Chapter 6

Poincaré Group

According to special relativity the space-time in the absence of gravity has a flat Minkowskian structure. The group of symmetries, which leaves distances between events in this Minkowskian space-time invariant, is named after the mathematician Henri Poincaré as the Poincaré group. In the following we work out its properties as a Lie group, which unifies mathematical structures of a group and a manifold as its group elements depend continuously and differentiably on certain parameters. In fact, the Poincaré group turns out to be a ten-parametric, non-abelian Lie group, which contains rotations in space, boosts between inertial systems, and translations in space-time. Thus, the elements of the Poincaré group depend continuously and differentiably on the rotation angles, the boost velocities and the translations. Furthermore, we discuss the Poincaré algebra, which amounts to restricting the Poincaré group to the tangent plane at the identity element, yielding the generators of rotations, boosts, and translations. And, conversely, the Lie theorem turns out to allow to reconstruct the full Poincaré group by evaluating an exponential function involving both the generators, i.e. the elements of the Lie algebra, and the group parameters. And, finally, we determine the Casimir operators of the Poincaré group, i.e. those operators commuting with all elements of the Poincaré algebra. Their eigenvalues turn out to characterize all irreducible representations of the Poincaré group, to one of which each elementary particle of the standard model has to belong. In this way, the Poincaré group characterizes the underlying symmetry of relativistic quantum field theory and, thus, represents its very backbone.

6.1 Special Relativity

Albert Einstein formulated the special relativity in 1905, which has changed since then the very concept of space and time in the absence of gravity. It is based on two basic postulates:

1. Postulate: The velocity of light is the same in all inertial systems.

2. Postulate: The fundamental laws of physics have the same form in all inertial systems.

On the one hand, this implies concrete physical consequences for fast moving particles, which are nowadays confirmed, for instance, in the Large Hadron Collider (LHC) at Cern on a daily basis. A prominent example is provided by the time dilatation, i.e. for an observer in an inertial frame of reference, a clock that is moving relative to it in another inertial frame of reference will be measured to tick slower than a clock that is at rest in its frame of reference. On the other hand, special relativity also unifies the fundamental description of space and time. In view of formalising the second postulate, a point in space-time, which is also called the Minskowski space, is characterized by the contravariant space-time four-vector

$$(x^\mu) = \left(x^0, x^1, x^2, x^3\right) = \left(ct, x^i\right) = (ct, \mathbf{x}) \,. \tag{6.1}$$

Here we use the convention that Greek (Latin) indices run from 0 to 3 (from 1 to 3). Furthermore, from the first postulate follows for a light ray in two different inertial systems:

$$(ct)^2 - \mathbf{x}^2 = (ct')^2 - \mathbf{x}'^2 \,. \tag{6.2}$$

This condition can be reformulated with the help of the covariant Minkowski metric

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.3}$$

sas the invariance of the scalar product of the space-time four-vectors x^μ and x'^μ in the respective inertial systems:

$$g_{\mu\nu}\, x^\mu x^\nu = g_{\mu\nu}\, x'^\mu x'^\nu \,. \tag{6.4}$$

Note that we use here the Einstein summation convention that one has to sum over all indices, which appear twice, i.e. once in form of an upper or contravariant index and once in form of a lower or covariant index. Apart from the contravariant space-time four-vector (6.1) we also introduce the covariant space-time four-vector

$$x_\mu = g_{\mu\nu}\, x^\nu \,. \tag{6.5}$$

Thus, the contravariant space-time four-vector x^ν is transformed via contraction with the covariant metric $g_{\mu\nu}$ to the corresponding covariant space-time four-vector x_μ. Inserting (6.1) and (6.3) in (6.6) the respective components of the covariant space-time four-vector turn out to be

$$(x_\mu) = (x_0, x_1, x_2, x_3) = \left(ct, -x^i\right) = (ct, -\mathbf{x}) \,. \tag{6.6}$$

With this the invariance of the scalar product (6.4) reduces to

$$x^\mu x_\mu = x'^\mu x'_\mu \,. \tag{6.7}$$

Furthermore, the obvious identity

$$g_{\mu\nu}\,\delta^{\nu}{}_{\kappa} = g_{\mu\kappa} \tag{6.8}$$

with the Kronecker symbol $\delta^{\nu}{}_{\kappa}$ means that the latter can be identified with the Minkowski metric $g^{\nu}{}_{\kappa}$, which consists of both the contravariant index ν and the covariant index κ:

$$(g^{\nu}{}_{\kappa}) = (\delta^{\nu}{}_{\kappa}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{6.9}$$

In addition we also define

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{6.10}$$

for which we read off with (6.3) and (6.9) the obvious identity

$$g^{\mu\nu}\,g_{\nu\kappa} = \delta^{\mu}{}_{\kappa} = g^{\mu}{}_{\kappa} \,. \tag{6.11}$$

Thus, (6.10) represents the contravariant Minkoswki metric. Due to (6.5) and (6.11) the covariant space-time four-vector x_ν is transformed via contraction with the contravariant metric $g^{\mu\nu}$ to the corresponding contravariant space-time four-vector x^μ:

$$g^{\mu\nu}\,x_\nu = g^{\mu\nu}\,g_{\nu\kappa}\,x^\kappa = \delta^{\mu}{}_{\kappa}\,x^\kappa = x^\mu \tag{6.12}$$

Thus, we can summarize that the co- and contravariant Minkowski metrices allow to pull down and up indices according to (6.5) and (6.12).

But the concept of four-vectors is much more general than the mere description of space-time four-vectors. Namely, a four-vector represents objects whose scalar products coincide in all inertial systems. Let us consider in view of another example the seminal energy-momentum dispersion relation of a relativistic particle, see Fig. 6.1, in two different inertial systems:

$$E^2 = M^2c^4 + \mathbf{p}^2c^2 \,, \qquad E'^2 = M'^2c'^4 + \mathbf{p}'^2c^2 \,. \tag{6.13}$$

Due to the equality of the rest masses M and M' in both inertial systems

$$M = M' \tag{6.14}$$

the energy-momentum dispersion relations (6.13) reduce to the identity

$$\left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = \left(\frac{E'}{c}\right)^2 - \mathbf{p}'^2 \,. \tag{6.15}$$

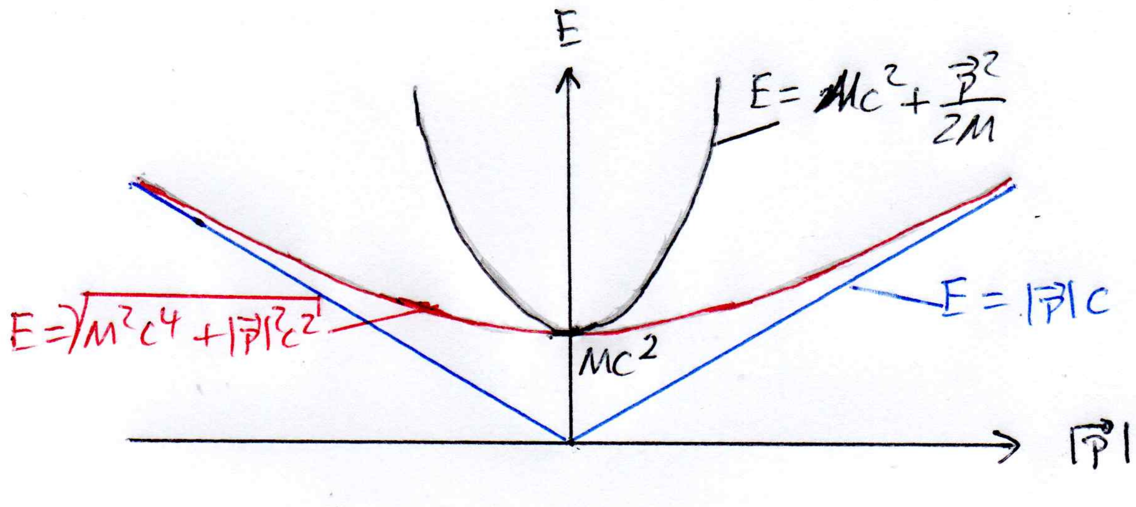


Figure 6.1: A relativistic energy-momentum dispersion (6.13) for massive (red) and massless (blue) particles in comparison with the non-relativistic limit (black).

Thus, introducing the contravariant momentum four-vector

$$(p^\mu) = \left(p^0, p^1, p^2, p^3\right) = \left(\frac{E}{c}, p^i\right) = \left(\frac{E}{c}, \mathbf{p}\right) \tag{6.16}$$

allows to formulate the identity (6.15) as the invariance of the scalar products of the contravariant momentum four-vectors p^μ and p'^μ:

$$g_{\mu\nu}\, p^\mu p^\nu = g_{\mu\nu}\, p'^\mu p'^\nu \,. \tag{6.17}$$

Defining in analogy to (6.5)

$$p_\mu = g_{\mu\nu}\, p^\nu \tag{6.18}$$

also the components of the covariant momentum four-vector

$$(p_\mu) = (p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -p^i\right) = \left(\frac{E}{c}, -\mathbf{p}\right) \,. \tag{6.19}$$

the invariance of the scalar product (6.17) can also be formulated as

$$p^\mu p_\mu = p'^\mu p'_\mu \,. \tag{6.20}$$

Furthermore, we conclude from (6.13), (6.14), and (6.19) that the scalar product of the four-momentum vector with itself is given by the rest mass M of the particle:

$$p^\mu p_\mu = M^2 c^2 \,. \tag{6.21}$$

6.2 Defining Representation of Lorentz Group

Now we study the consequences of the invariance of the scalar product of four-vectors with respect to a change from one inertial system to another. To this end we consider that the two inertial systems are connected via a linear coordinate transformation, which is mediated by a 4×4 matrix $\Lambda^\mu{}_\nu$:

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \, . \tag{6.22}$$

The invariance (6.4) then reads explicitly

$$g_{\mu\nu} \, x^\mu x^\nu = g_{\mu\nu} \, x'^\mu x'^\nu = g_{\mu\nu} \, \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \, x^\sigma x^\rho = g_{\sigma\rho} \, \Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu \, x^\mu x^\nu \, . \tag{6.23}$$

As (6.23) holds for arbitrary components x^μ of a space-time four-vector, we conclude the identity

$$g_{\mu\nu} = \Lambda^\sigma{}_\mu \, g_{\sigma\rho} \, \Lambda^\rho{}_\nu \, . \tag{6.24}$$

This represents the defining relation for Lorentz transformations Λ, which can be interpreted in two different ways. At first we write (6.24) in matrix notation

$$g = \Lambda^T \, g \, \Lambda \, , \tag{6.25}$$

where we have introduced the elements of the transposed matrix Λ^T according to

$$\left(\Lambda^T\right)_\mu{}^\sigma = g_{\mu\kappa} \left(\Lambda^T\right)^{\kappa\sigma} = g_{\mu\kappa} \Lambda^{\sigma\kappa} = \Lambda^{\sigma\kappa} g_{\kappa\mu} = \Lambda^\sigma{}_\mu \, . \tag{6.26}$$

Note that a left (right) index denotes the respective row (column) of the matrix, so we have concretely

$$\begin{pmatrix} \Lambda^T{}_0{}^0 & \Lambda^T{}_0{}^1 & \Lambda^T{}_0{}^2 & \Lambda^T{}_0{}^3 \\ \Lambda^T{}_1{}^0 & \Lambda^T{}_1{}^1 & \Lambda^T{}_1{}^2 & \Lambda^T{}_1{}^3 \\ \Lambda^T{}_2{}^0 & \Lambda^T{}_2{}^1 & \Lambda^T{}_2{}^2 & \Lambda^T{}_2{}^3 \\ \Lambda^T{}_3{}^0 & \Lambda^T{}_3{}^1 & \Lambda^T{}_3{}^2 & \Lambda^T{}_3{}^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^1{}_0 & \Lambda^2{}_0 & \Lambda^3{}_0 \\ \Lambda^0{}_1 & \Lambda^1{}_1 & \Lambda^2{}_1 & \Lambda^3{}_1 \\ \Lambda^0{}_2 & \Lambda^1{}_2 & \Lambda^2{}_2 & \Lambda^3{}_2 \\ \Lambda^0{}_3 & \Lambda^1{}_3 & \Lambda^2{}_3 & \Lambda^3{}_3 \end{pmatrix} . \tag{6.27}$$

The set $\mathcal{L}$ of all 4×4 matrices Λ, which transform the Minkowski matrix g according to (6.25) into the Minkowski metric g, defines the so-called Lorentz transformations. Note that another equivalent way to interpret the invariance (6.24) follows from contracting it with $g^{\nu\kappa}$. Taking into account (6.11) and (6.26) yields

$$\delta^\kappa{}_\mu = \delta_\mu{}^\kappa = \left(\Lambda^T\right)^\kappa{}_\sigma \, \Lambda^\sigma{}_\mu = \left(\Lambda^T \Lambda\right)^\kappa{}_\mu \, . \tag{6.28}$$

Thus, we conclude that Lorentz transformations Λ are also defined by the identity

$$\Lambda^T = \Lambda^{-1} \qquad \Longleftrightarrow \qquad \left(\Lambda^T\right)^\mu{}_\nu = \Lambda_\nu{}^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu \, . \tag{6.29}$$

By inspection we find that the set $\mathcal{L}$ fulfills all group axioms:

- At first we show that the closedness axiom is valid. Provided that Λ_1, Λ_2 belong to $\mathcal{L}$ we obtain from (6.25) that also $\Lambda_1\Lambda_2$ belongs to $\mathcal{L}$:

$$(\Lambda_1\Lambda_2)^T\; g\; (\Lambda_1\Lambda_2) = \Lambda_2^T \left(\Lambda_1^T\, g\, \Lambda_1\right) \Lambda_2 = \Lambda_2^T\, g\, \Lambda_2 = g\,. \tag{6.30}$$

- Then we take advantage of the associativity of matrix multiplication. For Λ_1, Λ_2, and Λ_3 belonging to $\mathcal{L}$ we conclude $(\Lambda_1\Lambda_2)\,\Lambda_3 = \Lambda_1\,(\Lambda_2\Lambda_3)$ from (6.25):

$$[(\Lambda_1\Lambda_2)\,\Lambda_3]^T\; g\; [(\Lambda_1\Lambda_2)\,\Lambda_3] = \Lambda_3^T \left[\Lambda_2^T \left(\Lambda_1^T\, g\, \Lambda_1\right) \Lambda_2\right] \Lambda_3 = g = [\Lambda_1\,(\Lambda_2\Lambda_3)]^T\; g\; [\Lambda_1\,(\Lambda_2\Lambda_3)]\,. \tag{6.31}$$

- The identity element is represented by the Kronecker symbol from (6.9):

$$\Lambda_e = I = \left(g^{\nu}{}_{\kappa}\right)\,. \tag{6.32}$$

  On the one hand we conclude that Λ_e belongs to $\mathcal{L}$ because of the identity

$$g = \Lambda_e^T\, g\, \Lambda_e\,. \tag{6.33}$$

  On the other hand we observe for any Λ belonging to $\mathcal{L}$:

$$\Lambda_e\Lambda = \Lambda\Lambda_e = \Lambda\,. \tag{6.34}$$

- And, finally, for each Λ from $\mathcal{L}$ we obtain for its determinant from (6.25):

$$\mathrm{Det}\, g = \mathrm{Det}\, \Lambda^T \cdot \mathrm{Det}\, g \cdot \mathrm{Det}\, \Lambda \qquad \Longrightarrow \qquad (\mathrm{Det}\, \Lambda)^2 = 1\,. \tag{6.35}$$

  We conclude then that Λ from $\mathcal{L}$ has a non-vanishing determinant, i.e. $\mathrm{Det}\,\Lambda \neq 0$, so there exists an inverse transformation Λ^{-1}. Furthermore, from (6.25) we yield:

$$\left(\Lambda^T\right)^{-1}\, g\, \Lambda^{-1} = g \qquad \Longrightarrow \qquad \left(\Lambda^{-1}\right)^T\, g\, \Lambda^{-1} = g\,. \tag{6.36}$$

  Thus, there exists an inverse Λ^{-1} from $\mathcal{L}$.

One denotes the set $\mathcal{L}$ of all Lorentz transformation as the Lorentz group or, more concretely, as the pseudo-orthogonal group $O(1,3)$ due to the concrete form of the covariant Minkowski metric (6.3). The Lorentz group $\mathcal{L}$ can be classified with respect to the following two properties:

- Due to (6.35) we read off that $\mathrm{Det}\,\Lambda = \pm 1$. A Lorentz transformation with $\mathrm{Det}\,\Lambda = +1$ ($\mathrm{Det}\,\Lambda = -1$) is denoted to be special (non-special).

- From (6.24) we conclude for $\mu = \nu = 0$ due to (6.3):

$$1 = g_{00} = \Lambda^{\sigma}{}_{0}\, g_{\sigma\rho}\, \Lambda^{\rho}{}_{0} = \left(\Lambda^{0}{}_{0}\right)^2 - \left(\Lambda^{i}{}_{0}\right)^2 \quad \Longrightarrow \quad \left(\Lambda^{0}{}_{0}\right)^2 = 1 + \left(\Lambda^{i}{}_{0}\right)^2 \geq 1\,. \tag{6.37}$$

  A Lorentz transformation Λ with $\Lambda^{0}{}_{0} \geq 1$ ($\Lambda^{0}{}_{0} \leq -1$) is called orthochronous (non-orthochronous).

branch	Det Λ	$\Lambda^0{}_0$	example
$\mathcal{L}_1$	+1	> 0	identity: diag (1,1,1,1)
$\mathcal{L}_2$	-1	> 0	space inversion: diag (1,-1,-1,-1)
$\mathcal{L}_3$	-1	< 0	time inversion: diag (-1,1,1,1)
$\mathcal{L}_4$	+1	< 0	space-time inversion: diag (-1,-1,-1,-1)

Table 6.1: Overview of the four branches of the Lorentz group.

Thus, we conclude that the Lorentz group consists of four different branches as indicated in Tab. 6.1. As the Lorentz transformations from the different branches can not be transformed into each other, the Lorentz group is not connected. Only the branch $\mathcal{L}_1$ of the special orthochronous Lorentz transformations represents a subgroup of the Lorentz group as performing consecutively two transformations from this branch does not allow to leave this branch. Therefore, in the following we deal with only this branch $\mathcal{L}_1$ and call these special orthochronous Lorentz transformations for the sake of simplicity as the Lorentz group.

6.3 Defining Representation of Lorentz Algebra

The set of all 4×4 matrices Λ is described in total by $4 \cdot 4 = 16$ degrees of freedom, where the invariance (6.24) leads to $4 \cdot 5/2 = 10$ restrictions. Therefore the dimension of the Lorentz group is

$$16 - 10 = 6\,. \tag{6.38}$$

Here we investigate, in particular, the elements of the Lorentz group in the vicinity of the unity element (6.32). All elements of the Lorentz group, which deviate infinitesimally from the unity element, can be represented as

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu\,. \tag{6.39}$$

Inserting (6.39) into the defining identity for Lorentz transformations (6.24), we obtain up to first order of the deviations $\omega^\mu{}_\nu$:

$$\begin{aligned}\Lambda^\sigma{}_\mu \Lambda^\rho{}_\nu\, g_{\sigma\rho} &= \left(g^\sigma{}_\mu + \omega^\sigma{}_\mu\right)\left(g^\rho{}_\nu + \omega^\rho{}_\nu\right) g_{\sigma\rho} \approx g^\sigma{}_\mu g^\rho{}_\nu g_{\sigma\rho} + \omega^\sigma{}_\mu g^\rho{}_\nu g_{\sigma\rho} + g^\sigma{}_\mu \omega^\rho{}_\nu g_{\sigma\rho} \\ &= g^\sigma{}_\mu g_{\sigma\nu} + \omega^\sigma{}_\mu g_{\sigma\nu} + \omega^\rho{}_\nu g_{\mu\rho} = g_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} = g_{\mu\nu}\,. \end{aligned} \tag{6.40}$$

Thus we conclude that the deviations of the Lorentz transformation from the unity element are represented by anti-symmetric 4×4 matrices:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0\,. \tag{6.41}$$

The set of all anti-symmetric 4×4 matrices are called the Lorentz algebra of the Lorentz group. The dimension of the Lorentz algebra is 6, which coincides with the dimension of the Lorentz group determined in (6.38). Using the anti-symmetry (6.41) the elements $\omega^{\mu}{}_{\nu}$ of the Lorentz algebra can be represented as

$$\omega^{\mu}{}_{\nu} = g^{\alpha\mu} g^{\beta}{}_{\nu} \omega_{\alpha\beta} = \frac{1}{2} \left(g^{\alpha\mu} g^{\beta}{}_{\nu} - g^{\beta\mu} g^{\alpha}{}_{\nu} \right) \omega_{\alpha\beta} \,. \tag{6.42}$$

Thus, all elements $\omega^{\mu}{}_{\nu}$ of the Lorentz algebra can be expanded with respect to basis elements as follows:

$$\omega^{\mu}{}_{\nu} = -\frac{i}{2} \left(L^{\alpha\beta} \right)^{\mu}{}_{\nu} \omega_{\alpha\beta} \,. \tag{6.43}$$

Here $\omega_{\alpha\beta}$ represent expansion coefficients and the representation matrices of the basis elements $L^{\alpha\beta}$ read:

$$\left(L^{\alpha\beta} \right)^{\mu}{}_{\nu} = i \left(g^{\alpha\mu} g^{\beta}{}_{\nu} - g^{\beta\mu} g^{\alpha}{}_{\nu} \right) \,. \tag{6.44}$$

The indices α, β characterize the respective basis elements $L^{\alpha\beta}$, whereas the indices μ, ν indicate the components $\left(L^{\alpha\beta} \right)^{\mu}{}_{\nu}$ of their respective 4×4 representation matrices. One calls (6.44) the defining representation of the Lorentz algebra as it was derived via (6.39) and (6.43) from the elements Λ of the Lorentz group acting on space-time. Its representation matrices (6.44) have obviously the properties to be anti-symmetric with respect to both pairs of indices α, β and μ, ν:

$$\left(L^{\beta\alpha} \right)^{\mu}{}_{\nu} = - \left(L^{\alpha\beta} \right)^{\mu}{}_{\nu} \,, \tag{6.45}$$

$$\left(L^{\alpha\beta} \right)_{\nu}{}^{\mu} = - \left(L^{\alpha\beta} \right)^{\mu}{}_{\nu} \,. \tag{6.46}$$

And now we determine the commutator between two basis elements $L^{\alpha\beta}$ and $L^{\gamma\delta}$. After a lengthy but straight-forward calculation, which we have relegated to the exercises, one obtains

$$\left[L^{\alpha\beta}, L^{\gamma\delta} \right]_{-} = i \left(g^{\alpha\delta} L^{\beta\gamma} + g^{\beta\gamma} L^{\alpha\delta} - g^{\alpha\gamma} L^{\beta\delta} - g^{\beta\delta} L^{\alpha\gamma} \right) \,. \tag{6.47}$$

This means that the Lorentz algebra is closed with respect to performing the commutator between two of its basis elements. Furthermore, the result (6.47) can be summarized according to

$$\left[L^{\alpha\beta}, L^{\gamma\delta} \right]_{-} = i C_{\epsilon\xi}^{\alpha\beta\gamma\delta} L^{\epsilon\xi} \,, \tag{6.48}$$

where the structure constants of the Lorentz algebra are given by

$$C_{\epsilon\xi}^{\alpha\beta\gamma\delta} = g^{\alpha\delta} g^{\beta}{}_{\epsilon} g^{\gamma}{}_{\xi} + g^{\beta\gamma} g^{\alpha}{}_{\epsilon} g^{\delta}{}_{\xi} - g^{\alpha\gamma} g^{\beta}{}_{\epsilon} g^{\delta}{}_{\xi} - g^{\beta\delta} g^{\alpha}{}_{\epsilon} g^{\gamma}{}_{\xi} \,. \tag{6.49}$$

6.4 Classification of Basis Elements

The basis elements $L^{\alpha\beta}$ of the Lorentz algebra can be sorted into two classes by specializing the indices α, β into spatial and spatio-temporal indices, respectively:

$$L_k = \frac{1}{2} \epsilon_{klm} L^{lm} \,, \tag{6.50}$$

$$M_k = L^{0k} \,. \tag{6.51}$$

Here ϵ_{klm} denotes the three-dimensional Levi-Cività tensor, which has the value $\epsilon_{123} = 1$ and is anti-symmetric with respect to two of its three indices:

$$\epsilon_{klm} = -\epsilon_{lkm} = -\epsilon_{mlk} = -\epsilon_{kml} \,. \tag{6.52}$$

According to (6.44) we obtain by taking into account (6.9) and (6.10) the following explicit representations for the basis elements (6.50):

$$\begin{aligned}
L_1 &= L^{23} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g^{22}g^3{}_3 \\ 0 & 0 & g^{33}g^2{}_2 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \\
L_2 &= L^{31} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g^{11}g^3{}_3 \\ 0 & 0 & 0 & 0 \\ 0 & -g^{33}g^1{}_1 & 0 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} , \\
L_3 &= L^{12} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -g^{11}g^2{}_2 & 0 \\ 0 & g^{22}g^1{}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{6.53}$$

Correspondingly, we yield for the basis elements (6.51):

$$\begin{aligned}
M_1 &= L^{01} = i \begin{pmatrix} 0 & g^{00}g^1{}_1 & 0 & 0 \\ -g^{11}g^0{}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \\
M_2 &= L^{02} = i \begin{pmatrix} 0 & 0 & g^{00}g^2{}_2 & 0 \\ 0 & 0 & 0 & 0 \\ -g^{22}g^0{}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \\
M_3 &= L^{03} = i \begin{pmatrix} 0 & 0 & 0 & g^{00}g^3{}_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -g^{33}g^0{}_0 & 0 & 0 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{6.54}$$

Specializing the commutator (6.47) to the respective spatial and temporal indices, we obtain corresponding commutator relations for the two classes of basis elements (6.50) and (6.51). To this end, however, one has to take into account the inversion of (6.50)

$$L^{ij} = \epsilon_{ijk} L_k \,, \tag{6.55}$$

which can be proven with the help of the contraction rule of the three-dimensional Levi-Cività symbol ϵ_{ijk}:

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \,. \tag{6.56}$$

With this we yield

$$[L_k, L_l]_- = i\epsilon_{klm} L_m \,, \tag{6.57}$$

$$[L_k, M_l]_- = i\epsilon_{klm} M_m \,, \tag{6.58}$$

$$[M_k, M_l]_- = -i\epsilon_{klm} L_m \,. \tag{6.59}$$

From the commutator (6.57) we read off that the basis elements (6.50) represent a subalgebra of the Lorentz algebra.

6.5 Lie Theorem

Considering the Lorentz group in the vicinity of the unity element (6.32), we recognize from (6.39) and (6.43) that there the basis elements $L^{\alpha\beta}$ appear:

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu - \frac{i}{2} \left(L^{\alpha\beta}\right)^\mu{}_\nu \, \omega_{\alpha\beta} \,. \tag{6.60}$$

Conversely, the Lie theorem states that the knowledge of the basis elements $L^{\alpha\beta}$ of the Lorentz algebra allows to determine each element of the Lorentz group by evaluating a matrix exponential function:

$$\Lambda = \exp\left\{ -\frac{i}{2} \, L^{\alpha\beta} \, \omega_{\alpha\beta} \right\} \,. \tag{6.61}$$

The statement of the Lie theorem suggests that the (basis) elements of the Lorentz algebra are called to be the (basis) generators of the Lorentz group. Corresponding to the decomposition of the basis generators $L^{\alpha\beta}$ into the two classes (6.50) and (6.51) also the expansion coefficients $\omega_{\alpha\beta}$ are decomposed into

$$\varphi^k = \frac{1}{2} \, \epsilon_{klm} \, \omega_{lm} \,, \tag{6.62}$$

$$\xi^k = \omega_{0k} \,. \tag{6.63}$$

By taking into account the anti-symmetric properties (6.41) and (6.45) as well as the definitions (6.51) and (6.55) the Lie theorem (6.61) reads

$$\Lambda = \exp\left\{ -\frac{i}{2} \, L^{kl} \omega_{kl} - \frac{i}{2} \, L^{0k} \omega_{0k} \right\} = \exp\Big\{ - i \boldsymbol{\varphi} \mathbf{L} - i \boldsymbol{\xi} \mathbf{M} \Big\} \,. \tag{6.64}$$

In the following we investigate further the Lie theorem (6.64) and show that $\boldsymbol{\xi} = \mathbf{0}$ corresponds to rotations and $\boldsymbol{\varphi} = \mathbf{0}$ to boosts, respectively. Thus, $\boldsymbol{\varphi}$ ($\boldsymbol{\xi}$) denote the vector of rotation angles (rapidities) and $\mathbf{L}$ ($\mathbf{M}$) represent the generators for the rotations (boosts).

6.6 Rotations

According to the Lie theorem (6.64) a general rotation with the vector of rotation angles $\boldsymbol{\varphi}$ is defined by the matrix exponential function

$$R(\boldsymbol{\varphi}) = \exp\Big\{ -i\boldsymbol{\varphi}\mathbf{L}\Big\}\,, \tag{6.65}$$

where the explicit representation matrices for the basis generators of rotations $\mathbf{L}$ are defined in (6.53). In the exercises (6.65) is evaluated, yielding the representation matrix of a rotation in the form

$$R_{00} = 1\,,\;\; R_{0j} = R_{j0} = 0\,,\;\; R_{jk}(\boldsymbol{\varphi}) = \frac{\varphi_i}{|\boldsymbol{\varphi}|}\,\epsilon_{ikj}\,\sin|\boldsymbol{\varphi}| + \frac{\varphi_j\varphi_k}{|\boldsymbol{\varphi}|^2}\,(1-\cos|\boldsymbol{\varphi}|) + \delta_{jk}\,\cos|\boldsymbol{\varphi}|\,. \tag{6.66}$$

Note that the 4×4 matrix defined by (6.66) fulfills two properties, which are characteristic for describing a rotation along the axis $\boldsymbol{\varphi}$ with the angle $|\boldsymbol{\varphi}|$. On the one hand the rotation axis $\boldsymbol{\varphi}$ is an eigenvalue of the rotation matrix $R(\boldsymbol{\varphi})$ with eigenvalue 1:

$$R(\boldsymbol{\varphi})\begin{pmatrix} 0 \\ \boldsymbol{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{\varphi} \end{pmatrix}. \tag{6.67}$$

On other hand the trace of the rotation matrix $R(\boldsymbol{\varphi})$ is related to the rotation angle $|\boldsymbol{\varphi}|$ via

$$\mathrm{Tr}\,R(\boldsymbol{\varphi}) = 2 + 2\cos|\boldsymbol{\varphi}|\,. \tag{6.68}$$

Furthermore, we note that the spatial components of a representation matrix of a rotation obey the orthonormality relation

$$R_{kl}(\boldsymbol{\varphi})R_{km}(\boldsymbol{\varphi}) = \delta_{lm}\,, \tag{6.69}$$

which follows from (6.28) and (6.29) but can also be proven by using the explicit expression (6.66).

Now we apply the rotation matrix (6.66) to a vector $\mathbf{x}$, which has a component parallel to the rotation axis

$$\mathbf{x}_{\parallel} = \frac{\boldsymbol{\varphi}\cdot\mathbf{x}}{|\boldsymbol{\varphi}|}\,\frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \tag{6.70}$$

and another one perpendicular to the rotation axis: $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$. For the rotated vector

$$x'_j = R_{jk}\,x_k \tag{6.71}$$

we then obtain the decomposition

$$\mathbf{x}' = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}\,\cos|\boldsymbol{\varphi}| + \frac{\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\times\mathbf{x}_{\perp}\,\sin|\boldsymbol{\varphi}|\,. \tag{6.72}$$

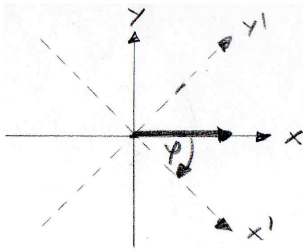
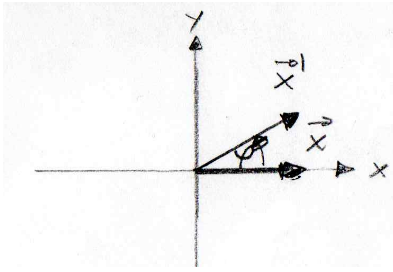
passive rotation	active rotation
vector is fixed	vector is rotated
coordinate system is rotated	coordinate system is fixed
	

Table 6.2: Passive and active rotations act in opposite directions.

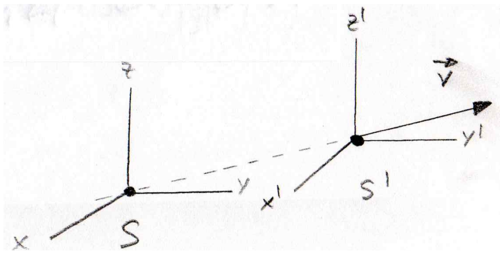


Figure 6.2: Inertial system S' moves with the velocity $\mathbf{v}$ relative to inertial system S.

Specializing (6.72) to a rotation around the axis $\boldsymbol{\varphi} = \varphi\, \mathbf{e}_z$ yields

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} . \tag{6.73}$$

Note that a coordinate transformation like the rotation in (6.73) allows for both a passive and an active interpretation, see Tab. 6.2. For instance, the transformation

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \mathbf{x}' = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{6.74}$$

can be interpreted either as the description of a fixed vector under the clockwise rotation of the coordinate system or an anti-clockwise rotation of the vector for a fixed coordinate system.

6.7 Boosts

According to the Lie theorem (6.64) a general boost with the vector of rapidities $\boldsymbol{\xi}$ is defined by the matrix exponential function

$$B(\boldsymbol{\xi}) = \exp\Big\{ -i\boldsymbol{\xi}\mathbf{M}\Big\} , \tag{6.75}$$

where the explicit representation matrices for the basis generators of boosts $\mathbf{M}$ are defined in (6.54). In the exercises (6.75) is evaluated, yielding the representation matrix of a boost in the form

$$B(\boldsymbol{\xi}) = \begin{pmatrix} \cosh|\boldsymbol{\xi}| & \dfrac{\xi_j}{|\boldsymbol{\xi}|}\sinh|\boldsymbol{\xi}| \\ \dfrac{\xi_i}{|\boldsymbol{\xi}|}\sinh|\boldsymbol{\xi}| & \delta_{ij} + \dfrac{\xi_i\xi_j}{|\boldsymbol{\xi}|^2}(\cosh|\boldsymbol{\xi}|-1) \end{pmatrix}. \tag{6.76}$$

We interpret the boost (6.76) passively in order to determine a relation between the rapidity $\boldsymbol{\xi}$ and the velocity $\mathbf{v}$, with which the inertial system S' is moving with respect to the inertial system S, see Fig. 6.2. To this end we observe that the coordinate origin of S' is described in both inertial systems S and S' with the following space-time four-vectors:

$$(x^\mu) = \begin{pmatrix} ct \\ \mathbf{v}t \end{pmatrix}, \qquad \Longrightarrow \qquad (x'^\mu) = \begin{pmatrix} ct' \\ \mathbf{0} \end{pmatrix}. \tag{6.77}$$

Thus, mapping the four-vector (x^μ) to (x'^μ) via the boost (6.77) according to

$$x'^\mu = B^\mu{}_\nu(\boldsymbol{\xi})\, x^\nu \tag{6.78}$$

we obtain from taking to account (6.76):

$$t' = t\cosh|\boldsymbol{\xi}| + \frac{\boldsymbol{\xi}\mathbf{v}t}{|\boldsymbol{\xi}|c}\sinh|\boldsymbol{\xi}|, \tag{6.79}$$

$$\mathbf{0} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\sinh|\boldsymbol{\xi}| + \frac{\mathbf{v}}{c} + \frac{\boldsymbol{\xi}\mathbf{v}}{|\boldsymbol{\xi}|c}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}(\cosh|\boldsymbol{\xi}|-1). \tag{6.80}$$

At first, we conclude from (6.80) that rapidity $\boldsymbol{\xi}$ and velocity $\mathbf{v}$ are anti-parallel with respect to each other:

$$\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} = -\frac{\mathbf{v}}{|\mathbf{v}|}. \tag{6.81}$$

Inserting (6.81) into (6.80) we conclude how the amounts of both the rapidity vector and the velocity vector are related:

$$\frac{|\mathbf{v}|}{c} = \sinh|\boldsymbol{\xi}| - \frac{|\mathbf{v}|}{c}(\cosh|\boldsymbol{\xi}|-1) \qquad \Longrightarrow \qquad \frac{|\mathbf{v}|}{c} = \tanh|\boldsymbol{\xi}|. \tag{6.82}$$

Thus, due to hyperbolic relations we obtain

$$\cosh|\boldsymbol{\xi}| = \frac{1}{\sqrt{1-\tanh^2|\boldsymbol{\xi}|}} = \gamma, \tag{6.83}$$

$$\sinh|\boldsymbol{\xi}| = \frac{\tanh|\boldsymbol{\xi}|}{\sqrt{1-\tanh^2|\boldsymbol{\xi}|}} = \frac{|\mathbf{v}|}{c}\gamma, \tag{6.84}$$

where we have introduced the Lorentz factor of special relativity as an abbreviation:

$$\gamma = \frac{1}{\sqrt{1-|\mathbf{v}|^2/c^2}}. \tag{6.85}$$

With (6.76) and (6.81)–(6.85) the representation matrix of a boost turns out to be

$$B(\mathbf{v}) = \begin{pmatrix} \gamma & -\dfrac{v_j}{c}\gamma \\ -\dfrac{v_i}{c}\gamma & \delta_{ij} + \dfrac{v_i v_j}{|\mathbf{v}|^2}(\gamma - 1) \end{pmatrix} . \tag{6.86}$$

Note that the components of a representation matrix of a boost obey the relation

$$B^{\mu}{}_{\nu}(\mathbf{v}) B_{\mu}{}^{\kappa}(\mathbf{v}) = \delta_{\nu}{}^{\kappa} , \tag{6.87}$$

which follows from (6.28) and (6.29) but can also be proven by using the explicit expression (6.76). And finally, as a concrete example, we read off from (6.79) and (6.81)–(6.85) the time dilatation

$$t' = t\,\gamma \left(1 - \frac{\mathbf{v}^2}{c^2}\right) = t\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} , \tag{6.88}$$

i.e. an observer in the inertial system S detects that the clock in the moving inertial system S' goes slower than the clock in S.

6.8 Scalar Field Representation

Let us consider a scalar field $\phi(x^\mu)$, which represents a tensor field of rank $n = 0$ as it is invariant with respect to any Lorentz transformation Λ. Within a passive interpretation of the Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad \Longleftrightarrow \qquad x^{\mu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x'^{\nu} \tag{6.89}$$

the four-vectors x^μ and x'^μ denote one and the same space-time point at the original and the transformed coordinate system S and S', respectively. Due to the invariance of the scalar field the original scalar field $\phi(x^\mu)$ in S must coincide with the transformed scalar field $\phi'(x'^\mu)$ in S':

$$\phi'(x'^{\mu}) = \phi(x^{\mu}) . \tag{6.90}$$

Expressing the original scalar field ϕ via the transformed coordinate system S' we obtain from (6.89) and (6.90):

$$\phi'(x'^{\mu}) = \phi\left(\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x'^{\nu}\right) . \tag{6.91}$$

In order to simplify our notation in view of the following considerations, we omit from now on the prime $'$ at the respective four-vectors:

$$\phi'(x^{\mu}) = \phi\left(\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\, x^{\nu}\right) . \tag{6.92}$$

Specializing (6.92) with the help of (6.39) and (6.42) to infinitesimal Lorentz transformations, we obtain up to first order in the expansion coefficients $\omega_{\alpha\beta}$:

$$\phi'(x^\mu) = \phi\left(x^\mu + \frac{i}{2}\,\omega_{\alpha\beta}\left(L^{\alpha\beta}\right)^\mu{}_\nu\,x^\nu\right) = \left(1 - \frac{i}{2}\,\omega_{\alpha\beta}\,\hat{L}^{\alpha\beta}\right)\phi(x^\mu)\,, \tag{6.93}$$

where the differential operators $\hat{L}^{\alpha\beta}$ are given by

$$\hat{L}^{\alpha\beta} = -\left(L^{\alpha\beta}\right)^\mu{}_\nu\,x^\nu\,\partial_\mu\,. \tag{6.94}$$

Due to the representation matrices (6.44) the differential operators turn out to be of the form

$$\hat{L}^{\alpha\beta} = i\left(x^\alpha\partial^\beta - x^\beta\partial^\alpha\right)\,. \tag{6.95}$$

Taking into account the definition of the four-momentum operator in quantum mechanics

$$\hat{p}^\alpha = i\hbar\,\partial^\alpha \tag{6.96}$$

Eq. (6.95) reduces to dimensionless angular momentum operators

$$\hat{L}^{\alpha\beta} = \frac{1}{\hbar}\left(x^\alpha\hat{p}^\beta - x^\beta\hat{p}^\alpha\right)\,. \tag{6.97}$$

Note that the components of the space-time four-vector and the momentum four-vector operator fulfill

$$\left[\hat{p}^\alpha, x^\beta\right]_- = i\hbar\,g^{\alpha\gamma}\,\left[\partial_\gamma, x^\beta\right]_- = i\hbar\,g^{\alpha\beta}\,. \tag{6.98}$$

Here we have taken into account that differentiating with respect to the components of a contravariant four-vector yields the components of a covariant four-vector:

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha}\,. \tag{6.99}$$

From (3.10) and (6.98) we get the following set of commutation relations:

$$\left[\hat{L}^{\alpha\beta}, x^\gamma\right]_- = -\left(L^{\alpha\beta}\right)^\gamma{}_\delta\,x^\delta\,, \tag{6.100}$$

$$\left[\hat{L}^{\alpha\beta}, \hat{p}^\gamma\right]_- = -\left(L^{\alpha\beta}\right)^\gamma{}_\delta\,\hat{p}^\delta\,. \tag{6.101}$$

Due to the commutation relations (6.100) and (6.101) one denotes the space-time four-vector x^λ and the momentum four-vector operator $\hat{p}^\lambda$ as vector operators. Correspondingly one considers an operator $\hat{O}^{\lambda_1\ldots\lambda_n}$ as a tensor operator of rank n if it transforms in each index $\lambda_1,\ldots,\lambda_n$ as a vector:

$$\left[\hat{L}^{\mu\nu}, \hat{O}^{\lambda_1\ldots\lambda_n}\right]_- = -\sum_{k=1}^{n}\left(L^{\mu\nu}\right)^{\lambda_k}{}_\kappa\,\hat{O}^{\lambda_1\ldots\lambda_{k-1}\kappa\lambda_{k+1}\ldots\lambda_n}\,. \tag{6.102}$$

The commutation relations (6.100) and (6.101) now allow to determine the commutation relations between the angular momentum operators (6.97) by taking into account (3.43):

$$\left[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}\right]_- = i\left(g^{\alpha\delta}\,\hat{L}^{\beta\gamma} + g^{\beta\gamma}\,\hat{L}^{\alpha\delta} - g^{\alpha\gamma}\,\hat{L}^{\beta\delta} - g^{\beta\delta}\,\hat{L}^{\alpha\gamma}\right)\,. \tag{6.103}$$

Comparing (6.47) with (6.103) we conclude that also the angular momentum operators $\hat{L}^{\alpha\beta}$ fulfill the commutation relations of the Lorentz algebra. Therefore, the angular momentum operators $\hat{L}^{\alpha\beta}$ are considered as a representation of the Lorentz algebra in the Hilbert space of scalar fields. Furthermore, with the help of the representation matrices (6.43) we can rewrite (6.103) according to

$$\left[\hat{L}^{\alpha\beta}, \hat{L}^{\gamma\delta}\right]_{-} = -\left(L^{\alpha\beta}\right)^{\gamma}{}_{\sigma}\,\hat{L}^{\sigma\delta} - \left(L^{\alpha\beta}\right)^{\delta}{}_{\sigma}\,\hat{L}^{\gamma\sigma}\,. \tag{6.104}$$

Thus, the angular momentum operators $\hat{L}^{\alpha\beta}$ represent in the sense of (6.102) tensor operators of rank 2.

6.9 Tensor/Spinor Field Representation

Now we consider a tensor or a spinor field $\psi^{\sigma}(x^{\mu})$, where the index σ stands for the respective tensor or spinor indices. Performing a Lorentz transformation one has to take into account that this affects both the space-time four-vector x^{μ} and the tensor or spinor components ψ^{σ}. Let us consider at first the concrete example of a four-vector $A^{\sigma}(x^{\mu})$, which represents a tensor field of rank $n = 1$ as one Lorentz matrix Λ is involved in transforming the tensor or spinor components ψ^{σ}:

$$A'^{\sigma}(x'^{\mu}) = \Lambda^{\sigma}{}_{\tau}\,A^{\tau}(x^{\mu})\,. \tag{6.105}$$

Reexpressing the space-time components of the original four-vector A^{τ} in S via the transformed coordinate system S', one yields

$$A'^{\sigma}(x^{\mu}) = \Lambda^{\sigma}{}_{\tau}\,A^{\tau}\left(\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\,x^{\nu}\right)\,, \tag{6.106}$$

where again the prime $'$ at the space-time four-vector has been omitted in order to simplify the notation. Afterwards we specialize (6.106) with the help of (6.39) and (6.42) to infinitesimal Lorentz transformations and obtain up to first order in the expansion coefficients $\omega_{\alpha\beta}$:

$$\begin{aligned} A'^{\sigma}(x^{\mu}) &= \left\{g^{\sigma}{}_{\tau} - \frac{i}{2}\left(L^{\alpha\beta}\right)^{\sigma}{}_{\tau}\,\omega_{\alpha\beta}\right\}\left\{A^{\tau}(x^{\mu}) + \frac{i}{2}\left(L^{\alpha\beta}\right)^{\mu}{}_{\nu}\,\omega_{\alpha\beta}\,x^{\nu}\,\partial_{\mu}\,A^{\tau}(x^{\mu})\right\} \\ \Longrightarrow\ A'^{\sigma}(x^{\mu}) &= \left\{g^{\sigma}{}_{\tau} - \frac{i}{2}\left(\hat{M}^{\alpha\beta}\right)^{\sigma}{}_{\tau}\,\omega_{\alpha\beta}\right\}A^{\tau}(x^{\mu})\,. \end{aligned} \tag{6.107}$$

Here the operator $\hat{M}^{\alpha\beta}$ turns out to be additive in the representation matrices (6.43) and the angular momentum operator (6.97):

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + L^{\alpha\beta}\,. \tag{6.108}$$

Thus, from (6.45) and (6.97) we read off the anti-symmetry

$$\hat{M}^{\alpha\beta} = -\hat{M}^{\beta\alpha}\,. \tag{6.109}$$

As both the representation matrices $L^{\alpha\beta}$ and the angular momentum operators $\hat{L}^{\alpha\beta}$ fulfill according to (6.47) and (6.103) the Lorentz algebra as well as they commute with each other

$$\left[L^{\alpha\beta}, \hat{L}^{\gamma\delta}\right]_- = 0\,, \tag{6.110}$$

we conclude that also the operators $\hat{M}^{\alpha\beta}$ fulfill the Lorentz algebra:

$$\left[\hat{M}^{\alpha\beta}, \hat{M}^{\gamma\delta}\right]_- = i\left(g^{\alpha\delta}\,\hat{M}^{\beta\gamma} + g^{\beta\gamma}\,\hat{M}^{\alpha\delta} - g^{\alpha\gamma}\,\hat{M}^{\beta\delta} - g^{\beta\delta}\,\hat{M}^{\alpha\gamma}\right)\,. \tag{6.111}$$

Now we return back to the general case of a tensor or spinor field $\psi^{\sigma}(x^{\mu})$. Performing an infinitesimal Lorentz transformation we have then in analogy to (6.107)

$$\psi'^{\sigma}(x^{\mu}) = \left\{g^{\sigma}{}_{\tau} - \frac{i}{2}\left(\hat{M}^{\alpha\beta}\right)^{\sigma}{}_{\tau}\,\omega_{\alpha\beta}\right\}\psi^{\tau}(x^{\mu})\,, \tag{6.112}$$

where the operator $\hat{M}^{\alpha\beta}$ has a decomposition similar to (6.108):

$$\hat{M}^{\alpha\beta} = \hat{L}^{\alpha\beta} + N^{\alpha\beta}\,. \tag{6.113}$$

Here the matrices $N^{\alpha\beta}$ are a tensor or spinor representation of the Lorentz algebra and, therefore, they must fulfill the commutator relation

$$\left[N^{\alpha\beta}, N^{\gamma\delta}\right]_- = i\left(g^{\alpha\delta}\,N^{\beta\gamma} + g^{\beta\gamma}\,N^{\alpha\delta} - g^{\alpha\gamma}\,N^{\beta\delta} - g^{\beta\delta}\,N^{\alpha\gamma}\right)\,. \tag{6.114}$$

Furthermore, both representations $\hat{L}^{\alpha\beta}$ and $N^{\alpha\beta}$ of the Lorentz algebra in Minkowski space and in the space of the tensor or spinor components are independent from each other, implying

$$\left[N^{\alpha\beta}, \hat{L}^{\gamma\delta}\right]_- = 0\,. \tag{6.115}$$

From this we then read off that also the operators $\hat{M}^{\alpha\beta}$ defined in (6.113) fulfill the commutation relation (6.111) of the Lorentz algebra. They are a representation of the Lorentz algebra in the Hilbert space of tensor or spinor fields. In addition, as $\hat{L}^{\alpha\beta}$ coincides with the orbital angular momentum (6.97), one can identify the representation $N^{\alpha\beta}$ of the Lorentz algebra in the space of the tensor or spinor components with the spin angular momentum and, thus, $\hat{M}^{\alpha\beta}$ with the total angular momentum.

6.10 Defining Representation of Poincaré Group

Poincaré transformations in Minkowski space are put together from a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ and a shift a^{μ}:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu}\,. \tag{6.116}$$

Whereas Lorentz transformations do not change the scalar product of four-vectors due to (6.4) and (6.17), Poincaré transformations (6.116) only leave distances between four-vectors invariant:

$$g_{\mu\nu}\,(x^{\mu} - y^{\mu})\,(x^{\nu} - y^{\nu}) = g_{\mu\nu}\,\left(x'^{\mu} - y'^{\mu}\right)\left(x'^{\nu} - y'^{\nu}\right)\,. \tag{6.117}$$

Therefore, Poincaré transformations are also called to be inhomogeneous Lorentz transformations.

We show now that the set $\mathcal{P}$ of all Poincaré transformations is a group. To this end we characterize an element from $\mathcal{P}$ with (Λ, a):

- At first we prove the closedness and assume, to this end, that both (Λ_1, a_1) and (Λ_2, a_2) belong to $\mathcal{P}$. Taking into account (6.116) we then conclude

$$\begin{aligned} x_2^\mu &= {\Lambda_2}^\mu{}_\nu \, x_1^\nu + a_2^\mu = {\Lambda_2}^\mu{}_\nu \left({\Lambda_1}^\nu{}_\kappa \, x^\kappa + a_1^\nu \right) + a_2^\mu = {\Lambda_2}^\mu{}_\nu \, {\Lambda_1}^\nu{}_\kappa \, x^\kappa + {\Lambda_2}^\mu{}_\nu \, a_1^\nu + a_2^\mu \\ \Longrightarrow & \qquad \Lambda^\mu{}_\nu = {\Lambda_2}^\mu{}_\nu \, {\Lambda_1}^\nu{}_\kappa \,, \qquad a^\mu = {\Lambda_2}^\mu{}_\nu \, a_1^\nu + a_2^\mu \,. \end{aligned} \tag{6.118}$$

  Thus, also

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda, a) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \tag{6.119}$$

  belongs to $\mathcal{P}$. One calls the multiplication rule (6.119) a semi-direct product of the Lorentz group $\mathcal{L}$ and the translation group $\mathcal{T}$. In case of a direct product one would have had the simpler multiplication rule:

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda, a) = (\Lambda_2\Lambda_1, a_1 + a_2)\,. \tag{6.120}$$

- In the next step we consider the associativity, so we consider that (Λ_1, a_1), (Λ_2, a_2) and (Λ_3, a_3) belong to $\mathcal{P}$. Thus, we obtain from (6.119)

$$(\Lambda_1, a_1)\left((\Lambda_2, a_2)(\Lambda_3, a_3)\right) = (\Lambda_1, a_1)(\Lambda_2\Lambda_3, \Lambda_2 a_3 + a_2) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 + a_1) \tag{6.121}$$

$$\left((\Lambda_1, a_1)(\Lambda_2, a_2)\right)(\Lambda_3, a_3) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)(\Lambda_3, a_3) = (\Lambda_1\Lambda_2\Lambda_3, \Lambda_1\Lambda_2 a_3 + \Lambda_1 a_2 + a_1) \tag{6.122}$$

  and deduce with this the associativity

$$(\Lambda_1, a_1)\left((\Lambda_2, a_2)(\Lambda_3, a_3)\right) = \left((\Lambda_1, a_1)(\Lambda_2, a_2)\right)(\Lambda_3, a_3)\,. \tag{6.123}$$

- Then we identify the unity element of $\mathcal{P}$ with $(\Lambda_e, a_e) = (I, 0)$ due to (6.32). Namely, with (Λ, a) from $\mathcal{P}$ we read off from (6.119)

$$(I, 0)(\Lambda, a) = (\Lambda, a) = (\Lambda, a)(I, 0)\,. \tag{6.124}$$

- And, finally, the inverse element of some element (Λ, a) belonging to $\mathcal{P}$ is given by $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$ from $\mathcal{P}$ as taking into account (6.119) leads to

$$(\Lambda, a)^{-1}(\Lambda, a) = (\Lambda^{-1}, -\Lambda^{-1}a)(\Lambda, a) = (\Lambda^{-1}\Lambda, \Lambda^{-1}a - \Lambda^{-1}a) = (I, 0)\,. \tag{6.125}$$

Similar to the Lorentz group also the Poincaré group is divided with the help of the values $\mathrm{Det}\,\Lambda$ and $\Lambda^0{}_0$ into the four branches $\mathcal{P}_i$ with $i = 1, 2, 3, 4$, see Tab. 6.1. In the following we restrict ourselves to consider the subgroup $\mathcal{P}_1$ of the Poincaré group $\mathcal{P}$, which is characterized by $\mathrm{Det}\,\Lambda > 0$ and $\Lambda^0{}_0 > 0$.

6.11 Tensor/Spinor Representation of Poincaré Algebra

Let us analyse a tensor or spinor field $\psi^\sigma(x^\mu)$, which is invariant with respect to a translation with an arbitrary four-vector a^μ. Within a passive interpretation of the translation

$$x'^\mu = x^\mu + a^\mu \qquad \Longleftrightarrow \qquad x^\mu = x'^\mu - a^\mu \tag{6.126}$$

both x^μ and x'^μ denote one and the same space-time point with respect to the original and the translated coordinate system S and S'. Due to the invariance of the tensor or spinor field its descriptions $\psi^\sigma(x^\mu)$ and $\psi'^\sigma(x'^\mu)$ in S and S' must coincide:

$$\psi'^\sigma(x'^\mu) = \psi^\sigma(x^\mu)\,. \tag{6.127}$$

Considering in (6.127) the original tensor or spinor field ψ^σ with respect to the transformed coordinate system S', we obtain from (6.126) and (6.127)

$$\psi'^\sigma(x^\mu) = \psi^\sigma(x^\mu - a^\mu)\,, \tag{6.128}$$

where we have omitted again the prime $'$ at the four-vectors in order to simplify the notation. For an infinitesimal translation $a^\mu = \epsilon^\mu$ we then have

$$\psi'^\sigma(x^\mu) = \psi^\sigma(x^\mu) - \epsilon^\alpha\,\partial_\alpha\,\psi^\sigma(x^\mu)\,. \tag{6.129}$$

Taking into account the momentum operator (6.96) this reduces to

$$\psi'^\sigma(x^\mu) = \left(1 + \frac{i}{\hbar}\,\epsilon_\alpha\,\hat{p}^\alpha\right)\psi^\sigma(x^\mu)\,. \tag{6.130}$$

Thus, the basis generators of the translations can be identified with the components of the momentum operator (6.96). Together with the basis generators of the Lorentz transformations, which are given by the total momentum operators (6.113), they span the Poincaré algebra. In order to characterize the Poincaré algebra completely, it remains to deduce the commutation relations between its basis generators $\hat{p}^\alpha$ and $\hat{M}^{\alpha\beta}$, which can be accomplished straight-forwardly. To this end we read off from (6.96) that the commutator between two basis generators of translations vanishes:

$$\left[\hat{p}^\alpha, \hat{p}^\beta\right]_- \;=\; 0\,. \tag{6.131}$$

Thus, the momentum operators $\hat{p}^\alpha$ represent a commutative subalgebra of the Poincaré algebra, which implies via the Lie theorem that the translations form an abelian subgroup of the Poincaré group. Afterwards, we consider the commutator between the generators $\hat{p}^\alpha$ and $\hat{M}^{\alpha\beta}$ themselves. Here we use that the representation of the basis generators of translations (6.96) and the representation $N^{\alpha\beta}$ of the Lorentz algebra in the space of the tensor or spinor components are independent from each other, implying

$$\left[\hat{p}^\alpha, N^{\beta\gamma}\right]_- \;=\; 0\,. \tag{6.132}$$

With this as well as (6.44), (6.101), and (6.113) we then obtain

$$\left[\hat{M}^{\alpha\beta}, \hat{p}^{\gamma}\right]_- \;=\; i\left(g^{\beta\gamma}\hat{p}^{\alpha} - g^{\alpha\gamma}\hat{p}^{\beta}\right) . \tag{6.133}$$

And we remark that the commutator relations between the total momentum (6.113) were already obtained in (6.111) and are characteristic of the Lorentz algebra. From (6.113) we read off due to the Lie theorem that the Lorentz algebra is a non-abelian subgroup of the Poincaré group.

Finally, the definition (6.102) of a tensor operators $\hat{O}^{\lambda_1,\ldots,\lambda_n}$ of rank n for the Lorentz algebra is straight-forwardly extended to the Poincaré algebra according to

$$\left[\hat{M}^{\mu\nu}, \hat{O}^{\lambda_1\ldots\lambda_n}\right]_- = -\sum_{k=1}^{n} \left(L^{\mu\nu}\right)^{\lambda_k}{}_{\kappa}\, \hat{O}^{\lambda_1\ldots\lambda_{k-1}\kappa\lambda_{k+1}\ldots\lambda_n} . \tag{6.134}$$

With the help of the representation matrices (6.44) the commutator relations (6.113) and (6.133) can then be rewritten as

$$\left[\hat{M}^{\alpha\beta}, \hat{p}^{\gamma}\right]_- \;=\; -\left(L^{\alpha\beta}\right)^{\gamma}{}_{\delta}\, \hat{p}^{\delta} , \tag{6.135}$$

$$\left[\hat{M}^{\alpha\beta}, \hat{M}^{\gamma\delta}\right]_- \;=\; -\left(L^{\alpha\beta}\right)^{\gamma}{}_{\sigma}\, \hat{M}^{\sigma\delta} - \left(L^{\alpha\beta}\right)^{\delta}{}_{\sigma}\, \hat{M}^{\gamma\sigma} . \tag{6.136}$$

Thus, according to (6.134), $\hat{p}^{\alpha}$ and $\hat{M}^{\alpha\beta}$ represent tensor operators of rank $n = 1$ and $n = 2$, respectively.

6.12 Casimir Operators of Poincaré Algebra

Those operators, which commute with all basis generators of a Lie algebra, are called Casimir operators. The first Casimir operator of the Poincaré algebra is given by the scalar product of the momentum operator with itself:

$$\hat{p}^2 = g_{\alpha\beta}\, \hat{p}^{\alpha}\hat{p}^{\beta} . \tag{6.137}$$

Taking into account (3.10) and (6.131) one can directly show that $\hat{p}^2$ commutes with all momentum operators:

$$\left[\hat{p}^2, \hat{p}^{\alpha}\right]_- = g_{\beta\gamma}\, \left[\hat{p}^{\beta}\hat{p}^{\gamma}, \hat{p}^{\alpha}\right]_- = g_{\beta\gamma}\, \left\{\hat{p}^{\beta}\left[\hat{p}^{\gamma}, \hat{p}^{\alpha}\right]_- + \left[\hat{p}^{\beta}, \hat{p}^{\alpha}\right]_-\, \hat{p}^{\gamma}\right\} = 0 . \tag{6.138}$$

Furthermore, $\hat{p}^2$ is per construction a Lorentz scalar and, thus, commutes with all generators of the Lorentz algebra $\hat{M}^{\alpha\beta}$ due to (3.10), (6.133), and (6.137):

$$\begin{aligned}\left[\hat{p}^2, \hat{M}^{\alpha\beta}\right]_- \;&=\; g_{\gamma\delta}\, \left[\hat{p}^{\gamma}\hat{p}^{\delta}, \hat{M}^{\alpha\beta}\right]_- = g_{\gamma\delta}\, \left\{\hat{p}^{\gamma}\left[\hat{p}^{\delta}, \hat{M}^{\alpha\beta}\right]_- + \left[\hat{p}^{\gamma}, \hat{M}^{\alpha\beta}\right]_- \hat{p}^{\delta}\right\} \\ &=\; i g_{\gamma\delta}\left\{\hat{p}^{\gamma}\left(g^{\alpha\delta}\hat{p}^{\beta} - g^{\beta\delta}\hat{p}^{\alpha}\right) + \left(g^{\alpha\gamma}\hat{p}^{\beta} - g^{\beta\gamma}\hat{p}^{\delta}\right)\hat{p}^{\delta}\right\} = 0 .\end{aligned} \tag{6.139}$$

In order to construct a second Casimir operator, we define now the Pauli-Lubanski operator

$$\hat{W}_\alpha = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta \hat{M}^{\gamma\delta}\,. \tag{6.140}$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ denotes the four-dimensional, totally anti-symmetric unity tensor, which is a relativistic extension of the three-dimensional Levi-Cività symbol used in (6.50). It has the value $\epsilon_{1234} = 1$ and is anti-symmetric with respect to two of its four indices:

$$\epsilon_{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\delta\gamma} = -\epsilon_{\alpha\delta\gamma\beta} = -\epsilon_{\alpha\gamma\beta\delta} = -\epsilon_{\delta\beta\gamma\alpha} = -\epsilon_{\gamma\beta\alpha\delta} = -\epsilon_{\beta\alpha\gamma\delta}\,. \tag{6.141}$$

The scalar product of the Pauli-Lubanski operator $\hat{W}_\alpha$ with the four-momentum operator $\hat{p}^\alpha$ vanishes due to (6.140) and (6.141):

$$\hat{W}_\alpha\,\hat{p}^\alpha = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta \hat{M}^{\gamma\delta}\hat{p}^\alpha = 0\,. \tag{6.142}$$

Furthermore, we read off from (3.102), (6.131), (6.133), (6.140), and (6.141) that the Pauli-Lubanski vector commutes with the four-momentum operator:

$$\begin{aligned}\left[\hat{W}^\alpha, \hat{p}^\sigma\right]_- &= g^{\alpha\alpha'}\left[\hat{W}_{\alpha'}, \hat{p}^\sigma\right]_- = \frac{1}{2}\,g^{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma\delta}\left[\hat{p}^\beta \hat{M}^{\gamma\delta}, \hat{p}^\sigma\right]_- = \frac{1}{2}\,g^{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma\delta}\left\{\hat{p}^\beta\left[\hat{M}^{\gamma\delta}, \hat{p}^\sigma\right]_-\right.\\ &\left.+\left[\hat{p}^\beta, \hat{p}^\sigma\right]_- \hat{M}^{\gamma\delta}\right\} = \frac{i}{2}\,g^{\alpha\alpha'}\epsilon_{\alpha'\beta\gamma\delta}\left(g^{\delta\sigma}\hat{p}^\beta\hat{p}^\gamma - g^{\gamma\sigma}\hat{p}^\beta\hat{p}^\delta\right) = 0\,.\end{aligned} \tag{6.143}$$

Now we determine the commutator of the Pauli-Lubanski operator with the basis generators of the Lorentz algebra. To this end we use (3.43), (6.113), (6.133), (6.140), and (6.141) and obtain at first:

$$\begin{aligned}&\left[\hat{M}^{\alpha\beta}, \hat{W}^\gamma\right]_- = g^{\gamma\delta}\left[\hat{M}^{\alpha\beta}, \hat{W}_\delta\right]_- = \frac{1}{2}\,g^{\gamma\delta}\epsilon_{\delta\rho\sigma\tau}\left[\hat{M}^{\alpha\beta}, \hat{p}^\rho \hat{M}^{\sigma\tau}\right]_-\\ &= \frac{1}{2}\,g^{\gamma\delta}\epsilon_{\delta\rho\sigma\tau}\left\{\left[\hat{M}^{\alpha\beta}, \hat{p}^\rho\right]_- \hat{M}^{\sigma\tau} + \hat{p}^\rho\left[\hat{M}^{\alpha\beta}, \hat{M}^{\sigma\tau}\right]_-\right\}\\ &= \frac{i}{2}\,g^{\gamma\delta}\left\{g^{\beta\rho}\epsilon_{\delta\rho\sigma\tau}\left(\hat{p}^\alpha\hat{M}^{\sigma\tau} - 2\hat{p}^\sigma M^{\alpha\tau}\right) - g^{\alpha\rho}\epsilon_{\delta\rho\sigma\tau}\left(\hat{p}^\beta\hat{M}^{\sigma\tau} - 2\hat{p}^\sigma M^{\beta\tau}\right)\right\}\,.\end{aligned} \tag{6.144}$$

In order to identify the right-hand side of (6.144) with known operators, several additional calculations are necessary. At first we apply the contraction rule for the ϵ-tensor

$$\begin{aligned}\epsilon_{\alpha\beta\gamma\delta}\,\epsilon^{\alpha'\beta'\gamma'\delta} &= \delta_\alpha^{\ \alpha'}\delta_\beta^{\ \beta'}\delta_\gamma^{\ \gamma'} + \delta_\alpha^{\ \beta'}\delta_\beta^{\ \gamma'}\delta_\gamma^{\ \alpha'} + \delta_\alpha^{\ \gamma'}\delta_\beta^{\ \alpha'}\delta_\gamma^{\ \beta'}\\ &\quad -\delta_\alpha^{\ \beta'}\delta_\beta^{\ \alpha'}\delta_\gamma^{\ \gamma'} - \delta_\alpha^{\ \alpha'}\delta_\beta^{\ \gamma'}\delta_\gamma^{\ \beta'} - \delta_\alpha^{\ \gamma'}\delta_\beta^{\ \beta'}\delta_\gamma^{\ \alpha'}\,,\end{aligned} \tag{6.145}$$

which is similar to (6.56), so that the relation (6.140) can be inverted in analogy to (6.50) and (6.55) due to the anti-symmetry (6.109):

$$\hat{W}_\alpha\,\epsilon^{\alpha\beta\gamma\delta} = \hat{p}^\beta\hat{M}^{\gamma\delta} + \hat{p}^\gamma\hat{M}^{\delta\beta} + \hat{p}^\delta\hat{M}^{\beta\gamma}\,. \tag{6.146}$$

Furthermore, we conclude from the contraction rule (6.145) the special case

$$\epsilon_{\alpha\beta\gamma\delta}\,\epsilon^{\alpha'\beta'\gamma\delta} = 2\left(\delta_\alpha^{\ \alpha'}\delta_\beta^{\ \beta'} - \delta_\alpha^{\ \beta'}\delta_\beta^{\ \alpha'}\right)\,, \tag{6.147}$$

so that (6.146) can be contracted with the ϵ-tensor. On the one hand we then obtain

$$\hat{W}_\alpha \, \epsilon^{\alpha\beta\gamma\delta} \, \epsilon_{\sigma\tau\gamma\delta} = 2 \left(\hat{W}_\sigma \, \delta_\tau{}^\beta - \hat{W}_\tau \, \delta_\sigma{}^\beta \right) , \tag{6.148}$$

whereas we read off from (6.146)

$$\hat{W}_\alpha \, \epsilon^{\alpha\beta\gamma\delta} \, \epsilon_{\sigma\tau\gamma\delta} = \hat{p}^\beta \hat{M}^{\gamma\delta} \, \epsilon_{\sigma\tau\gamma\delta} + \hat{p}^\gamma \hat{M}^{\delta\beta} \, \epsilon_{\sigma\tau\gamma\delta} + \hat{p}^\delta \hat{M}^{\beta\gamma} \, \epsilon_{\sigma\tau\gamma\delta} \, . \tag{6.149}$$

Thus, taking into account (6.109) and (6.141) we result in

$$\epsilon_{\sigma\tau\gamma\delta} \left(\hat{p}^\beta \hat{M}^{\gamma\delta} - 2\hat{p}^\gamma \hat{M}^{\beta\delta} \right) = 2 \left(\hat{W}_\sigma \, \delta_\tau{}^\beta - \hat{W}_\tau \, \delta_\sigma{}^\beta \right) . \tag{6.150}$$

Inserting then (6.150) into (6.144) determines the commutator of the Lubanski operator with the basis generators of the Lorentz algebra in the following form:

$$\left[\hat{M}^{\alpha\beta}, \hat{W}^\gamma \right]_- = i \left(g^{\beta\gamma} W^\alpha - g^{\alpha\gamma} W^\beta \right) . \tag{6.151}$$

With the help of the representation matrices (6.44) one recognizes that the Pauli-Lubaski operator represents a tensor operator of rank $n = 1$:

$$\left[\hat{M}^{\alpha\beta}, \hat{W}^\gamma \right]_- = - \left(L^{\alpha\beta} \right)^\gamma{}_\delta \, \hat{W}^\delta \, . \tag{6.152}$$

We consider now the scalar product of the Pauli-Lubanski operator with itself

$$\hat{W}^2 = g_{\alpha\beta} \hat{W}^\alpha \hat{W}^\beta \tag{6.153}$$

and show that it represents the second Casimir operator of the Poincaré algebra. At first, we yield for the commutator of $\hat{W}^2$ and $\hat{p}^\alpha$ due to (3.10) and (6.143)

$$\left[\hat{W}^2, \hat{p}^\alpha \right]_- = 0 \, . \tag{6.154}$$

In addition, we obtain that $\hat{W}^2$ also commutes with $\hat{M}^{\alpha\beta}$ by taking into account (3.43) and (6.151)

$$\left[\hat{W}^2, \hat{M}^{\alpha\beta} \right]_- = 0 \, . \tag{6.155}$$

Finally, the question arises how to physically interpret both Casimir operators of the Poincaré group. To this end we describe a particle with fixed four-momentum $p = (p^\mu)$ via a tensor or spinor field $\psi^\sigma(x)$ and the eigenvalue problem

$$\hat{p}^\mu \psi^\sigma(x) = p^\mu \psi^\sigma(x) \, . \tag{6.156}$$

Then the first Casimir operator (6.137) has an eigenvalue, which is determined by the rest mass M due to (6.21). Thus, in view of the second Casmir operator $\hat{W}^2$ it remains to interpret physically also the Pauli-Lubanski operator $\hat{W}_\alpha$. To this end we insert the decomposition (6.113) of the representation $\hat{M}^{\alpha\beta}$ of the Lorentz algebra in the Hilbert space of the tensor or spinor fields in the representation $\hat{L}^{\alpha\beta}$ of the Lorentz algebra in Minskowski space and the representation $N^{\alpha\beta}$ of the Lorentz algebra in the space of the tensor or spinor components into (6.140). Due to the anti-symmetry of the ϵ-tensor (6.141) this yields:

$$\hat{W}_\alpha = \frac{1}{6}\,\epsilon_{\alpha\beta\gamma\delta}\left(\hat{p}^\beta \hat{L}^{\gamma\delta} + \hat{p}^\gamma \hat{L}^{\delta\beta} + \hat{p}^\delta \hat{L}^{\beta\gamma}\right) + \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta} N^{\gamma\delta}\,. \tag{6.157}$$

Taking into account the definition of the orbital angular momentum operators (6.97) as well as the commutation relations (6.98) and (6.131) we observe that (6.157) reduces to

$$\hat{W}_\alpha = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\,\hat{p}^\beta\, N^{\gamma\delta}\,. \tag{6.158}$$

Thus, it turns out that the orbital angular momentum operator $\hat{L}^{\alpha\beta}$ does not contribute to the Pauli-Lubanski operator W_α. Describing again a particle with fixed four-momentum $p = (p^\mu)$ via a tensor or spinor field $\psi^\sigma(x)$, the eigenvalue problem with respect to the Pauli-Lubanski operator reads

$$\hat{W}_\alpha \psi^\sigma(x) = W_\alpha \psi^\sigma(x)\,, \tag{6.159}$$

where the eigenvector is given by the Pauli-Lubanski four-vector

$$W_\alpha = \frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\, p^\beta N^{\gamma\delta}\,. \tag{6.160}$$

Decomposing the basis generators $N^{\alpha\beta}$ of the Lorentz algebra in the space of tensor or spinor components in analogy to (6.50), (6.51) into two classes

$$S_k = \frac{1}{2}\,\epsilon_{klm} N^{lm}\,, \tag{6.161}$$

$$K_k = N^{0k}\,, \tag{6.162}$$

we introduce the two vectors

$$\mathbf{S} = (S_1, S_2, S_3) = \left(N^{23}, N^{31}, N^{12}\right)\,, \tag{6.163}$$

$$\mathbf{K} = (K_1, K_2, K_3) = \left(N^{01}, N^{02}, N^{03}\right)\,. \tag{6.164}$$

With this the covariant components of the Pauli-Lubanski four-vector (6.160) are defined similar to (6.6) and (6.19)

$$(W_\alpha) = (W_0, W_1, W_2, W_3) = \left(W_0, -W^i\right) = (W_0, -\mathbf{W})\,, \tag{6.165}$$

where the temporal and spatial components read

$$W_0 = \mathbf{p}\cdot\mathbf{S}\,, \tag{6.166}$$

$$\mathbf{W} = p^0\,\mathbf{S} + \mathbf{S}\times\mathbf{K}\,, \tag{6.167}$$

respectively. In the rest frame of the particle we have $p^0 = Mc$ and $\mathbf{p} = \mathbf{0}$, so that the temporal and spatial components of the Pauli-Lubanski vector (6.166) and (6.167) reduce to

$$W_0 = 0\,, \tag{6.168}$$

$$\mathbf{W} = Mc\,\mathbf{S}\,. \tag{6.169}$$

Analogously to the calculation of (6.57) we obtain from the commutation relation (6.114) a corresponding commutation relation for the vector components S_k:

$$[S_k, S_l]_- = i\epsilon_{klm} S_m \, . \tag{6.170}$$

Thus, we conclude that in the rest frame of the particle the Pauli-Lubanski four-vector represents the spin angular momentum of the particle. Therefore, $\hat{W}_\alpha$ in (6.140) is a relativistic generalization of the spin angular momentum.

6.13 Irreducible Representations of Poincaré Group

With the help of the eigenvalues of the Casimir operators (6.137) and (6.140) of the Poincaré algebra one can classify the irreducible representations of the Poincaré group. Note that they are infinite dimensional as they describe particles with an unbounded momentum. In contrast to that the defining representation of the Lorentz group was finite dimensional. The eigenvalue of the first Casimir operator (6.137) is characterized due to (6.21) by the rest mass M of the particle:

$$p^2 = M^2 c^2 \, . \tag{6.171}$$

Depending whether the rest mass M is non-zero or vanishes one distinguishes two different classes of representations.

6.13.1 Massive Representations

Let us consider first the case that the rest mass is non-zero, i.e. $M > 0$, which defines the massive representations. Then we remark that the second Casimir operator (6.140) has an eigenvalue, which is a Lorentz scalar, so it has in each inertial system the same value. In particular in the rest frame the eigenvalue of (6.140) reduces due to (6.168) to

$$W^2 = -\mathbf{W}^2 = -M^2 c^2 \, \mathbf{S}^2 \, . \tag{6.172}$$

As the components of the vector $\mathbf{S}$ obey the commutation relations (6.170) of the angular momentum algebra, the eigenvalues of (6.172) are given by

$$W^2 = -M^2 c^2 \, S(S+1) \, ; \qquad S = 0, 1/2, 1, 3/2, \ldots \, . \tag{6.173}$$

Such a massive representation is, thus, characterized by both the mass M and the spin S. As these are the fundamental properties of elementary particles, we have obtained the result that the elementary particles themselves can be identified with the irreducible representations of the Poincaré group. States within such a representation only differ in the third component of the spin vector, where $2S + 1$ different eigenvalues can occur.

6.13.2 Massless Representations

For a particle with a vanishing rest mass, i.e. $M = 0$, it is not possible to reach its rest frame by applying any Lorentz transformation. If this was possible, then this would have the unphysical consequence that the energy of the particle would vanish due to $p^0 = 0$. Therefore, massless particles need as a basic principle a different treatment.

Within a massless representation both four-vectors p^α and W^α have a vanishing scalar product with respect to each other due to (6.142):

$$p_\alpha W^\alpha = 0\,. \tag{6.174}$$

Furthermore, due to (6.171) and (6.172), they represent light-like four-vectors, i.e. they obey

$$p_\alpha p^\alpha = 0\,, \qquad W_\alpha W^\alpha = 0\,, \tag{6.175}$$

Decomposing (6.175) into its temporal and spatial components

$$\left(p^0\right)^2 = \mathbf{p}^2\,, \qquad \left(W^0\right)^2 = \mathbf{W}^2\,, \tag{6.176}$$

then we directly conclude from $p^\alpha \neq 0$ and $W^\alpha \neq 0$:

$$p^0 \neq 0\,, \qquad W^0 \neq 0\,. \tag{6.177}$$

Let us consider now the linear combination

$$A p^\alpha + B W^\alpha = 0\,. \tag{6.178}$$

Obviously, (6.178) does not only have the trivial solution $A = B = 0$ as we obtain from $\alpha = 0$ and from taking into account (6.177)

$$B = -\frac{p^0}{W^0}\,A\,. \tag{6.179}$$

Thus, both light-like four-vectors p^α and W^α are linear dependent. Therefore, for their respective operators $\hat{p}^\alpha$ and $\hat{W}^\alpha$ there must exist a proportionality factor operator $\hat{h}$ with the property

$$\hat{W}^\alpha = \hat{h}\,\hat{p}^\alpha\,. \tag{6.180}$$

Now we determine for this proportionality factor $\hat{h}$ the commutator relations with the generators of the Poincaré algebra. At first we get from (3.10), (6.131), (6.143), and (6.180)

$$\left[\hat{W}^\alpha, \hat{p}^\beta\right]_- = \left[\hat{h}\,\hat{p}^\alpha, \hat{p}^\beta\right]_- = \hat{h}\left[\hat{p}^\alpha, \hat{p}^\beta\right]_- + \left[\hat{h}, \hat{p}^\beta\right]_- \hat{p}^\alpha \quad \Longrightarrow \quad \left[\hat{h}, \hat{p}^\alpha\right]_- = 0\,. \tag{6.181}$$

In a similar way we determine from (3.43), (6.133), (6.151), and (6.180):

$$\left[\hat{M}^{\alpha\beta}, \hat{W}^\gamma\right]_- = \left[\hat{M}^{\alpha\beta}, \hat{h}\hat{p}^\gamma\right]_- = \left[\hat{M}^{\alpha\beta}, \hat{h}\right]_- \hat{p}^\gamma + \hat{h}\left[\hat{M}^{\alpha\beta}, \hat{p}^\gamma\right]_- \quad \Longrightarrow \quad \left[\hat{M}^{\alpha\beta}, \hat{h}\right]_- = 0\,. \tag{6.182}$$

This means that the proportionality factor $\hat{h}$ represents an additional Casimir operator. For the corresponding eigenvalues of $\hat{W}^\alpha$, $\hat{h}$, and $\hat{p}$ we then obtain from (6.180)

$$W^\alpha = h\, p^\alpha\,, \tag{6.183}$$

so we read off for the zeroth component $\alpha = 0$

$$h = \frac{W^0}{p^0}\,. \tag{6.184}$$

Thus, taking into account (6.166) and (6.176) the eigenvalue h of this additional Casimir operator $\hat{h}$ is given by

$$h = \frac{\mathbf{p}\mathbf{S}}{|\mathbf{p}|}\,, \tag{6.185}$$

which is intuitively accessible as the projection of the particle spin upon the direction of motion. Therefore, one calls $\hat{h}$ as the helicity operator. For a given spin S and momentum $\mathbf{p}$ the eigenvalue (6.185) of the helicity operator $\hat{h}$ has a fixed sign, i.e. either positive or negative, which is the same in all inertial systems.

One can define the helicity operator $\hat{h}$ also for massive particles, but then it does not represent a Casimir operator. This means, for instance, that then an appropriate Lorentz transformation can convert a state of positive helicity into another state with negative helicity. Thus, the helicity describes for a massive particle its state but not the massive particle itself. The latter is only possible for massless particles as they always move with light velocity.

6.13.3 Other Representations

From a mathematical point of view the Poincaré group does allow also for other classes of unitary representations. Among them is one with the constraint $p_\mu p^\mu = 0$ and a continuous spin. Another one obeys the constraint $p_\mu p^\mu < 0$ for particles moving with a velocity larger than the light velocity, which are known hypothetically as tachyons. But so far there is no experimental indication that these other representations of the Poincaré group are realised in nature by any elementary particle. But, although this is purely speculative, one of these other representations of the Poincaré group might indicate a solution for the virulent problem of our time that the physical nature of dark matter is yet unknown.