(t)\rangle \, . \tag{2.20}$$

Multiplying (2.14) from the left with the bra-vector $\langle \mathbf{x}_1, \dots, \mathbf{x}_n|$ leads for the n-particle wave function (2.20) to the n-particle Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) = \hat{H} \psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) \, . \tag{2.21}$$

Here the spatial representation of the Hamilton operator $\hat{H}$ follows due to (2.11), (2.15), and (2.18) from the Hamilton function H as follows:

$$\hat{H} = H\left(\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}_1}, \dots, \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}_n}; \mathbf{x}_1, \dots, \mathbf{x}_n\right) . \tag{2.22}$$

In case of the standard Hamilton function (2.8) we get

$$\hat{H} = \sum_{\nu=1}^{n} \left\{ -\frac{\hbar^2}{2M} \Delta_\nu + V_1(\mathbf{x}_\nu) \right\} + \frac{1}{2} \sum_{\nu=1}^{n} \sum_{\mu=1}^{n} V_2(\mathbf{x}_\nu - \mathbf{x}_\mu) \, . \tag{2.23}$$

As we have assumed here that both the 1- and the 2-particle potential V_1 and V_2 do not explicitly depend on time, one can perform for the n-particle wave function the separation ansatz

$$\psi(\mathbf{x}_1, \dots, \mathbf{x}_n; t) = \psi_E(\mathbf{x}_1, \dots, \mathbf{x}_n) \, e^{-iEt/\hbar} \, . \tag{2.24}$$

This reduces the time-dependent Schrödinger equation (2.21) to the time-independent Schrödinger equation:

$$\hat{H} \psi_E(\mathbf{x}_1, \dots, \mathbf{x}_n) = E \psi_E(\mathbf{x}_1, \dots, \mathbf{x}_n) \, , \tag{2.25}$$

where E denotes the energy eigenvalue.

2.2 Bosons and Fermions

The quantum mechanical laws summarized in the last section are only valid for identical particles, which are assumed to be distinguishable. But experimentally it has turned out that identical particles always happen to behave in the same way so that no objective measurement allows to distinguish one from the other. Thus, in the realm of quantum many-body theory the fundamental principle of the indistinguishability of identical particles has to be taken into account.

Physically relevant are only expectation values of observables. The principle of the indistinguishability of identical particles means in this context concretely that the expectation value of any operator $\hat{A}$ must not change when the enumeration of two particles is swapped within the n-particle wave function:

$$\begin{aligned} & \int d^3x_1 \cdot \ldots \cdot \int d^3x_n \, \psi^*(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n) \hat{A} \, \psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n) \\ = & \int d^3x_1 \cdot \ldots \cdot \int d^3x_n \, \psi^*(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) \hat{A} \, \psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) \, . \end{aligned} \tag{2.26}$$

From this definition of indistinguishability of identical particles we now derive various characteristic properties for both the operators $\hat{A}$ and the n-particle wave functions $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. Note that restricting the equality of expectation values (2.26) to just two particles is not a principle limitation as any permutation $\hat{P}$ can always be represented as a certain product of transpositions $\hat{P}_{jk}$

$$\hat{P} = \prod \hat{P}_{jk} \, . \tag{2.27}$$

Here the action of $\hat{P}_{jk}$ is defined by exchanging the particle coordinates j and k in the n-particle wave function:

$$\hat{P}_{jk} \psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n) = \psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) \, . \tag{2.28}$$

From (2.28) it is self-evident that the transposition $\hat{P}_{jk}$ is involutoric, i.e. applying it twice yields back the original n-particle wave function:

$$\hat{P}_{jk} \, \hat{P}_{jk} = 1 \qquad \Longrightarrow \qquad \hat{P}_{jk} = \hat{P}_{jk}^{-1} \, . \tag{2.29}$$

With the help of the transposition operator $\hat{P}_{jk}$ the defining equation (2.26) of the indistinguishability of identical particles can be converted from the spatial representation into the representation-free formulation:

$$\langle \psi | \hat{A} | \psi \rangle = \langle \hat{P}_{jk} \psi | \hat{A} | \hat{P}_{jk} \psi \rangle = \langle \psi | \hat{P}_{jk}^\dagger \hat{A} \hat{P}_{jk} | \psi \rangle \, . \tag{2.30}$$

From the trivial decomposition

$$\begin{aligned} \langle \phi | \hat{A} | \psi \rangle \; = \; & \frac{1}{4} \Big[\langle \phi + \psi | \hat{A} | \phi + \psi \rangle - \langle \phi - \psi | \hat{A} | \phi - \psi \rangle \\ & + i \langle \phi + i\psi | \hat{A} | \phi + i\psi \rangle - i \langle \phi - i\psi | \hat{A} | \phi - i\psi \rangle \Big] \end{aligned} \tag{2.31}$$

follows then together with (2.30) a useful identity for any matrix element:

$$\langle\phi|\hat{A}|\psi\rangle = \langle\phi|\hat{P}_{jk}^{\dagger}\hat{A}\hat{P}_{jk}|\psi\rangle\,. \tag{2.32}$$

Due to the arbitrariness of the states $|\phi\rangle$ and $|\psi\rangle$ we thus conclude the operator identity

$$\hat{A} = \hat{P}_{jk}^{\dagger}\hat{A}\hat{P}_{jk}\,. \tag{2.33}$$

Evaluating (2.33) for the special case $\hat{A} = \hat{P}_{jk}$ we read off due to (2.29) that the transposition operator $\hat{P}_{jk}$ turns out to be both hermitian

$$\hat{P}_{jk} = \hat{P}_{jk}^{\dagger} \tag{2.34}$$

and unitary

$$\hat{P}_{jk}^{-1} = \hat{P}_{jk}^{\dagger}\,. \tag{2.35}$$

Furthermore, we conclude from (2.33) and (2.35) that any operator $\hat{A}$ commutes with a transposition $\hat{P}_{jk}$:

$$\left[\hat{P}_{jk}, \hat{A}\right]_{-} = \hat{P}_{jk}\hat{A} - \hat{A}\hat{P}_{jk} = 0\,. \tag{2.36}$$

As the latter identity holds in particular for the Hamilton operator $\hat{A} = \hat{H}$ we know that there exist states, which are at the same time eigenstates of both the Hamilton operator $\hat{H}$ and all transposition operators $\hat{P}_{jk}$:

$$\hat{H}|\psi\rangle = E|\psi\rangle\,, \qquad \hat{P}_{jk}|\psi\rangle = p_{jk}|\psi\rangle\,. \tag{2.37}$$

Due to the hermiticity (2.34) of the transposition operators $\hat{P}_{jk}$ their respective eigenvalues p_{jk} must be real. And from the involutoric property (2.29) follows furthermore

$$p_{jk}^2 = 1 \tag{2.38}$$

Thus, the eigenvalues of the transposition operators $\hat{P}_{jk}$ are either $p_{jk} = 1$ or $p_{jk} = -1$. Moreover, it is reasonable that an n-particle wave function $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, which is an eigenfunction of all transposition operators $\hat{P}_{jk}$, must always have one and the same eigenvalue. In order to show this we consider the following identity:

$$\begin{aligned}
&\hat{P}_{1j}\hat{P}_{2k}\hat{P}_{12}\hat{P}_{2k}\hat{P}_{1j}\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n) \\
&= \hat{P}_{1j}\hat{P}_{2k}\hat{P}_{12}\psi(\mathbf{x}_j, \mathbf{x}_k, \ldots, \mathbf{x}_1, \ldots, \mathbf{x}_2, \ldots, \mathbf{x}_n)\hat{P}_{1j}\hat{P}_{2k}\psi(\mathbf{x}_k, \mathbf{x}_j, \ldots, \mathbf{x}_1, \ldots, \mathbf{x}_2, \ldots, \mathbf{x}_n) \\
&= \psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_n) = \hat{P}_{jk}\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_n)\,.
\end{aligned} \tag{2.39}$$

From this we conclude the operator identity:

$$\hat{P}_{jk} = \hat{P}_{1j}\hat{P}_{2k}\hat{P}_{12}\hat{P}_{2k}\hat{P}_{1j}\,, \tag{2.40}$$

so we obtain for the corresponding eigenvalues due to (2.38)

$$p_{jk} = (p_{1j})^2 \, (p_{2k})^2 \, p_{12} \qquad \Longrightarrow \qquad p_{jk} = p_{12} \, . \tag{2.41}$$

Therefore, identical particle possess either a symmetric ($\epsilon = +1$) or an anti-symmetric ($\epsilon = -1$) wave function with the property

$$\hat{P}_{jk}|\psi^\epsilon\rangle = \epsilon|\psi^\epsilon\rangle \, . \tag{2.42}$$

Using (2.34) and (2.42) we get that symmetric and anti-symmetric wave function are always orthogonal with respect to each other:

$$\begin{aligned} \langle\psi^-|\psi^+\rangle &= \langle\psi^-|\hat{P}_{jk}\psi^+\rangle = \langle\psi^-|\hat{P}_{jk}^\dagger\psi^+\rangle = \langle\hat{P}_{jk}\psi^-|\psi^+\rangle = -\langle\psi^-|\psi^+\rangle \\ &\Longrightarrow \qquad \langle\psi^-|\psi^+\rangle = 0 \, . \end{aligned} \tag{2.43}$$

Furthermore, it turns out that identical particles maintain their symmetry character for all times. To this end we state that the time evolution operator $\hat{U}(t_2, t_1)$ transforms an initial state of definite symmetry $|\psi^{\epsilon_1}(t_1)\rangle$ into a final state of definite symmetry $|\psi^{\epsilon_2}(t_2)\rangle$ via

$$|\psi^{\epsilon_2}(t_2)\rangle = \hat{U}(t_2, t_1)|\psi^{\epsilon_1}(t_1)\rangle \, . \tag{2.44}$$

Thus, taking (2.36) and (2.42) into account we conclude

$$\begin{aligned} \epsilon_2|\psi^{\epsilon_2}(t_2)\rangle = \hat{P}_{jk}|\psi^{\epsilon_2}(t_2)\rangle &= \hat{P}_{jk}\hat{U}(t_2, t_1)|\psi^{\epsilon_1}(t_1)\rangle = \hat{U}(t_2, t_1)\hat{P}_{jk}|\psi^{\epsilon_1}(t_1)\rangle = \epsilon_1|\psi^{\epsilon_1}(t_2)\rangle \\ &\Longrightarrow \qquad \epsilon_1 = \epsilon_2 \, . \end{aligned} \tag{2.45}$$

As a result we state that the Hilbert space of identical particles consists of either only symmetric or only anti-symmetric wave functions. In relativistic quantum field theory it is shown which Hilbert space is appropriate for which sort of particles. According to the spin-statistic theorem of Pauli identical particles with integer (half-integer) spin are bosons (fermions) and have symmetric (anti-symmetric) wave functions, see Tab. 1.2.

2.3 Non-Interacting Identical Particles

In general it is quite cumbersome to calculate n-particle wave functions by taking into account the symmetry property. We illustrate this by the example of non-interacting identical particles. According to (2.23), (2.25) and a vanishing 2-particle potential $V_2(\mathbf{x}_\nu - \mathbf{x}_\mu) = 0$ the following time-independent Schrödinger equation has to be solved:

$$\sum_{\nu=1}^{n} \left\{ -\frac{\hbar^2}{2M} \, \Delta_\nu + V_1(\mathbf{x}_\nu) \right\} \psi_E(\mathbf{x}_1, \ldots, \mathbf{x}_n) = E \, \psi_E(\mathbf{x}_1, \ldots, \mathbf{x}_n) \, . \tag{2.46}$$

In the following we assume that the 1-particle wave functions $\psi_{E_\alpha}(\mathbf{x})$ with the vector of quantum numbers α are known as solutions of the time-independent 1-particle Schrödinger equation

$$\left\{ -\frac{\hbar^2}{2M} \, \Delta + V_1(\mathbf{x}) \right\} \psi_{E_\alpha}(\mathbf{x}) = E_\alpha \, \psi_{E_\alpha}(\mathbf{x}) \, . \tag{2.47}$$

Thus they represent an orthonormal basis obeying both the orthonormality relation

$$\int d^3x\, \psi^*_{E_\alpha}(\mathbf{x})\psi_{E_{\alpha'}}(\mathbf{x}) = \delta_{\alpha,\alpha'} \tag{2.48}$$

and the completeness relation

$$\sum_\alpha \psi^*_{E_\alpha}(\mathbf{x})\psi_{E_\alpha}(\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')\,. \tag{2.49}$$

In case that the particles would be distinguishable, then a solution of the time-independent n-particle Schrödinger equation (2.46) factorizes into 1-particle wave functions:

$$\psi_E(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \psi_{E_{\alpha_1},\ldots,E_{\alpha_n}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \prod_{\nu=1}^n \psi_{E_{\alpha_\nu}}(\mathbf{x}_\nu) \tag{2.50}$$

and the total energy is the sum of the respective 1-particle energies

$$E = \sum_{\nu=1}^n E_{\alpha_\nu}\,. \tag{2.51}$$

Furthermore, the orthonormality and completeness relations of the 1-particle wave functions (2.48) and (2.49) imply corresponding relations for the n-particle wave functions

$$\int d^3x_1 \cdots \int d^3x_n\, \psi^*_{E_{\alpha_1},\ldots,E_{\alpha_n}}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\psi_{E_{\alpha'_1},\ldots,E_{\alpha'_n}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \prod_{\nu=1}^n \delta_{\alpha_\nu,\alpha'_\nu}\,, \tag{2.52}$$

$$\sum_{\alpha_1}\cdots\sum_{\alpha_n} \psi^*_{E_{\alpha_1},\ldots,E_{\alpha_n}}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\psi_{E_{\alpha_1},\ldots,E_{\alpha_n}}(\mathbf{x}'_1,\ldots,\mathbf{x}'_n) = \prod_{\nu=1}^n \delta(\mathbf{x}_\nu-\mathbf{x}'_\nu)\,. \tag{2.53}$$

But, as identical particles are indistinguishable, the n-particle wave functions must either be symmetric or anti-symmetric. To this end we introduce the (anti-)symmetrization operator

$$\hat{S}^\epsilon = \sum_{\hat{P}} \epsilon^p \hat{P}\,, \tag{2.54}$$

which consists of a sum over all permutation operators $\hat{P}$ and p denotes the number of transpositions of a certain permutation corresponding to the decomposition (2.27). Multiplying a permutation $\hat{P}$ in the sum (2.54) with a single transposition $\hat{P}_{jk}$, one obtains another permutation $\hat{P}' = \hat{P}_{jk}\hat{P}$ with $p' = p \pm 1$. This has due to $\epsilon = \pm 1$ the following consequence:

$$\hat{P}_{jk}\hat{S}^\epsilon = \sum_{\hat{P}} \epsilon^p \hat{P}_{jk}\hat{P} = \sum_{\hat{P}'} \epsilon^{p'\mp 1}\hat{P}' = \epsilon \sum_{\hat{P}'} \epsilon^{p'}\hat{P}' = \epsilon\hat{S}^\epsilon\,. \tag{2.55}$$

With the prescription

$$\psi^\epsilon_{\{E_\alpha\}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = N^\epsilon_{\{E_\alpha\}}\hat{S}^\epsilon \prod_{\nu=1}^n \psi_{E_{\alpha_\nu}}(\mathbf{x}_\nu) \tag{2.56}$$

we construct for each wave function (2.50) of n distinguishable particles a corresponding symmetric ($\epsilon = 1$) or anti-symmetric $\epsilon = -1$ n-particle wave function, which obeys (2.42) by taking

(2.55) into account. Due to the indistinguishability property the (anti-)symmetrized n-particle wave function (2.56) turns out to be independent of the concrete order of the 1-particle energies $E_{\alpha_1}, \ldots, E_{\alpha_1}$. In order to emphasize that within our notation, we have introduced in (2.56) the index $\{E_\alpha\}$.

At first, we remark that the (anti-)symmetrized n-particle wave function (2.56) obeys the time-independent Schrödinger equation (2.46) with the energy eigenvalue (2.51). This follows from (2.36) as well as the circumstance (2.27) that each permutation operator $\hat{P}$ in the sum (2.54) can be represented as a product of transposition operators:

$$\hat{H}|\psi_E\rangle = E|\psi_E\rangle \qquad \Longrightarrow \qquad \hat{H}\hat{S}^\epsilon|\psi_E\rangle = \hat{S}^\epsilon\hat{H}|\psi_E\rangle = E\hat{S}^\epsilon|\psi_E\rangle\,. \tag{2.57}$$

Furthermore, we read off from (2.54) and (2.56) an important observation for the anti-symmetric n-particle wave function, which is characterized by $\epsilon = -1$:

$$\psi^-_{\{E_\alpha\}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = N^-_{\{E_\alpha\}} \sum_{\hat{P}} (-1)^p\, \psi_{E_{\alpha_1}}(\mathbf{x}_{P(1)}) \cdots \psi_{E_{\alpha_n}}(\mathbf{x}_{P(n)})\,. \tag{2.58}$$

Thus, the anti-symmetric n-particle wave function can be represented in form of a Slater determinant:

$$\psi^-_{\{E_\alpha\}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = N^-_{\{E_\alpha\}} \begin{vmatrix} \psi_{E_{\alpha_1}}(\mathbf{x}_1) & \psi_{E_{\alpha_1}}(\mathbf{x}_2) & \cdots & \psi_{E_{\alpha_1}}(\mathbf{x}_n) \\ \vdots & \vdots & & \vdots \\ \psi_{E_{\alpha_n}}(\mathbf{x}_1) & \psi_{E_{\alpha_n}}(\mathbf{x}_2) & \cdots & \psi_{E_{\alpha_n}}(\mathbf{x}_n) \end{vmatrix}\,. \tag{2.59}$$

In the case of an equality of two rows, i.e. $\alpha_j = \alpha_k$, or two columns, i.e. $\mathbf{x}_j = \mathbf{x}_k$, the anti-symmetric n-particle wave function (2.59) vanishes and with this the probability to have such a wave function. This just represents the fundamental Pauli exclusion principle that two fermions can not be neither in the same state nor at the same space point. A corresponding restriction does not exist for bosons. This means that there can be more than one boson in one state or at one space point. In order not to overload the following combinatorial considerations, we consider from now on only those bosonic wavefunctions, where a state or a space point is occupied at most by one boson.

It remains to determine the normalization constant $N^\epsilon_{\{E_\alpha\}}$ in (2.56). To this end we apply (2.27), that each permutation operator $\hat{P}$ can be represented by transpositions $\hat{P}_{jk}$, and conclude that iterating (2.55) yields

$$\hat{P}\hat{S}^\epsilon = \epsilon^p \hat{S}^\epsilon\,. \tag{2.60}$$

Taking into account (2.54) the scalar product between two (anti-)symmetric n-particle wave functions (2.56) reads at first

$$\langle\psi^\epsilon_{\{E_\alpha\}}|\psi^\epsilon_{\{E_{\alpha'}\}}\rangle = N^\epsilon_{\{E_\alpha\}} \sum_{\hat{P}} \epsilon^p \langle\psi_{E_{\alpha_1}} \cdots \psi_{E_{\alpha_n}}|\hat{P}^\dagger\psi^\epsilon_{\{E_{\alpha'}\}}\rangle\,. \tag{2.61}$$

Due to (2.27) and (2.34) as well as (2.56) and (2.60) this reduces to

$$\langle \psi^\epsilon_{\{E_\alpha\}} | \psi^\epsilon_{\{E_{\alpha'}\}} \rangle = N^\epsilon_{\{E_\alpha\}} \sum_{\hat{P}} \epsilon^{2p} \langle \psi_{E_{\alpha_1}} \cdots \psi_{E_{\alpha_n}} | \psi^\epsilon_{\{E_{\alpha'}\}} \rangle \,. \tag{2.62}$$

As we have $\epsilon = \pm 1$, the summand turns out to be independent of the respective permutations $\hat{P}$, so the sum reduces to the factor $n!$, which is the number of all possible permutations. Taking into account again (2.56) and (2.60) we get

$$\langle \psi^\epsilon_{\{E_\alpha\}} | \psi^\epsilon_{\{E_{\alpha'}\}} \rangle = N^\epsilon_{\{E_\alpha\}} N^\epsilon_{\{E_{\alpha'}\}} n! \sum_{\hat{P}'} \epsilon^{p'} \langle \psi_{E_{\alpha_1}} \cdots \psi_{E_{\alpha_n}} | \hat{P}' \psi_{E_{\alpha'_1}} \cdots \psi_{E_{\alpha'_n}} \rangle \,. \tag{2.63}$$

And with the orthonormality (2.52) of the 1-particle wavefunctions we finally obtain for the scalar product the expression

$$\langle \psi^\epsilon_{\{E_\alpha\}} | \psi^\epsilon_{\{E_{\alpha'}\}} \rangle = N^\epsilon_{\{E_\alpha\}} N^\epsilon_{\{E_{\alpha'}\}} n! \sum_{\hat{P}'} \epsilon^{p'} \delta_{\alpha_1, \alpha'_{P'(1)}} \cdots \delta_{\alpha_1, \alpha'_{P'(n)}} \,. \tag{2.64}$$

We now demand the orthonormality relation

$$\langle \psi^\epsilon_{\{E_\alpha\}} | \psi^\epsilon_{\{E_{\alpha'}\}} \rangle = \delta^\epsilon_{\alpha_1, \ldots, \alpha_n; \alpha'_1, \ldots \alpha'_n} \tag{2.65}$$

with the (anti-)symmetrized Kronecker symbol

$$\delta^\epsilon_{\alpha_1, \ldots, \alpha_n; \alpha'_1, \ldots \alpha'_n} = \sum_{\hat{P}} \epsilon^{p} \, \delta_{\alpha_1, \alpha'_{P(1)}} \cdots \delta_{\alpha_1, \alpha'_{P(n)}} \,. \tag{2.66}$$

As we restrict ourselves both for bosons and fermions to the case that all single-particle states differ from each other, i.e. $\alpha_\mu \neq \alpha_\nu$ for $\mu \neq \nu$, in (2.65) and (2.66) only the identity permutation $\hat{P} = 1$ survives, which fixes the normalization constant $N^\epsilon_{\{E_\alpha\}}$ according to

$$N^\epsilon_{\{E_\alpha\}} = \frac{1}{\sqrt{n!}} \,. \tag{2.67}$$

Finally, we show that one can span the whole Hilbert space of (anti-)symmetrized n-particle wave functions with (2.56). To this end we start from the completeness relation (2.53) of the n-particle wave function and apply twice the (anti-)symmetrization operator (2.54), once upon the space coordinates $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and once upon the space coordinates $\mathbf{x}'_1, \ldots, \mathbf{x}'_n$:

$$\begin{aligned} \sum_{\hat{P}} \sum_{\hat{P}'} \epsilon^{p+p'} \sum_{\alpha_1} \cdots \sum_{\alpha_n} \psi^*_{E_{\alpha_1}}(\mathbf{x}_{P(1)}) \cdots \psi^*_{E_{\alpha_n}}(\mathbf{x}_{P(n)}) \psi_{E_{\alpha_1}}(\mathbf{x}'_{P'(1)}) \cdots \psi_{E_{\alpha_n}}(\mathbf{x}'_{P'(n)}) \\ = \sum_{\hat{P}} \sum_{\hat{P}'} \epsilon^{p+p'} \delta(\mathbf{x}_{P(1)} - \mathbf{x}'_{P'(1)}) \cdots \delta(\mathbf{x}_{P(n)} - \mathbf{x}'_{P'(n)}) \,. \end{aligned} \tag{2.68}$$

At the left-hand side the space coordinates $\mathbf{x}_{P(1)}, \ldots, \mathbf{x}_{P(n)}$ and $\mathbf{x}'_{P'(1)}, \ldots, \mathbf{x}'_{P'(n)}$ are rearranged in their respective standard order $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $\mathbf{x}'_1, \ldots, \mathbf{x}'_n$. As a consequence the quantum numbers $\alpha_1, \ldots \alpha_n$ are rearranged to $\alpha_{P(1)}, \ldots \alpha_{P(n)}$ and $\alpha_{P'(1)}, \ldots \alpha_{P'(n)}$, respectively. A corresponding reordering on the right-hand side from $\mathbf{x}_{P(1)}, \ldots, \mathbf{x}_{P(n)}$ to $\mathbf{x}_1, \ldots, \mathbf{x}_n$ rearranges then $\mathbf{x}'_{P'(1)}, \ldots, \mathbf{x}'_{P'(n)}$ to $\mathbf{x}'_{P'(P(1))}, \ldots, \mathbf{x}'_{P'(P(n))}$, yielding

$$\sum_{\alpha_1} \cdots \sum_{\alpha_n} \Big\{ \sum_{\hat{P}} \epsilon^p \psi^*_{E_{\alpha_{P(1)}}}(\mathbf{x}_1) \cdots \psi^*_{E_{\alpha_{P(n)}}}(\mathbf{x}_n) \Big\} \Big\{ \sum_{\hat{P}'} \epsilon^{p'} \psi^*_{E_{\alpha_{P'(1)}}}(\mathbf{x}'_1) \cdots \psi^*_{E_{\alpha_{P'(n)}}}(\mathbf{x}'_n) \Big\}$$
$$= \sum_{\hat{P}} \sum_{\hat{P}'} \epsilon^{p+p'} \delta(\mathbf{x}_1 - \mathbf{x}'_{P'(P(1))}) \cdots \delta(\mathbf{x}_n - \mathbf{x}'_{P'(P(n))}) \, . \tag{2.69}$$

At the left-hand side we now use (2.54), (2.56), and (2.67), whereas at the right-hand side the sum over all permutations $\hat{P}'$ is substituted by an equivalent sum over all permutations $\hat{Q} = \hat{P}'\hat{P}$ with $q = p' + p$, so that afterwards the sum over $\hat{P}$ can straight-forwardly be performed. With this we finally obtain the completeness relation

$$\sum_{\alpha_1} \cdots \sum_{\alpha_n} \psi^{\epsilon *}_{\{E_\alpha\}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \psi^{\epsilon}_{\{E_\alpha\}}(\mathbf{x}'_1, \ldots, \mathbf{x}'_n) = \delta^\epsilon(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{x}'_1, \ldots, \mathbf{x}'_n) \, , \tag{2.70}$$

where we have introduced analogous to (2.66) the (anti-)symmetrized delta function

$$\delta^\epsilon(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{x}'_1, \ldots, \mathbf{x}'_n) = \sum_{\hat{Q}} \epsilon^q \delta(\mathbf{x}_1 - \mathbf{x}'_{Q(1)}) \cdots \delta(\mathbf{x}_n - \mathbf{x}'_{Q(n)}) \, . \tag{2.71}$$

The considerations of the present section have the purpose to generate a basis of the Hilbert space of indistinguishable identical particles via a(n) (anti-)symmetrization of the known basis of the Hilbert space of distinguishable identical particles. So far the starting point has been the eigenvalue problem (2.46) of the underlying Hamilton operator. But another basis results from considering the eigenvalue problem (2.15) of the coordinate operators as the starting point. Then the eigenfunctions $|\mathbf{x}_1, \ldots, \mathbf{x}_n\rangle$ with the continuous eigenvalues $\mathbf{x}_1, \ldots, \mathbf{x}_n$ span the Hilbert space of distinguishable identical particles. The subsequent (anti-)symmetrization is performed analogous to (2.54), (2.56), and (2.67), yielding another basis in the Hilbert space of indistinguishable identical particles:

$$|\mathbf{x}_1, \ldots, \mathbf{x}_n\rangle^\epsilon = \frac{1}{\sqrt{n!}} \sum_{\hat{P}} \epsilon^p |\mathbf{x}_{P(1)}, \ldots, \mathbf{x}_{P(n)}\rangle \, . \tag{2.72}$$

Both the orthonormality relation and the completeness relation corresponding to (2.65) and (2.70) read then

$$^\epsilon\langle \mathbf{x}_1, \ldots, \mathbf{x}_n | \mathbf{x}'_1, \ldots, \mathbf{x}'_n \rangle^\epsilon = \delta^\epsilon(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{x}'_1, \ldots, \mathbf{x}'_n) \, , \tag{2.73}$$
$$\int d^3x_1 \cdots \int d^3x_n \, |\mathbf{x}_1, \ldots, \mathbf{x}_n\rangle^\epsilon \, {}^\epsilon\langle \mathbf{x}_1, \ldots, \mathbf{x}_n| = 1 \, . \tag{2.74}$$

For the purpose of illustration we consider the spatial representation for two particles. The basis for two distinguishable identical particles reads in coordinate representation according to (2.16) and (2.20)

$$\psi_{\mathbf{x}_1, \mathbf{x}_2}(\mathbf{z}_1, \mathbf{z}_2) = \langle \mathbf{z}_1, \mathbf{z}_2 | \mathbf{x}_1, \mathbf{x}_2 \rangle = \delta(\mathbf{z}_1 - \mathbf{x}_1)\delta(\mathbf{z}_2 - \mathbf{x}_2) \, . \tag{2.75}$$

Correspondingly, the coordinate representation for two indistinguishable particles follows from (2.20):

$$\psi^{\epsilon}_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{z}_1,\mathbf{z}_2) = \langle \mathbf{z}_1,\mathbf{z}_2|\mathbf{x}_1,\mathbf{x}_2\rangle^{\epsilon}\,, \tag{2.76}$$

which reduces due to (2.16) and (2.72) to

$$\psi^{\epsilon}_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{z}_1,\mathbf{z}_2) = \frac{1}{\sqrt{2}}\Big\{\delta(\mathbf{z}_1-\mathbf{x}_1)\delta(\mathbf{z}_2-\mathbf{x}_2)+\epsilon\delta(\mathbf{z}_1-\mathbf{x}_2)\delta(\mathbf{z}_2-\mathbf{x}_1)\Big\}\,. \tag{2.77}$$

Note that (2.77) also follows from an (anti-)symmetrization (2.72) from (2.75) as defined by (2.54), (2.56), and (2.67). With this one obtains for the orthonormality relation (2.73) by taking into account (2.71)

$$\int d^3z_1 \int d^3z_2\, \psi^{\epsilon *}_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{z}_1,\mathbf{z}_2)\psi^{\epsilon}_{\mathbf{x}'_1,\mathbf{x}'_2}(\mathbf{z}_1,\mathbf{z}_2) = \delta^{\epsilon}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}'_1,\mathbf{x}'_2) \tag{2.78}$$

and correspondingly the completeness relation (2.74) reads together with (2.73) and (2.75)

$$\int d^3x_1 \int d^3x_2\, \psi^{\epsilon *}_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{z}_1,\mathbf{z}_2)\psi^{\epsilon}_{\mathbf{x}_1,\mathbf{x}_2}(\mathbf{z}'_1,\mathbf{z}'_2) = \delta^{\epsilon}(\mathbf{z}_1,\mathbf{z}_2;\mathbf{z}'_1,\mathbf{z}'_2)\,. \tag{2.79}$$