Chapter 9

Dirac Field

In particle physics, the Dirac equation is a relativistic wave equation, which was derived by the British physicist Paul Dirac in 1928 by unifying the principles of both the quantum theory and the theory of special relativity. It describes massive spin-1/2 particles such as electrons and quarks. Historically, it was validated by accounting for the fine details of the hydrogen spectrum in a rigorous way. The equation also implies the existence of a new form of matter, the so-called anti-matter, previously unsuspected as well as unobserved. In 1932 the positron as the anti-particle of the electron was the first anti-matter to be detected in the cosmic radiation by Carl David Anderson.

The wave function in the Dirac theory consists of four complex fields, which are called a spinor as it transforms differently with respect to Lorentz transformations than a vector. For instance, one needs a rotation around a fixed axis by 720° in order to recover the original spinor instead of 360° for a vector. In the non-relativistic limit one obtains the Pauli two-component wave function, whereas the Schrödinger equation deals only with a wave function of one complex field. Moreover, in the limit of zero mass, the Dirac equation reduces to the Weyl equation, which was supposed to describe massless neutrinos for decades.

In the following we derive at first the Dirac theory group theoretically by systematically working out the spinor representation of the Lorentz group. Although this derivation does not correspond to the historic one of Paul Dirac and is technically more involved, it has several advantages. On the one hand it emphasizes the Lorentz invariance as one of the fundamental building blocks of any quantum field theory and explains as a side effect why a four-component Dirac spinor is needed to describe a massive spin 1/2 particle. On the other hand it enables to construct plane wave solutions by boosting trivial plane wave solutions in the rest frame to a uniformly moving reference frame as an elegant alternative to plainly solving the underlying Dirac equation.

Then we show the invariance of the Dirac theory with respect to discrete symmetries like charge conjugation, parity transformation, and time inversion. With this we prove exemplarily the seminal CPT theorem, which represents a fundamental property of physical laws. It states

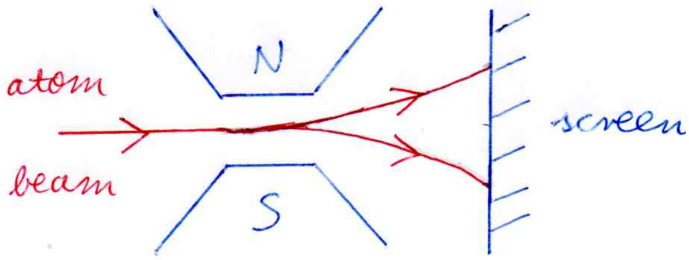

Figure 9.1: Set-up of the Stern-Gerlach experiment: a beam of silver or hydrogen atoms is split into two parts due to an inhomogeneous magnetic field .

that a mirror universe, where also all matter is replaced by anti-matter, would evolve under exactly the same physical laws. As a consequence the masses and life-times of particles and antiparticles are exactly equal.

Afterwards, we discuss how to quantize the Dirac theory within the realm of the canonical field quantization. With this we are able to deal with many massive spin 1/2 particles, whose description naturally also contains their respective antiparticles. And, finally, we determine the Dirac propagator, which describes the free motion of massive spin 1/2 particles and becomes important for a perturbative treatment of the light-matter interaction in terms of Feynman diagrams.

9.1 Pauli Matrices

The Stern-Gerlach experiment from 1922 involves sending a beam of silver or hydrogen atoms through an inhomogeneous magnetic field and observing their deflection. As each silver or hydrogen atom is in the ground state, its valence electron is in the $5s^1$ or the $1s^1$ state. Although the atoms should then not have any angular momentum, the beam is split into two parts, see Fig. 9.1. The reason for this is the spin angular momentum $s = 1/2$ of the valence electron, which leads to a residual magnetic moment of the atom and, thus, to a deflection in the applied inhomogeneous magnetic field. In order to mathematically describe the multiplicity of $2s+1 = 2$ spin degrees of freedom, Wolfgang Pauli introduced three complex 2×2 matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.1}$$

It is straight-forward to prove that the three Pauli matrices fulfill the following anti-commutators:

$$\left[\sigma^k, \sigma^l\right]_+ = 2\,\delta_{kl}\, I\,, \tag{9.2}$$

where I denotes the 2×2 unit matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9.3}$$

Here (9.2) means that the Pauli matrices represent a Clifford algebra with $N = 3$. Namely, a Clifford algebra with N generators $\xi^1, \ldots, \xi^N$ is defined by the anti-commutators

$$\left[\xi^k, \xi^l\right]_+ = 2\,\delta_{kl}\,. \tag{9.4}$$

But one can also convince oneself that the Pauli matrices additionally obey the commutators

$$\left[\sigma^k, \sigma^l\right]_- = 2i\,\epsilon_{klm}\,\sigma^m\,. \tag{9.5}$$

Here (9.5) means that the Pauli matrices also represent a Lie algebra with $N = 3$ generators. Namely, a Lie algebra with N generators $\xi^1, \ldots, \xi^N$ is defined by the commutators

$$\left[\xi^k, \xi^l\right]_- = i\,C_{klm}\,\xi^m\,, \tag{9.6}$$

where C_{klm} denote the structure constants of the Lie algebra. By adding (9.2) and (9.5) we result in the important calculation rule

$$\sigma^k \sigma^l = \delta_{kl} I + i\epsilon_{klm}\,\sigma^m\,, \tag{9.7}$$

which allows to simplify products of Pauli matrices.

9.2 Spinor Representation of Lorentz Algebra

With the help of the Pauli matrices can construct two different representations of the Lorentz algebra. At first, we remark that the matrices

$$L_k = \frac{1}{2}\,\sigma^k \tag{9.8}$$

obey the cummutator relations (2.44) of the generators of rotations. Furthermore, one can identify the generators of boosts via

$$M_k = \pm\,\frac{i}{2}\,\sigma^k\,, \tag{9.9}$$

where both signs are possible. In fact with the identifications (9.8), (9.9) both commutator relations (2.45), (2.46) are valid. With this we define the following two representations of the Lorentz algebra:

$$D^{(1/2,0)}: \qquad (L_k, M_k) \;=\; \left(\frac{1}{2}\,\sigma^k, -\frac{i}{2}\,\sigma^k\right)\,, \tag{9.10}$$

$$D^{(0,1/2)}: \qquad (L_k, M_k) \;=\; \left(\frac{1}{2}\,\sigma^k, \frac{i}{2}\,\sigma^k\right)\,. \tag{9.11}$$

A general representation of the Lorentz algebra is characterized by $D^{(s_1,s_2)}$, where both quantum numbers s_1, s_2 can have all possible half-integer or integer values $0, 1/2, 1, 3/2, 2, \ldots$. It turns out that the space corresponding to the representation $D^{(s_1,s_2)}$ contains particles, whose spin lies in the interval $[|s_1 - s_2|, s_1 + s_2]$. In particular, particles with a single fixed spin s therefore belong to the representation $D^{(s,0)}$ or $D^{(0,s)}$. The trivial representation $D^{(0,0)}$ for a spinless particle assigns to each generator of the Lorentz algebra the number 1.

According to the Lie theorem of Section 6.5 the evaluation of the matrix-valued exponential function

$$D(\Lambda) = e^{-i\mathbf{L}\boldsymbol{\varphi} - i\mathbf{M}\boldsymbol{\xi}} \tag{9.12}$$

yields a representation of the Lorentz group, which corresponds to the representation of the Lorentz algebra. In both cases (9.10) and (9.11) we obtain from (9.12):

$$D^{(1/2,0)}(\Lambda) = \exp\left(-\frac{i}{2}\,\boldsymbol{\sigma\varphi} - \frac{1}{2}\,\boldsymbol{\sigma\xi}\right), \tag{9.13}$$

$$D^{(0,1/2)}(\Lambda) = \exp\left(-\frac{i}{2}\,\boldsymbol{\sigma\varphi} + \frac{1}{2}\,\boldsymbol{\sigma\xi}\right). \tag{9.14}$$

In the following we evaluate the respective matrix-valued exponential functions (9.13), (9.14) both for rotations $\boldsymbol{\xi} = \mathbf{0}$ and for boosts $\boldsymbol{\varphi} = \mathbf{0}$.

9.3 Spinor Representation of Rotations

According to (9.13) and (9.14) the spinor representation of rotations is given in both cases by

$$D\left(R(\boldsymbol{\varphi})\right) = \exp\left(-\frac{i}{2}\,\boldsymbol{\sigma\varphi}\right). \tag{9.15}$$

Due to the hermiticity of the Pauli matrices (9.1)

$$\left(\sigma^k\right)^\dagger = \sigma^k \tag{9.16}$$

the representation matrices of the rotations are unitary:

$$D\left(R(\boldsymbol{\varphi})\right)^\dagger = D\left(R(\boldsymbol{\varphi})\right)^{-1}. \tag{9.17}$$

Considering the Taylor series of the exponential function in (9.15) we evaluate separately the even and the odd terms:

$$D\left(R(\boldsymbol{\varphi})\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{(\boldsymbol{\sigma\varphi})^{2n}}{2^{2n}} - i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{(\boldsymbol{\sigma\varphi})^{2n+1}}{2^{2n+1}}. \tag{9.18}$$

Applying the calculational rule (9.7) we obtain

$$(\boldsymbol{\sigma\varphi})^2 = \varphi_k\varphi_l\sigma^k\sigma^l = \varphi_k\varphi_l\left(\delta_{kl}\,I + i\epsilon_{klm}\,\sigma^m\right) = \boldsymbol{\varphi}^2\,I, \tag{9.19}$$

so that (9.18) leads to

$$D\left(R(\boldsymbol{\varphi})\right) = \left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)^{2n}\right\} I - i\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{|\boldsymbol{\varphi}|}{2}\right)^{2n+1}\right\}\frac{\boldsymbol{\sigma\varphi}}{|\boldsymbol{\varphi}|}. \tag{9.20}$$

Taking into account the Taylor series of the trigonometric functions, one finally yields the spinor representation matrices for rotations

$$D\left(R(\boldsymbol{\varphi})\right) = I\,\cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) - i\,\frac{\boldsymbol{\sigma}\boldsymbol{\varphi}}{|\boldsymbol{\varphi}|}\,\sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right)\,, \tag{9.21}$$

which are, indeed, unitary (9.17) due to (9.16). Note that both representations $D^{(1/2,0)}$ and $D^{(0,1/2)}$ yield the same representation matrices for rotations. Furthermore, we observe that one needs in (9.21) a rotation of 4π in order to recover the identity, which is a characteristic property for a spinor representation.

9.4 Spinor Representation of Boosts

According to (9.13) and (9.14) the representation of the boosts reads

$$D\left(B(\boldsymbol{\xi})\right) = \exp\left(\mp\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right)\,. \tag{9.22}$$

Due to the hermiticity of the Pauli matrices in (9.16) also the representation matrices of the boosts are hermitian:

$$D\left(B(\boldsymbol{\xi})\right)^{\dagger} = D\left(B(\boldsymbol{\xi})\right)\,. \tag{9.23}$$

The Taylor series of the matrix exponential function (9.22) is evaluated separately for even and odd terms:

$$D\left(B(\boldsymbol{\xi})\right) = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\frac{(\boldsymbol{\sigma}\boldsymbol{\xi})^{2n}}{2^{2n}} \mp \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\frac{(\boldsymbol{\sigma}\boldsymbol{\xi})^{2n+1}}{2^{2n+1}}\,. \tag{9.24}$$

With the help of (9.19) this changes to

$$D\left(B(\boldsymbol{\xi})\right) = \left\{\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\left(\frac{|\boldsymbol{\xi}|}{2}\right)^{2n}\right\} I \mp \left\{\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\left(\frac{|\boldsymbol{\xi}|}{2}\right)^{2n+1}\right\}\frac{\boldsymbol{\sigma}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,. \tag{9.25}$$

Taking into account the Taylor series of hyperbolic functions, one gets from (9.25) for the representation matrices (9.22) of the boosts

$$D\left(B(\boldsymbol{\xi})\right) = \exp\left(\mp\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = I\,\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right) \mp \frac{\boldsymbol{\sigma}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,\sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\,. \tag{9.26}$$

As a reminder we note again that the upper and the lower sign stands for the representation $D^{(1/2,0)}$ and $D^{(0,1/2)}$, respectively. Furthermore, we remark that the representation matrices (9.26) are, indeed, hermitian (9.23) due to (9.16).

In order to simplify (9.26) further we consider now a particle of mass M in the rest frame, so that its contravariant four-momentum vector is given by

$$(p_R^{\mu}) = (Mc, \mathbf{0})\,. \tag{9.27}$$

Performing an active boost into the inertial frame the contravariant four-momentum vector (9.27) changes to

$$p^\mu = B^\mu{}_\nu(\boldsymbol{\xi}) p_R^\nu \,, \tag{9.28}$$

where the respective matrix elements of the boost $B^\mu{}_\nu(\boldsymbol{\xi})$ were already determined in Section 6.7 in terms of the the rapidity $\boldsymbol{\xi}$. Using (6.76) we thus obtain

$$(p^\mu) = \left(p^0, \mathbf{p}\right) = \left(Mc\cosh|\boldsymbol{\xi}|, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\, Mc\sinh|\boldsymbol{\xi}|\right) . \tag{9.29}$$

Combining (9.29) with the hyperbolic Pythagoras

$$\cosh^2\alpha - \sinh^2\alpha = 1 \tag{9.30}$$

and the hyperbolic addition theorems

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh(\alpha)\sinh\beta \,, \tag{9.31}$$

$$\sinh(\alpha+\beta) = \sinh(\alpha)\cosh\beta + \cosh\alpha\sinh\beta \,, \tag{9.32}$$

the following relations are derived:

$$\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right) = \sqrt{\frac{\cosh|\boldsymbol{\xi}|+1}{2}} = \sqrt{\frac{p^0+Mc}{2Mc}} \,, \tag{9.33}$$

$$\sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right) = \sqrt{\frac{\cosh|\boldsymbol{\xi}|-1}{2}} = \sqrt{\frac{p^0-Mc}{2Mc}} \,, \tag{9.34}$$

$$\sinh(|\boldsymbol{\xi}|) = 2\sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right) = \frac{\sqrt{(p^0-Mc)(p^0+Mc)}}{Mc} \,. \tag{9.35}$$

Using (9.33)–(9.35), the representation matrix (9.26) of the boost can be expressed by the components of the contravariant four-momentum vector (9.29):

$$D\left(B(\boldsymbol{\xi})\right) = \exp\left(\mp\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\xi}\right) = I\sqrt{\frac{p^0+Mc}{2Mc}} \mp \frac{\boldsymbol{\sigma}\mathbf{p}}{Mc}\sqrt{\frac{p^0-Mc}{2Mc}}\,\frac{Mc}{\sqrt{(p^0-Mc)(p^0+Mc)}}$$

$$\Longrightarrow \qquad D\left(B(\boldsymbol{\xi})\right) = \frac{(p^0+Mc)I \mp \boldsymbol{\sigma}\mathbf{p}}{\sqrt{2Mc(p^0+Mc)}} \,. \tag{9.36}$$

In the following it turns out to be technically advantageous to extend the three Pauli matrices σ^k by the unit matrix

$$\sigma^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.37}$$

to a four-vector of Pauli matrices:

$$(\sigma^\mu) = (\sigma^0, \sigma^k) \,. \tag{9.38}$$

Then (9.36) implies that the boost of the representation $D^{(1/2,0)}$ can be concisely written as

$$D^{(1/2,0)}\left(B(\boldsymbol{\xi})\right) = \exp\left(-\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = \frac{p\sigma + Mc}{\sqrt{2Mc(p^0 + Mc)}}\,, \tag{9.39}$$

where the scalar product between the four-vector of Pauli matrices (9.38) and the four-momentum vector is used:

$$p\sigma = p_\mu \sigma^\mu = p^0\sigma^0 - \mathbf{p}\boldsymbol{\sigma}\,. \tag{9.40}$$

Furthermore, we introduce the spatially inverted four-vector

$$\tilde{x} = (\tilde{x}^0, \tilde{x}^k) = (x^0, -x^k) \tag{9.41}$$

and, correspondingly, also the spatially inverted four-vector of Pauli matrices

$$\tilde{\sigma} = (\tilde{\sigma}^0, \tilde{\sigma}^k) = (\sigma^0, -\sigma^k)\,. \tag{9.42}$$

With this we read off from (9.36) that the boost of the representation $D^{(0,1/2}$ is given by

$$D^{(0,1/2)}\left(B(\boldsymbol{\xi})\right) = \exp\left(+\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = \frac{p\tilde{\sigma} + Mc}{\sqrt{2Mc(p^0 + Mc)}} \tag{9.43}$$

due to the scalar product

$$p\tilde{\sigma} = p_\mu \tilde{\sigma}^\mu = p^0\sigma^0 + \mathbf{p}\boldsymbol{\sigma}\,. \tag{9.44}$$

For various later calculations it turns out to be useful to express the boost representations (9.39) and (9.43) as the square root of the same expression with a doubled rapidity. Indeed, taking into account (9.26) and (9.29) we obtain

$$\exp\left(\mp\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = \sqrt{\exp\left(\mp\boldsymbol{\sigma}\boldsymbol{\xi}\right)} = \sqrt{\cosh|\boldsymbol{\xi}| \mp \frac{\boldsymbol{\sigma}\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\sinh|\boldsymbol{\xi}|} = \sqrt{\frac{p^0}{Mc} \mp \frac{\mathbf{p}\boldsymbol{\sigma}}{Mc}}\,. \tag{9.45}$$

Thus, together (9.40) and (9.44) we conclude

$$\exp\left(-\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = \sqrt{\frac{p\sigma}{Mc}}\,, \tag{9.46}$$

$$\exp\left(+\frac{1}{2}\,\boldsymbol{\sigma}\boldsymbol{\xi}\right) = \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,. \tag{9.47}$$

Whenever we will use later on the spinor representations for boosts (9.46) and (9.47) we have to keep in mind that they present efficient shortcut notations for the more involved concrete expressions (9.39) and (9.43).

9.5 Lorentz Invariant Combinations of Weyl Spinors

So far we have constructed with $D^{(1/2,0)}$ and $D^{(0,1/2)}$ the smallest non-trivial representations of the Lorentz group. Now we define the corresponding Weyl spinors $\xi^{\alpha}(x)$ and $\eta^{\dot{\alpha}}(x)$ of type $(1/2,0)$ and $(0,1/2)$ upon which the representation matrices of the Lorentz group act. The different transformation properties of the Weyl spinors $\xi_{\alpha}(x)$ and $\eta^{\dot{\alpha}}(x)$ under a Lorentz transformation are expressed by using lower non-dotted and upper dotted indices, respectively:

$$\xi_{\alpha}(x) \qquad \longrightarrow \qquad \xi'_{\alpha}(x') = D^{(1/2,0)}(\Lambda)_{\alpha}{}^{\beta}\,\xi_{\beta}(x)\,, \tag{9.48}$$

$$\eta_{\dot{\alpha}}(x) \qquad \longrightarrow \qquad \eta'^{\dot{\alpha}}(x') = D^{(0,1/2)}(\Lambda)^{\dot{\alpha}}{}_{\dot{\beta}}\,\eta^{\dot{\beta}}(x)\,. \tag{9.49}$$

In the following we aim for constructing a Lorentz invariant action on the basis of using these Weyl spinors. To this end we restrict ourselves to consider quadratic terms in the Weyl spinors and their first partial derivatives.

At first, we only deal with quadratic terms in the Weyl spinors without any first partial derivative, which are needed to describe massive particles. In this case there are in total four different combinations of two Weyl spinors

$$\xi^{\dagger}\xi\,, \quad \eta^{\dagger}\eta\,, \quad \eta^{\dagger}\xi\,, \quad \xi^{\dagger}\eta\,, \tag{9.50}$$

which are converted by a Lorentz transformation Λ into

$$\begin{aligned} &\xi^{\dagger}D^{(1/2,0)}(\Lambda)^{\dagger}D^{(1/2,0)}(\Lambda)\xi\,, \quad \eta^{\dagger}D^{(0,1/2)}(\Lambda)^{\dagger}D^{(0,1/2)}(\Lambda)\eta\,, \\ &\eta^{\dagger}D^{(0,1/2)}(\Lambda)^{\dagger}D^{(1/2,0)}(\Lambda)\xi\,, \quad \xi^{\dagger}D^{(1/2,0)}(\Lambda)^{\dagger}D^{(0,1/2)}(\Lambda)\eta\,, \end{aligned} \tag{9.51}$$

respectively. In case of a rotation $\Lambda = R$ the representation matrices $D^{(1/2,0)}(R)$ and $D^{(0,1/2)}(R)$ coincide according to (9.13) and (9.14). Furthermore, we conclude from the unitarity (9.17) of these representation matrices that all four transformed combinations (9.51) are identical to the original combinations (9.50). But in case of a boost $\Lambda = B$ we read off from (9.13) and (9.14) that the representation matrices $D^{(1/2,0)}(B)$ and $D^{(0,1/2)}(B)$ are just inverse with respect to each other:

$$D^{(1/2,0)}\left(B\right) = D^{(0,1/2)}\left(B\right)^{-1}\,. \tag{9.52}$$

In combination with the hermiticity (9.23) of these representation matrices it follows then that only the last two of the transformed combinations (9.50) match with their corresponding original combinations (9.50). In summary, we conclude that a Lorentz invariant action without space-time derivatives is only possible by combining the two Weyl spinors ξ and η.

In order to describe a particle, which moves in space-time, the action must also contain first partial derivatives of the Weyl spinors. To this end we consider at first spatial derivatives and form all possible combinations of two Weyl spinors

$$\xi^{\dagger}\sigma^{k}\partial_{k}\xi\,, \quad \eta^{\dagger}\sigma^{k}\partial_{k}\eta\,, \quad \eta^{\dagger}\sigma^{k}\partial_{k}\xi\,, \quad \xi^{\dagger}\sigma^{k}\partial_{k}\eta\,. \tag{9.53}$$

They are converted by a Lorentz transformation Λ into

$$\begin{aligned}
\xi^\dagger D^{(1/2,0)}(\Lambda)^\dagger\, \sigma^k\, D^{(1/2,0)}(\Lambda)\, \partial'_k \xi\,, \quad \eta^\dagger D^{(0,1/2)}(\Lambda)^\dagger\, \sigma^k\, D^{(0,1/2)}(\Lambda)\, \partial'_k \eta\,, \\
\eta^\dagger D^{(0,1/2)}(\Lambda)^\dagger\, \sigma^k\, D^{1/2,0)}(\Lambda)\, \partial'_k \xi\,, \quad \xi^\dagger D^{1/2,0)}(\Lambda)^\dagger\, \sigma^k\, D^{(0,1/2)}(\Lambda)\, \partial'_k \eta\,.
\end{aligned} \tag{9.54}$$

In case of a rotation $\Lambda = R$, the representation matrices $D^{(1/2,0)}(R)$ and $D^{(0,1/2)}(R)$ are identical, so that due to (9.54) only the expression

$$D(R)^\dagger\, \sigma^k\, D(R) \tag{9.55}$$

has to be examined in detail. Using (9.21) we arrive at first at

$$\begin{aligned}
D(R)^\dagger\, \sigma^k\, D(R) &= \left\{ \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) + i\frac{\boldsymbol{\sigma\varphi}}{|\boldsymbol{\varphi}|} \sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \right\} \sigma^k \left\{ \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) - i\frac{\boldsymbol{\sigma\varphi}}{|\boldsymbol{\varphi}|} \sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \right\} \\
&= \cos^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \sigma^k + i \sin\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \cos\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \frac{\varphi_l}{|\boldsymbol{\varphi}|} \left[\sigma^l, \sigma^k\right]_- + \sin^2\left(\frac{|\boldsymbol{\varphi}|}{2}\right) \frac{\varphi_l \varphi_m}{|\boldsymbol{\varphi}|^2}\, \sigma^l \sigma^k \sigma^m\,.
\end{aligned} \tag{9.56}$$

In the last term the product of three Pauli matrices appears, which can be simplified by successively applying the calculation rule (9.7) and by taking into account the contraction rule of the three-dimensional Levi-Cività symbol (6.56):

$$\begin{aligned}
\sigma^l \sigma^k \sigma^m &= \left(\delta_{lk} + i\epsilon_{lkn}\sigma^n\right) \sigma^m = \delta_{lk}\sigma^m + i\epsilon_{lkn}\sigma^n\sigma^m \\
&= \delta_{lk}\sigma^m + i\epsilon_{lkn}(\delta_{nm} + i\epsilon_{nmp}\sigma^p) = \delta_{lk}\sigma^m + i\epsilon_{lkm} - (\delta_{ml}\delta_{kp} - \delta_{lp}\delta_{km})\sigma^p\,.
\end{aligned} \tag{9.57}$$

With this we end up with the result

$$\sigma^l \sigma^k \sigma^m = i\epsilon_{lkm} + \delta_{lk}\sigma^m + \delta_{km}\sigma^l - \delta_{lm}\sigma^k\,. \tag{9.58}$$

Inserting (9.5) and (9.58) in (9.56) and using trigonometric relations then yields

$$D(R)^\dagger \sigma^k D(R) = \sigma^k \cos|\boldsymbol{\varphi}| + \epsilon_{klm} \frac{\varphi_l}{|\boldsymbol{\varphi}|}\, \sigma^m \sin|\boldsymbol{\varphi}| + \frac{\varphi_k \boldsymbol{\sigma\varphi}}{|\boldsymbol{\varphi}|^2} \left(1 - \cos|\boldsymbol{\varphi}|\right)\,. \tag{9.59}$$

This result can be concisely summarized as

$$D(R)^\dagger \sigma^k D(R) = R_{kl}\, \sigma^l\,, \tag{9.60}$$

where R_{kl} coincides with the representation matrix of rotations in three-dimensional space as already determined in (6.66). As the partial derivatives in (9.54) also transform like a vector

$$\partial_k \qquad \longrightarrow \qquad \partial'_k = R_{kl}\, \partial_l \tag{9.61}$$

and the representation matrix R is orthonormal due to (6.69), all combinations (9.54) turn out to be invariant under rotations:

$$D(R)^\dagger\, \sigma^k\, D(R)\, \partial'_k = R_{kl}\, \sigma^l\, R_{km}\, \partial_m = \delta_{lm}\, \sigma^l\, \partial_m = \sigma^k\, \partial_k\,. \tag{9.62}$$

Now the question arises, how the combinations of two Weyl spinors (9.53) can be extended to relativistic invariant combinations. To this end we remember that the Pauli matrices σ^k can be extended to four-vectors in two different ways, namely in the form of the four-vector of Pauli matrices σ^μ in (9.38) and in the form of the spatially inverted four-vector of Pauli matrices $\tilde{\sigma}^\mu$ in (9.42). Therefore we consider now the following eight combinations of two Weyl spinors:

$$\begin{aligned} &\xi^\dagger\sigma^\mu\partial_\mu\xi\,, \quad \eta^\dagger\sigma^\mu\partial_\mu\eta\,, \quad \eta^\dagger\sigma^\mu\partial_\mu\xi\,, \quad \xi^\dagger\sigma^\mu\partial_\mu\eta\,, \\ &\xi^\dagger\tilde{\sigma}^\mu\partial_\mu\xi\,, \quad \eta^\dagger\tilde{\sigma}^\mu\partial_\mu\eta\,, \quad \eta^\dagger\tilde{\sigma}^\mu\partial_\mu\xi\,, \quad \xi^\dagger\tilde{\sigma}^\mu\partial_\mu\eta\,. \end{aligned} \tag{9.63}$$

Here the additional term $\sigma^0\partial_0$ with the time derivative appears, which is trivially invariant under rotations

$$D(R)^\dagger\,\sigma^0\,D(R)\partial_0' = D(R)^\dagger D(R)\,\partial_0' = \partial_0 = \sigma^0\partial_0\,. \tag{9.64}$$

Thus it does not destroy the above discussed rotational invariance of the spatial derivative terms.

With this it remains to investigate, which of the eight combinations (9.63) are invariant under boost transformations. To this end expressions of the form

$$D(B)^\dagger\,\sigma^\mu\,D(B)\,, \qquad D(B)^\dagger\,\tilde{\sigma}^\mu\,D(B) \tag{9.65}$$

appear, where both representations (9.22) can occur in the left and the right factor, respectively. Let us first consider the case $\mu = 0$. In the case that the two representations in the left and right factor of (9.65) are different, then (9.65) is identical to σ^0 due to (9.23), (9.37), and (9.52). As this does not correspond to the transformation behavior, which is characteristic for boosts, we conclude that the 3rd, the 4th, the 7th, and the 8th combination in (9.63) is not invariant under boosts. In the case that both representations in the left and right factor of (9.65) are identical, then we obtain on the one hand for $\mu = 0$ together with (9.23), (9.26), and (9.37):

$$D(B)^\dagger\sigma^0 D(B) = D(B)^2 = \cosh|\boldsymbol{\xi}| \mp \frac{\boldsymbol{\sigma\xi}}{|\boldsymbol{\xi}|}\,\sinh|\boldsymbol{\xi}|\,. \tag{9.66}$$

On the other hand we get for $\mu = k$ due to (9.23) and (9.26)

$$\begin{aligned} D(B)^\dagger\sigma^k D(B) &= \left\{\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right) \mp \frac{\boldsymbol{\sigma\xi}}{|\boldsymbol{\xi}|}\,\sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\right\}\sigma^k\left\{\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right) \mp \frac{\boldsymbol{\sigma\xi}}{|\boldsymbol{\xi}|}\,\sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\right\} \\ &= \cosh^2\left(\frac{|\boldsymbol{\xi}|}{2}\right)\sigma^k \mp \sinh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\cosh\left(\frac{|\boldsymbol{\xi}|}{2}\right)\frac{\xi_l}{|\boldsymbol{\xi}|}\left[\sigma^l,\sigma^k\right]_+ + \sinh^2\left(\frac{|\boldsymbol{\xi}|}{2}\right)\frac{\xi_l\xi_m}{|\boldsymbol{\xi}|^2}\,\sigma^l\sigma^k\sigma^m\,. \end{aligned} \tag{9.67}$$

Inserting (9.2) and (9.58) in (9.67) and using hyperbolic relations then yields

$$D(B)^\dagger\left(\mp\sigma^k\right)D(B) = \mp\sigma^k + \frac{\xi_k}{|\boldsymbol{\xi}|}\,\sinh|\boldsymbol{\xi}| + \frac{\xi_k}{|\boldsymbol{\xi}|}\frac{(\mp\boldsymbol{\sigma})\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\,\left(\cosh|\boldsymbol{\xi}| - 1\right)\,. \tag{9.68}$$

The two results (9.66) and (9.68) can be concisely summarized by

$$D^{(1/2,0)}(B)^\dagger\,\tilde{\sigma}^\mu\,D^{(1/2,0)}(B) \;=\; B^\mu{}_\nu\,\tilde{\sigma}^\nu\,, \tag{9.69}$$

$$D^{(0,1/2)}(B)^\dagger\,\sigma^\mu\,D^{(0,1/2)}(B) \;=\; B^\mu{}_\nu\,\sigma^\nu\,, \tag{9.70}$$

where $B^{\mu}{}_{\nu}$ coincides with the representation matrix of boost in the four-dimensional space-time as already determined in (6.76). As the partial derivatives in (9.63) also transform like a covariant four-vector

$$\partial_\mu \qquad \longrightarrow \qquad \partial'_\mu = B_\mu{}^\nu \, \partial_\nu \, , \tag{9.71}$$

and the representation matrix B fulfills the property (6.87), we can prove due to (9.69) and (9.70) the following invariances:

$$\xi^\dagger \tilde{\sigma}^\mu \partial_\mu \xi \longrightarrow \xi^\dagger D^{(1/2,0)}(B)^\dagger \tilde{\sigma}^\mu D^{(1/2,0)}(B) \partial'_\mu \xi = \xi^\dagger B^\mu{}_\nu \tilde{\sigma}^\nu B_\mu{}^\kappa \partial_\kappa \xi = \xi^\dagger \delta_\nu{}^\kappa \tilde{\sigma}^\nu \partial_\kappa \xi = \xi^\dagger \tilde{\sigma}^\nu \partial_\nu \xi \, , \tag{9.72}$$

$$\eta^\dagger \sigma^\mu \partial_\mu \eta \longrightarrow \eta^\dagger D^{(0,1/2)}(B)^\dagger \sigma^\mu D^{(0,1/2)}(B) \partial'_\mu \eta = \eta^\dagger B^\mu{}_\nu \sigma^\nu B_\mu{}^\kappa \partial_\kappa \eta = \eta^\dagger \delta_\nu{}^\kappa \sigma^\nu \partial_\kappa \eta = \eta^\dagger \sigma^\nu \partial_\nu \eta \, . \tag{9.73}$$

For the two remaining combinations $\eta^\dagger \tilde{\sigma}^\mu \partial_\mu \eta$ and $\xi^\dagger \sigma^\mu \partial_\mu \xi$ in (9.63) a boost invariance can not be proved, because both $\tilde{\sigma}^\mu$ and σ^μ transform due to (9.69) and (9.70)) as a four-vector under the representations $D^{(1/2,0)}(B)$ and $D^{(0,1/2)}(B)$, respectively.

9.6 Dirac Action

From the considerations of the previous section follows the most general Lorentz-invariant action for describing a massive spin 1/2 particle

$$\mathcal{A} = \mathcal{A}[\xi(\bullet), \xi^\dagger(\bullet); \eta(\bullet), \eta^\dagger(\bullet)] \, , \tag{9.74}$$

which contains only quadratic terms in the Weyl spinors and their first partial derivatives:

$$\mathcal{A} = \frac{1}{c} \int d^4x \, \mathcal{L}\big(\xi(x), \partial_\mu \xi(x); \xi^\dagger(x), \partial_\mu \xi^\dagger(x); \eta(x), \partial_\mu \eta(x); \eta^\dagger(x) \, , \partial_\mu \eta^\dagger(x)\big) \, . \tag{9.75}$$

Here the Lagrange density

$$\mathcal{L} = A i \xi^\dagger \tilde{\sigma}^\mu \partial_\mu \xi + B i \eta^\dagger \sigma^\mu \partial_\mu \eta + C \xi^\dagger \eta + D \eta^\dagger \xi \, , \tag{9.76}$$

contains constants A, B, C, D, which are not yet defined. Below in Section 9.8 we show that the additional demand for an invariance of the Lagrange density under parity transformation leads to the fact, that both Weyl spinors ξ and η have to appear on equal footing. This reduces (9.76) to

$$\mathcal{L} = A \left(i \xi^\dagger \tilde{\sigma}^\mu \partial_\mu \xi + i \eta^\dagger \sigma^\mu \partial_\mu \eta - m \xi^\dagger \eta - m \eta^\dagger \xi \right) \, . \tag{9.77}$$

The still undetermined parameters A, m define the physical dimension of the action and are only fixed at a later stage by considering the non-relativistic limit. Due to the non-zero rest mass M of the particle, the action (9.77) necessarily contains both Weyl spinors ξ and η. Only in the case that the rest mass of the particle vanishes, a Lorentz-invariant action can be formed with just one of the two Weyl spinors, as is discussed below in Section 9.9.

Due to the action (9.77) the Weyl spinors ξ and η satisfy the equations of motion

$$\frac{\delta \mathcal{A}}{\delta \xi^\dagger(x)} = \frac{\partial \mathcal{L}}{\partial \xi^\dagger(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \xi^\dagger(x)\right)} = A\Big\{i\tilde{\sigma}^\mu \partial_\mu \xi(x) - m\eta(x)\Big\} = 0\,, \tag{9.78}$$

$$\frac{\delta \mathcal{A}}{\delta \eta^\dagger(x)} = \frac{\partial \mathcal{L}}{\partial \eta^\dagger(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \eta^\dagger(x)\right)} = A\Big\{i\sigma^\mu \partial_\mu \eta(x) - m\xi(x)\Big\} = 0\,. \tag{9.79}$$

In order to combine these two equations of motion one needs the calculation rules

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = 2g^{\mu\nu} I, \tag{9.80}$$

$$\tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu = 2g^{\mu\nu} I\,, \tag{9.81}$$

which can be explicitly shown by specializing μ, ν to spatial and temporal indices. To this end one has to take into account the Clifford algebra property (9.2), the definitions (9.37), (9.38), and (9.42), as well as the components of the Minkowski metric in (6.3):

$$\sigma^0 \tilde{\sigma}^0 + \sigma^0 \tilde{\sigma}^0 = 2\sigma^0 = 2I = 2g^{00} I\,, \tag{9.82}$$

$$\sigma^0 \tilde{\sigma}^k + \sigma^k \tilde{\sigma}^0 = -\sigma^0 \sigma^k + \sigma^k \sigma^0 = 0 = 2g^{0k} I\,, \tag{9.83}$$

$$\sigma^k \tilde{\sigma}^l + \sigma^l \tilde{\sigma}^k = -\sigma^k \sigma^l - \sigma^l \sigma^k = -2\delta_{kl} I = 2g^{kl} I\,. \tag{9.84}$$

Multiplying (9.78) with $i\sigma^\nu \partial_\nu$ and using (9.79) or, vice versa, multiplying (9.79) with $i\tilde{\sigma}^\nu \partial_\nu$ and using (9.78), we obtain due to (9.80) and (9.81)

$$-\sigma^\nu \tilde{\sigma}^\nu \partial_\nu \partial_\mu \xi(x) - mi\sigma^\nu \partial_\nu \eta(x) = -g^{\mu\nu} \partial_\mu \partial_\nu \xi(x) - m^2 \xi(x) = 0\,, \tag{9.85}$$

$$-\tilde{\sigma}^\nu \sigma^\nu \partial_\nu \partial_\mu \eta(x) - mi\tilde{\sigma}^\nu \partial_\nu \xi(x) = -g^{\mu\nu} \partial_\mu \partial_\nu \eta(x) - m^2 \eta(x) = 0\,. \tag{9.86}$$

Thus, both Weyl spinors ξ and η satisfy the Klein-Gordon equation of a particle (7.19), provided that the parameter m is identified according to

$$m = \frac{Mc}{\hbar}\,, \tag{9.87}$$

i.e. being inversely proportional to the Compton wave length (7.21).

Since the description of a massive spin 1/2 particle necessarily involves both Weyl spinors ξ and η, it is suggestive to combine them to a Dirac spinor:

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}. \tag{9.88}$$

In view of that we rewrite the Lagrange density (9.77)

$$\mathcal{L} = A\left\{(\xi^\dagger, \eta^\dagger)\begin{pmatrix} \tilde{\sigma}^\mu & O \\ O & \sigma^\mu \end{pmatrix} i\partial_\mu \begin{pmatrix} \xi \\ \eta \end{pmatrix} - (\xi^\dagger, \eta^\dagger)\begin{pmatrix} O & mI \\ mI & O \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right\}, \tag{9.89}$$

where we used the 2×2 unit matrix (9.3) and introduced in addition the 2×2 zero matrix

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.90}$$

Furthermore, we define the Dirac adjoint of the Dirac spinor (9.88) according to

$$\overline{\psi}(x) = \left(\eta^\dagger(x), \xi^\dagger(x)\right) = \psi^\dagger(x) \begin{pmatrix} O & I \\ I & O \end{pmatrix} \leftrightarrow \psi^\dagger(x) = (\xi^\dagger(x), \eta^\dagger(x)) = \overline{\psi}(x) \begin{pmatrix} O & I \\ I & O \end{pmatrix} . \tag{9.91}$$

With this the Lagrange density (9.89) changes into

$$\mathcal{L} = A \left\{ \overline{\psi} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} \tilde{\sigma}^\mu & O \\ O & \sigma^\mu \end{pmatrix} i\partial_\mu \psi - \overline{\psi} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} O & mI \\ mI & O \end{pmatrix} \psi \right\} , \tag{9.92}$$

which finally reduces to

$$\mathcal{L} = A\overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \, . \tag{9.93}$$

Here we have introduced the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} , \tag{9.94}$$

which turn out to obey the property of a Clifford algebra, see Eq. (9.4), due to the calculational rules (9.80) and (9.81):

$$\begin{aligned} [\gamma^\mu, \gamma^\nu]_+ &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} \begin{pmatrix} O & \sigma^\nu \\ \tilde{\sigma}^\nu & O \end{pmatrix} + \begin{pmatrix} O & \sigma^\nu \\ \tilde{\sigma}^\nu & O \end{pmatrix} \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} \\ &= \begin{pmatrix} \sigma^\mu\tilde{\sigma}^\nu + \sigma^\nu\tilde{\sigma}^\mu & O \\ O & \tilde{\sigma}^\nu\sigma^\mu + \tilde{\sigma}^\nu\sigma^\mu \end{pmatrix} = 2g^{\mu\nu} \begin{pmatrix} I & O \\ O & I \end{pmatrix} . \end{aligned} \tag{9.95}$$

The action (9.74), (9.75) can, thus, be interpreted as a functional of the Dirac spinor $\psi(x)$ and the Dirac adjoint Dirac spinor $\overline{\psi}(x)$:

$$\mathcal{A}[\psi(\bullet); \overline{\psi}(\bullet)] = \frac{1}{c} \int d^4x \, \mathcal{L}\left(\psi(x), \partial_\mu\psi(x); \overline{\psi}(x); \partial_\mu\overline{\psi}(x)\right) \, . \tag{9.96}$$

The equation of motion of the Dirac spinor is thus given by

$$\frac{\delta \mathcal{A}}{\delta \overline{\psi}(x)} = \frac{\partial \mathcal{L}}{\partial \overline{\psi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi}(x))} = A\Big\{ i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) \Big\} = 0 \, . \tag{9.97}$$

This reduces to

$$\left(i\partial\!\!\!/ - m\right)\psi(x) = 0 \tag{9.98}$$

with introducing the Feynman dagger as another widespread shortcut notation

$$\partial\!\!\!/ = \gamma^\mu \partial_\mu \, . \tag{9.99}$$

9.7 Spinor Representation of Lorentz Group

By construction the Dirac action (9.93), (9.96) is invariant under Lorentz transformations. Nevertheless we now aim for proving this again from a different point of view by studying the representation of the Lorentz group in the space of the Dirac spinors. To this end we deduce from the representations of the Lorentz group in the space of the Weyl spinors in (9.48) and (9.49)

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \qquad \longrightarrow \qquad \psi'(x') = \begin{pmatrix} \xi'(x') \\ \eta'(x') \end{pmatrix} = D(\Lambda)\psi(x)\,. \tag{9.100}$$

Here the representation matrices $D(\Lambda)$ for the Dirac spinor are composed of the respective representation matrices $D^{(1/2,0)}(\Lambda)$ and $D^{(0,1/2)}(\Lambda)$ for the Weyl spinors:

$$D(\Lambda) = \begin{pmatrix} D^{(1/2,0)}(\Lambda) & O \\ O & D^{(0,1/2)}(\Lambda) \end{pmatrix}. \tag{9.101}$$

Furthermore, we note that the relation (9.91) between the Dirac adjoint Dirac spinor $\overline{\psi}$ and the adjoint Dirac spinor $\psi^\dagger$ simplifies due to (9.37) and (9.94):

$$\overline{\psi}(x) = \psi^\dagger(x)\gamma^0 \qquad \Longleftrightarrow \qquad \psi^\dagger(x) = \overline{\psi}(x)\gamma^0\,. \tag{9.102}$$

Due to (9.100) and (9.102) the Lorentz transformation of the Dirac adjoint Dirac spinor reads

$$\overline{\psi}'(x') = \psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)D(\Lambda)^\dagger\gamma^0 = \overline{\psi}(x)\overline{D}(\Lambda)\,. \tag{9.103}$$

Here we have introduced the Dirac adjoint representation matrices

$$\overline{D}(\Lambda) = \gamma^0 D(\Lambda)^\dagger \gamma^0\,, \tag{9.104}$$

for which we obtain due to (9.37), (9.94), and (9.101) the explicit result

$$\overline{D}(\Lambda) = \begin{pmatrix} D^{(0,1/2)}(\Lambda)^\dagger & O \\ O & D^{(1/2,0)}(\Lambda)^\dagger \end{pmatrix}. \tag{9.105}$$

Thus, taking into account (9.13), (9.14), (9.16), (9.101), and (9.105) we conclude

$$\overline{D}(\Lambda) = D(\Lambda)^{-1}\,. \tag{9.106}$$

Furthermore, we note that we showed in Section 9.5

$$D^{(1/2,0)}(\Lambda)^\dagger\, \tilde{\sigma}^\mu\, D^{(1/2,0)}(\Lambda) = \Lambda^\mu{}_\nu \tilde{\sigma}^\mu\,, \tag{9.107}$$

$$D^{(0,1/2)}(\Lambda)^\dagger\, \sigma^\mu\, D^{(0,1/2)}(\Lambda) = \Lambda^\mu{}_\nu \sigma^\mu \tag{9.108}$$

for $\Lambda = R$ and $\Lambda = B$ in (9.62), (9.64) and (9.69), (9.70), respectively. But since every Lorentz transformation can be understood as a successive execution of a boost and a rotation

$$\Lambda = BR\,, \tag{9.109}$$

the corresponding representation matrices factorize, i.e. we have

$$D^{(1/2,0)}(\Lambda) = D^{(1/2,0)}(B)D^{(1/2,0)}(R)\,, \qquad D^{(0,1/2)}(\Lambda) = D^{(0,1/2)}(B)D^{(0,1/2)}(R)\,. \qquad (9.110)$$

With this we can show that (9.107) and (9.108) are even valid for any Lorentz transformation. At first we obtain for the representation $D^{(1/2,0)}$

$$\begin{aligned} D^{(1/2,0)}(\Lambda)^\dagger\,\tilde{\sigma}^\mu\,D^{(1/2,0)}(\Lambda) &= D^{(1/2,0)}(R)^\dagger\,D^{(1/2,0)}(B)^\dagger\,\tilde{\sigma}^\mu\,D^{(1/2,0)}(B)\,D^{(1/2,0)}(R) \\ &= B^\mu{}_\nu\,D^{(1/2,0)}(R)^\dagger\,\tilde{\sigma}^\nu\,D^{(1/2,0)}(R) = B^\mu{}_\nu\,R^\nu{}_\kappa\,\tilde{\sigma}^\kappa = \Lambda^\mu{}_\nu\,\tilde{\sigma}^\nu\,, \qquad (9.111) \end{aligned}$$

and, correspondingly, we get for the representation $D^{(0,1/2)}$

$$\begin{aligned} D^{(0,1/2)}(\Lambda)^\dagger\,\sigma^\mu\,D^{(0,1/2)}(\Lambda) &= D^{(0,1/2)}(R)^\dagger\,D^{(0,1/2)}(B)^\dagger\,\sigma^\mu\,D^{(0,1/2)}(B)\,D^{(0,1/2)}(R) \\ &= B^\mu{}_\nu\,D^{(0,1/2)}(R)^\dagger\,\sigma^\nu\,D^{(0,1/2)}(R) = B^\mu{}_\nu\,R^\nu{}_\kappa\,\sigma^\kappa = \Lambda^\mu{}_\nu\,\sigma^\nu\,. \qquad (9.112) \end{aligned}$$

Note that we have used (9.109) in the last step of both (9.111) and (9.112). The two transformation laws (9.107) and (9.108) can now be combined into one for the Dirac matrices (9.94). Taking into account (9.101) and (9.105) a direct multiplication of the involved 4×4 matrices yields

$$\begin{aligned} \overline{D}(\Lambda)\gamma^\mu D(\Lambda) &= \begin{pmatrix} D^{(0,1/2)}(\Lambda)^\dagger & O \\ O & D^{(1/2,0)}(\Lambda)^\dagger \end{pmatrix} \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} \begin{pmatrix} D^{(1/2,0)}(\Lambda) & O \\ O & D^{(0,1/2)}(\Lambda) \end{pmatrix} \\ &= \begin{pmatrix} O & D^{(0,1/2)}(\Lambda)^\dagger\sigma^\mu D^{(0,1/2)}(\Lambda) \\ D^{(1/2,0)}(\Lambda)^\dagger\tilde{\sigma}^\mu D^{(1/2,0)}(\Lambda) & O \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} O & \sigma^\nu \\ \tilde{\sigma}^\nu & O \end{pmatrix} = \Lambda^\mu{}_\nu\gamma^\nu\,. \qquad (9.113) \end{aligned}$$

After these preparations the invariance of the Dirac action can be shown as follows. At first we obtain for the Lorentz transformation of the action (9.93), (9.96) due to (9.100), (9.103), and the property $d^4x' = d^4x$ of special Lorentz transformations:

$$\begin{aligned} \mathcal{A}' &= \frac{A}{c}\int d^4x'\,\overline{\psi}'(x')\left(i\gamma^\mu\partial'_\mu - m\right)\psi'(x') \\ &= \frac{A}{c}\int d^4x\,\overline{\psi}(x)\Big\{i\overline{D}(\Lambda)\gamma^\mu D(\Lambda)\partial'_\mu - m\overline{D}(\Lambda)D(\Lambda)\Big\}\psi(x)\,. \qquad (9.114) \end{aligned}$$

Using (9.106) and (9.113) as well as taking into account that the partial derivatives in (9.114) transform like a covariant four vector

$$\partial_\mu \qquad \longrightarrow \qquad \partial'_\mu = \Lambda_\mu{}^\nu\,\partial_\nu \qquad (9.115)$$

we get

$$\mathcal{A}' = \frac{A}{c}\int d^4x\;\overline{\psi}(x)\,(i\Lambda^\mu{}_\nu\Lambda_\mu{}^\kappa\gamma^\nu\partial_\kappa - m)\,\psi(x)\,. \qquad (9.116)$$

From (6.28) we then conclude that the Lorentz transformed action (9.116) coincides with the original action (9.93), (9.96).

Let us further investigate the representation (9.101) of the Lorentz group in the space of the Dirac spinors. To this end we use (9.13) as well as (9.14) and bring it to the following form:

$$D(\Lambda) = \exp\left\{ -i \begin{pmatrix} \sigma^k/2 & O \\ O & \sigma^k/2 \end{pmatrix} \varphi_k - i \begin{pmatrix} -i\sigma^k/2 & O \\ O & i\sigma^k/2 \end{pmatrix} \xi_k \right\} . \tag{9.117}$$

Comparing this with a covariant formulation of the Lie theorem as in (6.61)–(6.64)

$$D(\Lambda) = \exp\left\{ -\frac{i}{2}\,\omega_{\mu\nu} S^{\mu\nu} \right\} = \exp\left\{ -\frac{i}{2}\,\epsilon_{kij} S^{ij} \varphi^k - i S^{0k} \xi^k \right\} , \tag{9.118}$$

the representation matrices for the generators of the boosts are given by

$$D(M_k) = S^{0k} = \begin{pmatrix} -i\sigma^k/2 & O \\ O & i\sigma^k/2 \end{pmatrix} , \tag{9.119}$$

while the representation matrices for the generators of the rotations follow from

$$D(L_k) = S^k = \frac{1}{2}\epsilon_{kij} S^{ij} = \begin{pmatrix} \sigma^k/2 & O \\ O & \sigma^k/2 \end{pmatrix} \tag{9.120}$$

and read

$$S^{ij} = \epsilon_{ijk} \begin{pmatrix} \sigma^k/2 & O \\ O & \sigma^k/2 \end{pmatrix} . \tag{9.121}$$

According to (6.161) we read off that (9.120) just represents the spin vector for spin 1/2 particles. Furthermore, the two results (9.119) and (9.121) can be summarized in a covariant form with the help of the Dirac matrices (9.94) as follows:

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]_- . \tag{9.122}$$

Indeed, whereas Eq. (9.119) follows directly from (9.122), the corresponding derivation of (9.120) needs to take into account the Lie algebra property of the Pauli matrices (9.5).

Now we aim for determining the commutator between two representation matrices $S^{\mu\nu}$ of the Lorentz algebra in the space of the Dirac spinors. To this end we apply the calculation rule (3.94), the definition (9.121) as well as the Clifford algebra property of the Dirac matrices in (9.95) and calculate at first the commutator

$$\left[S^{\mu\nu}, \gamma^\lambda\right]_- = i\left(g^{\nu\lambda}\gamma^\mu - g^{\mu\lambda}\gamma^\nu\right) . \tag{9.123}$$

Then we use (3.10) and (9.121)–(9.123) for obtaining

$$\left[S^{\mu\nu}, S^{\kappa\lambda}\right]_- = i\left(g^{\mu\lambda}S^{\nu\kappa} + g^{\nu\kappa}S^{\mu\lambda} - g^{\mu\kappa}S^{\nu\lambda} - g^{\nu\lambda}S^{\mu\kappa}\right) . \tag{9.124}$$

Thus, we read off from (9.124) that the representation matrices $S^{\mu\nu}$ satisfy, indeed, the usual commutation relations of the Lorentz algebra, see Eqs. (6.48) and (6.49). Furthermore, (9.123) and (9.124) show that γ^λ and $S^{\kappa\lambda}$ represent a tensor operator of rank $n = 1$ and $n = 2$ in the sense of (6.102).

9.8 Parity Transformation

Due to a parity transformation P the four-vector x is mapped to the spatially inverted four-vector $\tilde{x}$ introduced in (9.41):

$$x'_P = Px = \tilde{x}\,. \tag{9.125}$$

Performing a parity transformation P two times in a row, the original four-vector is reproduced. Thus, the parity transformation P is involutoric:

$$P^2 = 1 \qquad \Longleftrightarrow \qquad P^{-1} = P\,. \tag{9.126}$$

The representation matrix for such a parity transformation reads as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9.127}$$

Furthermore, it can be straight-forwardly shown that the representation matrix of the parity transformation (9.127) commutates with the matrix representations for the generators of rotations (6.53)

$$P^{-1} L_k P = L_k \qquad \Longleftrightarrow \qquad [P, L_k]_- = 0 \tag{9.128}$$

and anti-commutates with the matrix representations for the generators of boosts (6.54)

$$P^{-1} M_k P = -M_k \qquad \Longleftrightarrow \qquad [P, M_k]_+ = 0\,. \tag{9.129}$$

For instance, we have

$$P^{-1} L_1 P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = L_1\,, \tag{9.130}$$

$$P^{-1} M_1 P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -M_1\,. \tag{9.131}$$

Performing a parity transformation upon a Dirac spinor yields

$$\psi(x) \longrightarrow \psi'_P(x) = D(P)\psi(\tilde{x})\,, \tag{9.132}$$

where $D(P)$ denotes the corresponding representation matrix of the parity transformation in the space of Dirac spinors. Thus, $D(P)$ must possess the same properties as P. For instance, due to (9.126), $D(P)$ must be involutoric:

$$D(P)^2 = 1\,. \tag{9.133}$$

Furthermore, $D(P)$ must satisfy both a commutator and an anti-commutator relation with the representation matrices $D(L_k)$ and $D(M_k)$ of the rotations and boosts in the space of Dirac spinors, respectively, which are analogous to (9.128) and (9.129):

$$D(P)^{-1}D(L_k)D(P) = D(L_k)\,, \tag{9.134}$$

$$D(P)^{-1}D(M_k)D(P) = -D(M_k)\,. \tag{9.135}$$

We now determine the representation matrix $D(P)$ from the requirement that the Dirac equation is invariant under a parity transformation. To this end we rewrite at first the Dirac equation (9.97) by the applying the substitution $x \to \tilde{x}$:

$$\left(i\gamma^\mu \tilde{\partial}_\mu - m\right)\psi(\tilde{x}) = 0\,. \tag{9.136}$$

Then we replace $\psi(\tilde{x})$ in (9.136) with $\psi'_P(x)$ according to (9.132) and use the property of the scalar product that $\tilde{\gamma}^\mu \partial_\mu = \gamma^\mu \tilde{\partial}_\mu$ holds, yielding

$$\left\{iD(P)\tilde{\gamma}^\mu D(P)^{-1}\partial_\mu - m\right\}\psi'_P(x) = 0\,. \tag{9.137}$$

Thus, Eq. (9.137) reduces to the Dirac equation for the parity transformed mirrored Dirac spinor $\psi'_P(x)$, i.e.

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi'_P(x) = 0\,, \tag{9.138}$$

provided that the representation matrix $D(P)$ satisfies the condition

$$D(P)\tilde{\gamma}^\mu D(P)^{-1} = \gamma^\mu\,. \tag{9.139}$$

Let us define the representation matrix $D(P)$ according to

$$D(P) = \gamma^0\,. \tag{9.140}$$

Then the involution property (9.133) is valid

$$D(P)^2 = (\gamma^0)^2 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix} \tag{9.141}$$

and the condition (9.139) is fulfilled due to the Clifford algebra (9.95):

$$\gamma^0\tilde{\gamma}^0\gamma^0 = (\gamma^0)^3 = \gamma^0\,, \tag{9.142}$$

$$\gamma^0\tilde{\gamma}^k\gamma^0 = -\gamma^0\gamma^k\gamma^0 = \gamma^k\,. \tag{9.143}$$

Furthermore, taking into account (9.94), (9.119), (9.120) as well as (9.140) both the commutators (9.134) and the anti-commutators (9.135) can straight-forwardly be shown:

$$D(P)^{-1}D(L_k)D(P) = \begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} \sigma^k/2 & O \\ O & \sigma^k/2 \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix} = D(L_k)\,, \tag{9.144}$$

$$D(P)^{-1}D(M_k)D(P) = \begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} -i\sigma^k/2 & O \\ O & i\sigma^k/2 \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix} = -D(M_k)\,. \tag{9.145}$$

Additionally, we read off from the definition of γ^0 in (9.94) that a parity transformation (9.132) has the effect of interchanging the Weyl spinors ξ and η in the Dirac spinor (9.88):

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} \longrightarrow \psi'_P(x) = \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} \xi(\tilde{x}) \\ \eta(\tilde{x}) \end{pmatrix} = \begin{pmatrix} \eta(\tilde{x}) \\ \xi(\tilde{x}) \end{pmatrix} . \tag{9.146}$$

Thus, in a theory, where both $\psi(x)$ and $\psi'_p(x)$ represent physically realized states, one needs both Weyl spinors ξ and η. And from the Lorentz invariance considerations in Section 9.5 follows then that the corresponding action must necessarily have a mass term. Furthermore, we conclude from (9.146) that in a parity transformation invariant theory both Weyl spinors ξ and η have to appear on equal footing.

9.9 Neutrinos

A neutrino is an elementary particle with spin 1/2, which interacts only via the weak force and gravity. Historically, the neutrino was postulated first by Wolfgang Pauli in 1930 as an additional particle being involved in the beta decay of a neutron into a proton and an electron in order explain the conservation of energy, momentum, and angular momentum. The neutrino is so named because it is electrically neutral and because its rest mass is so small that it was long thought to be zero, leading to the suffix -ino. Therefore, in accordance with previous experimental results, neutrinos were considered for decades to be massless spin 1/2-particles, which are described by a single Weyl spinor ξ or η. According to (9.76), their Lagrangian density is then given by either

$$\mathcal{L} = Ai\xi^\dagger \tilde{\sigma}^\mu \partial_\mu \xi \tag{9.147}$$

or by

$$\mathcal{L} = Ai\eta^\dagger \sigma^\mu \partial_\mu \eta\,. \tag{9.148}$$

Like in the Maxwell theory also the Lagrangians (9.147) and (9.148) of the Weyl theory do not contain a Planck constant but still represent a valid first-quantized theory due to the vanishing rest mass. In both cases, the Lagrangian density is invariant under Lorentz transformations according to Section 9.5 but not invariant under parity transformations due to Section 9.8. In order to describe neutrinos also with a Dirac spinor ψ, one must project out the upper or the lower Weyl spinor ξ or η. To this end one introduces the matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{9.149}$$

for which we obtain due to the definition of the Dirac matrices in (9.94)

$$\gamma^5 = i \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} O & \sigma^1 \\ -\sigma^1 & I \end{pmatrix} \begin{pmatrix} O & \sigma^2 \\ -\sigma^2 & O \end{pmatrix} \begin{pmatrix} O & \sigma^3 \\ -\sigma^3 & O \end{pmatrix} = i \begin{pmatrix} \sigma^1\sigma^2\sigma^3 & O \\ O & -\sigma^1\sigma^2\sigma^3 \end{pmatrix} . \tag{9.150}$$

Here the product of the Pauli matrices (9.1) turns out to be

$$\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \tag{9.151}$$

so that (9.150) reduces to

$$\gamma^5 = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}. \tag{9.152}$$

Thus, we read off that also γ^5 is involutoric:

$$\left(\gamma^5\right)^2 = 1 \qquad \Longrightarrow \qquad \left(\gamma^5\right)^{-1} = \gamma^5\,, \tag{9.153}$$

Furthermore, with the help of the γ^5 matrix we can construct projection matrices

$$P_u = \frac{1}{2}\left(1-\gamma^5\right) \ = \ \begin{pmatrix} I & O \\ O & O \end{pmatrix}, \tag{9.154}$$

$$P_l = \frac{1}{2}\left(1+\gamma^5\right) \ = \ \begin{pmatrix} O & O \\ O & I \end{pmatrix}, \tag{9.155}$$

which possess the desired effect:

$$P_u\psi \ = \ \begin{pmatrix} I & O \\ O & O \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \tag{9.156}$$

$$P_l\psi \ = \ \begin{pmatrix} O & O \\ O & I \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \tag{9.157}$$

Thus, we read off that the Weyl spinors ξ and η represent in form of $(1\mp\gamma^5)\psi/2$ eigenstates of the matrix γ^5 with the eigenvalues ∓ 1:

$$\gamma^5\,\frac{1}{2}\left(1\mp\gamma^5\right)\psi = \mp\frac{1}{2}\left(1\mp\gamma^5\right)\psi\,. \tag{9.158}$$

As the neutrino states can be classified according to the eigenvalues of the matrix γ^5, it is of special importance. One calls γ^5 the chirality operator and speaks of left (-1) or right $(+1)$ chirality for the states $(1\mp\gamma^5)\psi/2$.

We note that the chirality operator γ^5 from (9.149) can also be written as

$$\gamma^5 = \frac{i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\,\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda\,. \tag{9.159}$$

Indeed, due to the anti-symmetry (6.141) of the ϵ-tensor only $4! = 24$ terms contribute to (9.159), where each term consists of a product of 4 different Dirac matrices. Furthermore, all 24 terms agree due to the anti-symmetry $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$ following from the Clifford algebra (9.95) and due to the anti-symmetry (6.141) of the ϵ-tensor, yielding (9.149). Since the Dirac matrices γ^μ transform according to (9.113) like a contravariant four-vector under Lorentz transformations, Eq. (9.159) has due to (9.106) the consequence that the chirality operator γ^5 is Lorentz invariant:

$$\begin{aligned}\overline{D}(\Lambda)\gamma^5 D(\Lambda) &= \frac{i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\left\{\overline{D}(\Lambda)\gamma^\mu D(\Lambda)\right\}\left\{\overline{D}(\Lambda)\gamma^\nu D(\Lambda)\right\}\left\{\overline{D}(\Lambda)\gamma^\kappa D(\Lambda)\right\}\left\{\overline{D}(\Lambda)\gamma^\lambda D(\Lambda)\right\} \\ &= \frac{i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\kappa{}_{\kappa'}\Lambda^\lambda{}_{\lambda'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\kappa'}\gamma^{\lambda'} = \frac{i}{24}\,\epsilon_{\mu'\nu'\kappa'\lambda'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\kappa'}\gamma^{\lambda'} = \gamma^5\,. \quad (9.160)\end{aligned}$$

Here we used the Weierstraß expansion of the determinant a 4×4-matrix $\Lambda = (\Lambda^\mu{}_\nu)$

$$(\text{Det }\Lambda)\epsilon_{\mu'\nu'\kappa'\lambda'} = \epsilon_{\mu\nu\kappa\lambda}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\kappa{}_{\kappa'}\Lambda^\lambda{}_{\lambda'}\,, \quad (9.161)$$

where the property Det $\Lambda = 1$ of the special Lorentz transformations implies that the four-dimensional Levi-Cività tensor has the same components in all inertial systems:

$$\epsilon'_{\mu\nu\kappa\lambda} = \epsilon_{\mu\nu\kappa\lambda}\,. \quad (9.162)$$

With the help of (9.154)–(9.157) the two neutrino Lagrangians (9.147) and (9.148) can be expressed by Dirac spinors:

$$\mathcal{L} = Ai\overline{\psi}(x)\gamma^\mu\partial_\mu\frac{1}{2}(1\mp\gamma^5)\psi(x)\,. \quad (9.163)$$

In fact, taking into account (9.88), (9.92), and (9.94) an explicit calculation yields for the upper Weyl spinor

$$\begin{aligned}\mathcal{L} &= Ai\overline{\psi}(x)\gamma^\mu\partial_\mu\,\frac{1}{2}\left(1-\gamma^5\right)\psi(x) = Ai(\xi^\dagger,\eta^\dagger)\begin{pmatrix} O & I \\ I & O\end{pmatrix}\begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O\end{pmatrix}\partial_\mu\,\frac{1}{2}\left(1-\gamma^5\right)\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\ &= Ai(\xi^\dagger,\eta^\dagger)\begin{pmatrix}\tilde{\sigma}^\mu & O \\ O & \sigma^\mu\end{pmatrix}\partial_\mu\begin{pmatrix}\xi \\ 0\end{pmatrix} = Ai\xi^\dagger\tilde{\sigma}^\mu\partial_\mu\xi \quad (9.164)\end{aligned}$$

and, correspondingly, for the lower Weyl spinor

$$\begin{aligned}\mathcal{L} &= Ai\overline{\psi}(x)\gamma^\mu\partial_\mu\,\frac{1}{2}(1+\gamma^5)\psi(x) = Ai(\xi^\dagger,\eta^\dagger)\begin{pmatrix} O & I \\ I & O\end{pmatrix}\begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O\end{pmatrix}\partial_\mu\,\frac{1}{2}(1+\gamma^5)\begin{pmatrix}\xi \\ \eta\end{pmatrix} \\ &= Ai(\xi^\dagger,\eta^\dagger)\begin{pmatrix}\tilde{\sigma}^\mu & O \\ O & \sigma^\mu\end{pmatrix}\partial_\mu\begin{pmatrix}0 \\ \eta\end{pmatrix} = Ai\eta^\dagger\sigma^\mu\partial_\mu\eta\,. \quad (9.165)\end{aligned}$$

The two neutrino Lagrangians (9.163) are manifestly Lorentz-invariant due to (9.100), (9.103), (9.113), and (9.160). Furthermore, we have due to (9.133), (9.139), (9.149), and (9.159)

$$\begin{aligned}&D(P)^{-1}\gamma^5 D(P) = \frac{i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\left\{D(P)^{-1}\gamma^\mu D(P)\right\}\left\{D(P)^{-1}\gamma^\nu D(P)\right\}\left\{D(P)^{-1}\gamma^\kappa D(P)\right\} \\ &\times\left\{D(P)^{-1}\gamma^\lambda D(P)\right\} = \frac{i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\tilde{\gamma}^\mu\tilde{\gamma}^\nu\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda = \frac{-i}{24}\,\epsilon_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = -\gamma^5\,, \quad (9.166)\end{aligned}$$

so that a parity transformation maps the two neutrino Lagrangians (9.163) into each other.

We remark that the Lagrangians (9.163) were proposed for the first time by the mathematician Hermann Weyl in 1929 to describe massless spin 1/2-particles. But since the neutrino

Lagrangians (9.163) are not invariant under parity transformations and at that time only interactions like the electromagnetic or the strong one were known, which are invariant under parity transformations, the Lagrangians (9.163) were not considered to be physical for a long time. Only in 1956 it was shown by Chien-Shiung Wu in a β-decay experiment of ${}^{60}_{27}\text{Co}$ that the weak interaction is not invariant under parity transformations and, thus, violates parity. Since this discovery neutrinos were assumed to be described by the Lagrangians (9.163) for decades. But in 1987 one managed to resolve the flavour of sun neutrinos in the Kamiokande experiment and one showed that they oscillate between the electron, the myuon, and the tauon flavour. From this observation it was concluded that neutrinos must have finite masses although their precise values have not yet been determined. Therefore, the Lagrangians (9.163) have been abandoned for describing neutrinos. But, due to their charge neutrality, until today it has not yet been finally decided how to describe theoretically neutrinos as massive spin 1/2 particles. Currently there exist two alternative descriptions, which go back to proposals of Paul Dirac and Ettore Majorana, respectively. In the first case neutrinos and anti-matter neutrinos are considered to be different particles, whereas in the second case they are assumed to be one and the same particle masquerading as two. An experimental decision between both possible theoretical descriptions is still lacking.

Subsequently, we consider the Weyl equation, that is, i.e. the equation of motion for massless spin 1/2 particles, which follows from (9.163):

$$\frac{\delta\mathcal{A}}{\delta\overline{\psi}(x)} = \frac{\partial\mathcal{L}}{\partial\overline{\psi}(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi}(x))} = Ai\gamma^\mu\partial_\mu \frac{1}{2}\left(1 \mp \gamma^5\right)\psi(x) = 0\,. \tag{9.167}$$

In the case of a particle with a fixed four-momentum vector $p = (p^\mu)$

$$\psi(x) = \psi\, e^{-ipx} \tag{9.168}$$

the Weyl equation (9.167) changes into

$$\boldsymbol{\gamma}\mathbf{p}\,\frac{1}{2}\left(1 \mp \gamma^5\right)\psi = \gamma^0 p^0\,\frac{1}{2}\left(1 \mp \gamma^5\right)\psi\,. \tag{9.169}$$

Multiplying (9.169) from the left by $\gamma^5\gamma^0$, we obtain due to (9.94) and (9.152)

$$\gamma^5\gamma^0\gamma^k = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} O & \sigma^k \\ -\sigma^k & O \end{pmatrix} = \begin{pmatrix} \sigma^k & O \\ O & \sigma^k \end{pmatrix}, \tag{9.170}$$

thus, taking into account the spin operator (9.120) the result is

$$\frac{\mathbf{Sp}}{|p^0|}\,\frac{1}{2}\left(1 \mp \gamma^5\right) = \frac{1}{2}\,\mathrm{sgn}(p^0)\,\gamma^5\,\frac{1}{2}\left(1 \mp \gamma^5\right)\,. \tag{9.171}$$

Due to the energy-momentum dispersion relation $p^0 = \pm|\mathbf{p}|$ the eigenstates $(1 \mp \gamma^5)\psi/2$ of the chirality operator γ^5 with the eigenvalues ∓ 1, see Eq. (9.158), are also eigenstates of the helicity operator with the eigenvalues $\mp\mathrm{sgn}(p^0)/2$. Thus, we conclude that chirality and helicity are identical for massless spin 1/2 particles.

9.10 Charge conjugation

The Lagrange density (9.93) of the Dirac field is also invariant with respect to another discrete symmetry transformation, where the components of the Dirac spinor $\psi(x)$ are replaced by the components of the complex conjugate Dirac spinor $\psi^*(x)$. In order to perform such a symmetry transformation we make the ansatz

$$\psi'_C(x) = C\overline{\psi}^T(x) = C\gamma^0\psi^*(x)\,, \tag{9.172}$$

where the row spinor $\overline{\psi}(x)$ from (9.91) goes over into the corresponding column spinor $\overline{\psi}^T(x)$ by transposition and we have used that $(\gamma^0)^T = \gamma^0$ due to (9.94). Furthermore, C denotes a complex 4×4-matrix which mixes these components and is defined by the fact that the transformed Dirac spinor (9.172) obeys the same Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\,\psi'_C(x) = 0 \tag{9.173}$$

as the original Dirac spinor $\psi(x)$ in (9.97). Inserting (9.172) into (9.173) and multiplying from the left by C^{-1}, then we obtain at first

$$iC^{-1}\gamma^\mu C\partial_\mu\overline{\psi}^T(x) - m\overline{\psi}^T(x) = 0\,, \tag{9.174}$$

which changes due to a subsequent transposition T into

$$i\partial_\mu\overline{\psi}(x)\left(C^{-1}\gamma^\mu C\right)^T - m\overline{\psi}(x) = 0\,. \tag{9.175}$$

This equation of motion is now compared with the Dirac equation for the Dirac adjoint Dirac spinor $\overline{\psi}(x)$. In order to derive it we start from the Dirac equation (9.97) and go over to the adjoint, yielding

$$-i\partial_\mu\psi^\dagger(x)(\gamma^\mu)^\dagger - m\psi^\dagger(x) = 0\,. \tag{9.176}$$

Taking into account the Clifford algebra (9.95) for $\mu = \nu = 0$ and (9.102) changes (9.176) into

$$-i\partial_\mu\overline{\psi}(x)\gamma^0(\gamma^\mu)^\dagger\gamma^0 - m\overline{\psi}(x) = 0\,. \tag{9.177}$$

Here we note that the Dirac matrices (9.94) have due to (9.16) the property

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} O & \tilde{\sigma}^\mu \\ \sigma^\mu & O \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix} = \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} = \gamma^\mu\,, \tag{9.178}$$

so that the Dirac equation for the Dirac-adjoint spinor (9.177) reduces to

$$i\partial_\mu\overline{\psi}(x)\gamma^\mu + m\overline{\psi}(x) = 0\,. \tag{9.179}$$

We remark that this equation of motion for the Dirac adjoint Dirac spinor $\overline{\psi}(x)$ corresponds to the Euler-Lagrange equation of the Dirac Lagrange density (9.92):

$$\frac{\delta\mathcal{A}}{\delta\psi(x)} = \frac{\partial\mathcal{L}}{\partial\psi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi(x))} = A\left\{i\partial_\mu\overline{\psi}(x)\gamma^\mu + m\overline{\psi}(x)\right\} = 0\,. \tag{9.180}$$

The comparison of (9.175) and (9.179) then leads to the following equation for determining the matrix C:

$$\left(C^{-1}\gamma^{\mu}C\right)^T = -\gamma^{\mu} \qquad \Longrightarrow \qquad C^{-1}\gamma^{\mu}C = -\left(\gamma^{\mu}\right)^T . \tag{9.181}$$

In order to solve (9.181) we make the following diagonal ansatz for the matrix C

$$C = \begin{pmatrix} c & O \\ O & -c \end{pmatrix} , \qquad C^{-1} = \begin{pmatrix} c^{-1} & O \\ O & -c^{-1} \end{pmatrix} . \tag{9.182}$$

With this we obtain from (9.94) for the left-hand side of (9.181)

$$\begin{pmatrix} c^{-1} & O \\ O & -c^{-1} \end{pmatrix} \begin{pmatrix} O & \sigma^{\mu} \\ \tilde{\sigma}^{\mu} & O \end{pmatrix} \begin{pmatrix} c & O \\ O & -c \end{pmatrix} = \begin{pmatrix} O & -c^{-1}\sigma^{\mu}c \\ -c^{-1}\tilde{\sigma}^{\mu}c & O \end{pmatrix} , \tag{9.183}$$

so we conclude from (9.181)

$$c^{-1}\sigma^{\mu}c = (\tilde{\sigma}^{\mu})^T , \qquad c^{-1}\tilde{\sigma}^{\mu}c = (\sigma^{\mu})^T . \tag{9.184}$$

Splitting both equations (9.184) into $\mu = 0$ and $\mu = k$, they yield the conditions

$$c^{-1}\sigma^0 c = (\sigma^0)^T , \tag{9.185}$$

$$c^{-1}\sigma^k c = -(\sigma^k)^T . \tag{9.186}$$

Here the transposed Pauli matrices (9.1) and (9.37) are given by

$$\left(\sigma^0\right)^T = \sigma^0 , \qquad \left(\sigma^1\right)^T = \sigma^1 , \qquad \left(\sigma^2\right)^T = -\sigma^2 , \qquad \left(\sigma^3\right)^T = \sigma^3 . \tag{9.187}$$

Let us now define the matrix c according to

$$c = -i\sigma^2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{9.188}$$

As it has the properties

$$c^{\dagger} = c^{-1} = c^T = -c = -c^* , \tag{9.189}$$

we read off that (9.185) and (9.186) are, indeed, fulfilled due to (9.2) and (9.187)–(9.189)

$$c^{-1}\sigma^0 c = i\sigma^2\sigma^0(-i\sigma^2) = \sigma^2\sigma^0\sigma^2 = (\sigma^2)^2 = \sigma^0 = (\sigma^0)^T , \tag{9.190}$$

$$c^{-1}\sigma^1 c = i\sigma^2\sigma^1(-i\sigma^2) = \sigma^2\sigma^1\sigma^2 = -(\sigma^2)^2\sigma^1 = -\sigma^1 = -(\sigma^1)^T , \tag{9.191}$$

$$c^{-1}\sigma^2 c = i\sigma^2\sigma^2(-i\sigma^2) = (\sigma^2)^2\sigma^2 = \sigma^2 = -(\sigma^2)^T , \tag{9.192}$$

$$c^{-1}\sigma^3 c = i\sigma^2\sigma^3(-i\sigma^2) = \sigma^2\sigma^3\sigma^2 = -(\sigma^2)^2\sigma^3 = -\sigma^3 = -(\sigma^3)^T . \tag{9.193}$$

Thus, in conclusion, taking into account (9.188) and (9.189) the matrix C defined in (9.182) has the properties

$$C^{\dagger} = C^{-1} = C^T = -C = -C^* \tag{9.194}$$

and can be represented as a product of Dirac matrices (9.94):

$$i\gamma^0\gamma^2 = i\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} O & \sigma^2 \\ -\sigma^2 & O \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & O \\ O & i\sigma^2 \end{pmatrix} = \begin{pmatrix} c & O \\ O & -c \end{pmatrix} = C\,. \qquad (9.195)$$

Moreover, taking into account (9.94), (9.182), (9.189), and (9.194), it follows that also the discrete symmetry transformation (9.172) is involutoric:

$$\psi''_C(x) = C\gamma^0\psi'^*_C(x) = C\gamma^0C^*(\gamma^0)^*\psi(x) = C\gamma^0C\gamma^0\psi(x) = \begin{pmatrix} c & O \\ O & -c \end{pmatrix}\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} c & O \\ O & -c \end{pmatrix}$$
$$\times\begin{pmatrix} O & I \\ I & O \end{pmatrix}\psi(x) = \begin{pmatrix} O & c \\ -c & O \end{pmatrix}\begin{pmatrix} O & c \\ -c & O \end{pmatrix}\psi(x) = \begin{pmatrix} -c^2 & O \\ O & -c^2 \end{pmatrix}\psi(x) = \psi(x)\,. \qquad (9.196)$$

And, finally, we investigate how the discrete symmetry transformation (9.172) affects the four-vector current density of the Dirac field invariant. Multiplying the equations of motion (9.97) and (9.177) for $\psi(x)$ and $\overline{\psi}(x)$ with $\overline{\psi}(x)$ and $\psi(x)$, respectively, we yield

$$i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\overline{\psi}(x)\psi(x) = 0\,, \qquad (9.197)$$
$$i\partial_\mu\overline{\psi}(x)\gamma^\mu\psi(x) + m\overline{\psi}(x)\psi(x) = 0\,, \qquad (9.198)$$

so we read off the continuity equation

$$i\partial_\mu\Big\{\overline{\psi}(x)\gamma^\mu\psi(x)\Big\} = 0 \qquad \Longrightarrow \qquad \partial_\mu j^\mu(x) = 0\,. \qquad (9.199)$$

Here the four-vector current density $j^\mu(x)$ is fixed except for a constant K:

$$j^\mu(x) = K\,\overline{\psi}(x)\gamma^\mu\psi(x)\,. \qquad (9.200)$$

Thus, the conserved charge reads due to (9.94) and (9.200)

$$Q = \int d^3x\, j^0(\mathbf{x},t) = K\int d^3x\,\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,. \qquad (9.201)$$

In order to apply the discrete symmetry transformation (9.172) to the four-vector current density (9.200), we need to know how the Dirac adjoint Dirac spinor (9.102) is transformed. Thus, applying (9.94), (9.182), (9.189), and (9.194) we yield

$$\overline{\psi}'_C(x) = \psi'^\dagger_C(x)\gamma^0 = \psi^T(x)(\gamma^0)^\dagger C^\dagger\gamma^0 = -\psi^T(x)\gamma^0C\gamma^0 = -\psi^T(x)\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} c & O \\ O & -c \end{pmatrix}$$
$$\times\begin{pmatrix} O & I \\ I & O \end{pmatrix} = -\psi^T(x)\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} O & c \\ -c & O \end{pmatrix} = -\psi^T(x)\begin{pmatrix} -c & O \\ O & c \end{pmatrix} = \psi^T(x)C\,. \qquad (9.202)$$

Transforming the four-vector current density (9.200) with (9.172) and (9.202) we then conclude at first

$$j'^\mu_C(x) = K\overline{\psi}'_C\gamma^\mu\psi'_C(x) = K\psi^T(x)C\gamma^\mu C\gamma^0\psi^*(x)\,. \qquad (9.203)$$

As each individual component of the transformed four-vector current density (9.203) coincides with its transposition, i.e. $j_C'^{\mu}(x) = (j_C'^{\mu}(x))^T$, it follows from (9.94), (9.102), (9.181), (9.194), and (9.200) that

$$j_C'^{\mu}(x) = K\psi^{\dagger}(x)(\gamma^0)^T(C\gamma^{\mu}C)^T\psi(x) = K\psi^{\dagger}(x)\gamma^0\gamma^{\mu}\psi(x) = K\overline{\psi}(x)\gamma^{\mu}\psi(x) = j^{\mu}(x)\,. \quad (9.204)$$

Thus, we conclude that the discrete symmetry transformation (9.172) turns out not to change the four-vector current density. Note that the physical meaning of the discrete symmetry transformation (9.172) as a charge conjugation becomes clear only after having implemented the second quantization of the Dirac field, as then the four-vector density operator changes its sign in contrast to (9.204),

9.11 Time Inversion

Performing a time inversion T, the space-time four-vector x is mapped into the time-inverted space-time four-vector $-\tilde{x}$:

$$x_T' = Tx = -\tilde{x}\,. \quad (9.205)$$

Executing a time inversion T successively twice, one reproduces the original state, so the time inversion T is also involutoric:

$$T^2 = 1 \qquad \Longleftrightarrow \qquad T^{-1} = T\,. \quad (9.206)$$

The representation matrix for such a time inversion reads as follows

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (9.207)$$

Thus, we conclude that the representation matrix of the time inversion (9.207) commutates with the matrix representations for the generators of rotations (6.53)

$$T^{-1}L_kT = L_k \quad (9.208)$$

and anti-commutates with the matrix representations for the generators of boosts (6.54)

$$T^{-1}M_kT = -M_k\,. \quad (9.209)$$

For instance, we have

$$T^{-1}L_1T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L_1 , \qquad (9.210)$$

$$T^{-1}M_1T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -M_1 . \qquad (9.211)$$

As the time inversion is more intriguing to interpret, we investigate at first its consequences for the Schrödinger equation

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M} \Delta \right) \psi(\mathbf{x}, t) = 0 . \qquad (9.212)$$

Obviously, the time inverted wave function

$$\psi'_T(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t) \qquad (9.213)$$

also obeys the Schrödinger equation:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M} \Delta \right) \psi'_T(\mathbf{x}, t) = 0 . \qquad (9.214)$$

In analogy to (9.213) we now perform the time inversion for a Dirac spinor via

$$\psi(x) \longrightarrow \psi'_T(x) = D(T)\psi^*(-\tilde{x}) , \qquad (9.215)$$

where $D(T)$ stands for the representation matrix of the time inversion in the space of Dirac spinors. Then $D(T)$ must also fulfill the involutoric property (9.206)

$$D(T)^2 = 1 \qquad (9.216)$$

and we expect that also the commutator and anti-commutator relations (9.208) and (9.209) are satisfied by the representation matrices $D(L_k)$ and $D(M_k)$ of rotations and boosts in the space of Dirac spinors, respectively:

$$D(T)^{-1}D(L_k)D(T) = D(L_k) , \qquad (9.217)$$

$$D(T)^{-1}D(M_k)D(T) = -D(M_k) . \qquad (9.218)$$

In analogy with (9.214), we also require that the time inverted Dirac spinor (9.215) satisfies the Dirac equation (9.97):

$$(i\gamma^\mu \partial_\mu - m) \, \psi'_T(x) = 0 . \qquad (9.219)$$

Inserting (9.215) into (9.219), we obtain

$$-i\left\{D(T)^{-1}\gamma^\mu D(T)\right\}^* \partial_\mu \psi(-\tilde{x}) - m\psi(-\tilde{x}) = 0\,. \tag{9.220}$$

Comparing (9.220) with the time-inverted Dirac equation (9.97)

$$-i\tilde{\gamma}^\mu \partial_\mu \psi(-\tilde{x}) - m\psi(-\tilde{x}) = 0\,, \tag{9.221}$$

where we used $\gamma^\mu \tilde{\partial}_\mu = \tilde{\gamma}^\mu \partial_\mu$, the representation matrix $D(T)$ of the time inversion is determined by the equation

$$D(T)^{-1}\gamma^\mu D(T) = (\tilde{\gamma}^\mu)^*\,. \tag{9.222}$$

On the one hand we calculate the conjugate complex of the Dirac matrices (9.94) by taking into account the Pauli matrices (9.1), yielding

$$\begin{aligned}
\left(\gamma^0\right)^* &= \begin{pmatrix} O & \sigma^0 \\ \sigma^0 & O \end{pmatrix}, \qquad \left(\gamma^1\right)^* = \begin{pmatrix} O & \sigma^1 \\ -\sigma^1 & O \end{pmatrix}, \\
\left(\gamma^2\right)^* &= \begin{pmatrix} O & -\sigma^2 \\ \sigma^2 & O \end{pmatrix}, \qquad \left(\gamma^3\right)^* = \begin{pmatrix} O & \sigma^3 \\ -\sigma^3 & O \end{pmatrix}.
\end{aligned} \tag{9.223}$$

On the other hand we obtain for the quantities $(\tilde{\gamma}^\mu)^T$:

$$\begin{aligned}
\left(\tilde{\gamma}^0\right)^T &= \left(\gamma^0\right)^T = \begin{pmatrix} O & (\sigma^0)^T \\ (\sigma^0)^T & O \end{pmatrix} = \begin{pmatrix} O & \sigma^0 \\ \sigma^0 & O \end{pmatrix}, \\
\left(\tilde{\gamma}^1\right)^T &= -\left(\gamma^1\right)^T = -\begin{pmatrix} O & -(\sigma^1)^T \\ (\sigma^1)^T & O \end{pmatrix} = \begin{pmatrix} O & \sigma^1 \\ -\sigma^1 & O \end{pmatrix}, \\
\left(\tilde{\gamma}^2\right)^T &= -\left(\gamma^2\right)^T = -\begin{pmatrix} O & -(\sigma^2)^T \\ (\sigma^2)^T & O \end{pmatrix} = \begin{pmatrix} O & -\sigma^2 \\ \sigma^2 & O \end{pmatrix}, \\
\left(\tilde{\gamma}^3\right)^T &= -\left(\gamma^3\right)^T = -\begin{pmatrix} O & -(\sigma^3)^T \\ (\sigma^3)^T & O \end{pmatrix} = \begin{pmatrix} O & \sigma^3 \\ -\sigma^3 & O \end{pmatrix}.
\end{aligned} \tag{9.224}$$

Thus, from (9.223) and (9.224) we read off the following identity

$$(\gamma^\mu)^* = (\tilde{\gamma}^\mu)^T \qquad \Longrightarrow \qquad (\gamma^\mu)^\dagger = \tilde{\gamma}^\mu\,. \tag{9.225}$$

Inserting (9.225) into (9.222) then results in

$$D(T)^{-1}\gamma^\mu D(T) = (\gamma^\mu)^T\,. \tag{9.226}$$

Now we take into account the property (9.181), which relates the Dirac matrices γ^μ with the representation matrix C of charge conjugation in the space of Dirac spinors. With this the equation (9.226) for determining $D(T)$ leads to

$$D(T)^{-1}\gamma^\mu D(T) = -C^{-1}\gamma^\mu C \qquad \Longrightarrow \qquad \left\{D(T)C^{-1}\right\}^{-1}\gamma^\mu\left\{D(T)C^{-1}\right\} = -\gamma^\mu\,. \tag{9.227}$$

A solution of (9.227) is given by

$$D(T)C^{-1} = -i\gamma^5 \tag{9.228}$$

together with its inverted matrix following from (9.153)

$$\left\{D(T)C^{-1}\right\}^{-1} = i\gamma^5 \,, \tag{9.229}$$

as is verified by an explicit calculation due to (9.94) and (9.152):

$$\gamma^5\gamma^\mu\gamma^5 = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}\begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix}\begin{pmatrix} -I & O \\ O & I \end{pmatrix} = -\begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} = -\gamma^\mu \,. \tag{9.230}$$

Note that (9.228) represents a quite subtle relation, which involves with the matrices γ^5, C, and $D(T)$ technical ingredients of all three discrete transformation, i.e. the parity, the charge conjugation, and the time inversion. Thus, taking into account (9.152) and (9.182), the representation matrix $D(T)$ follows from (9.228)

$$D(T) = -i\gamma^5 C = -i\begin{pmatrix} -I & O \\ O & I \end{pmatrix}\begin{pmatrix} c & O \\ O & -c \end{pmatrix} = i\begin{pmatrix} c & O \\ O & c \end{pmatrix} , \tag{9.231}$$

which has due to (9.189) the properties

$$D(T) = D(T)^{-1} = D(T)^\dagger = -D(T)^* = -D(T)^T \,. \tag{9.232}$$

According to (9.232) the representation matrix $D(T)$ satisfies the involutoric property (9.216), but the time inversion of the Dirac spinor is not involutoric due to (9.215) and (9.232):

$$\psi''_T(x) = D(T)\psi'^*_T(-\tilde{x}) = D(T)D(T)^*\psi(x) = -\psi(x) \,. \tag{9.233}$$

This behavior of Dirac spinors under time inversion corresponds to that under a rotation, where we read off from (9.21) and (9.101) that the original Dirac spinor is only recovered after a rotation with the angle 4π. Furthermore, we obtain for the commutators of $D(T)$ with the generators of rotation $D(L_k)$ due to (9.16), (9.120), (9.186), and (9.189)

$$\begin{aligned} D(T)^{-1}D(L_k)D(T) &= -\begin{pmatrix} c & O \\ O & c \end{pmatrix}\begin{pmatrix} \sigma^k/2 & O \\ O & \sigma^k/2 \end{pmatrix}\begin{pmatrix} c & O \\ O & c \end{pmatrix} = \begin{pmatrix} -c\sigma^k c/2 & O \\ O & -c\sigma^k c/2 \end{pmatrix} \\ &= \begin{pmatrix} (\sigma^k)^T/2 & O \\ O & (\sigma^k)^T/2 \end{pmatrix} = \begin{pmatrix} (\sigma^k)^*/2 & O \\ O & (\sigma^k)^*/2 \end{pmatrix} = D(L_k)^* \,, \end{aligned} \tag{9.234}$$

and, correspondingly, the commutators of $D(T)$ with $D(M_k)$ yield with (9.119) and (9.231)

$$\begin{aligned} D(T)^{-1}D(M_k)D(T) &= \frac{i}{2}\begin{pmatrix} c & O \\ O & c \end{pmatrix}\begin{pmatrix} \sigma^k & O \\ O & -\sigma^k/2 \end{pmatrix}\begin{pmatrix} c & O \\ O & c \end{pmatrix} = \frac{i}{2}\begin{pmatrix} c\sigma^k c & O \\ O & -c\sigma^k c \end{pmatrix} \\ &= \frac{-i}{2}\begin{pmatrix} (\sigma^k)^T & O \\ O & -(\sigma^k)^T \end{pmatrix} = \frac{-i}{2}\begin{pmatrix} (\sigma^k)^* & 0 \\ 0 & -(\sigma^k)^* \end{pmatrix} = -D(M_k)^* \,. \end{aligned} \tag{9.235}$$

The results (9.234) and (9.235) do not match the original expectations (9.217) and (9.218). Instead, they indicate that the time inversion represents an anti-linear operation as is further discussed in the exercises in the context of the second quantization of the Dirac field.

9.12 Dirac Representation

The representation (9.94) of the Dirac matrices used so far is called the chiral representation or the Weyl representation, as then the chirality operator γ^5 is diagonal according to (9.152). From a group-theoretical point of view this representation has the advantage that the representation matrices of the Lorentz transformation in the space of the Dirac spinors have a block diagonal shape according to (9.101), i.e. both Weyl spinors are treated on equal footing. Another common representation of the Dirac matrices is the so-called Dirac representation or the standard representation

$$\psi_D(x) = S_D \psi(x)\,, \tag{9.236}$$

where the transformation matrix S_D is given by

$$S_D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \tag{9.237}$$

with the inverse

$$S_D^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = S_D^T\,. \tag{9.238}$$

Thus, the transformation matrix S_D is orthonormal or, more precisely, unitary. For the Dirac adjoint Dirac spinor $\overline{\psi}(x)$ one obtains in the Dirac representation from (9.91), (9.94), (9.237), and (9.238):

$$\begin{aligned} \overline{\psi}_D(x) &= \psi_D^\dagger(x)\gamma^0 = \psi^\dagger(x) S_D^\dagger \gamma^0 = \overline{\psi}(x)\gamma^0 S_D^\dagger \gamma^0 = \overline{\psi}(x)\frac{1}{\sqrt{2}} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \\ &= \overline{\psi}(x)\frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \overline{\psi}(x) S_D^{-1}\,. \end{aligned} \tag{9.239}$$

In the same way one obtains for the Dirac matrices γ^μ in the Dirac representation

$$\gamma_D^0 = S_D \gamma^0 S_D^{-1} = \frac{1}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} O & I \\ I & O \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \tag{9.240}$$

$$\gamma_D^k = S_D \gamma^k S_D^{-1} = \frac{1}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} O & \sigma^k \\ -\sigma^k & O \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} O & \sigma^k \\ -\sigma^k & O \end{pmatrix}. \tag{9.241}$$

And, correspondingly, the chirality operator (9.152) in the Dirac representation turns out to be no longer diagonal:

$$\gamma_D^5 = S_D \gamma^5 S_D^{-1} = \frac{1}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} -I & O \\ O & I \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \begin{pmatrix} O & I \\ I & O \end{pmatrix}. \tag{9.242}$$

Conversely, the Dirac matrix γ^0 is not diagonal in the Weyl representation (9.94), while it is diagonal in the Dirac representation (9.240). Furthermore, the generators of the rotations in the spinor space (9.120) are invariant under the change of representation

$$D(L_k)_D = S_D D(L_k) S_D^{-1} = \frac{1}{4}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} \sigma^k & O \\ O & \sigma^k \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma^k & O \\ O & \sigma^k \end{pmatrix}, \quad (9.243)$$

whereas the generators of the boosts in the spinor space (9.119) result in the Dirac representation to be given by

$$D(M_k)_D = S_D D(M_k) S_D^{-1} = \frac{i}{4}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} -\sigma^k & O \\ O & \sigma^k \end{pmatrix}\begin{pmatrix} I & -I \\ I & I \end{pmatrix} = \frac{i}{2}\begin{pmatrix} O & -\sigma^k \\ \sigma^k & O \end{pmatrix}. \quad (9.244)$$

9.13 Non-Relativistic Limit

The Dirac representation has the advantage that the non-relativistic limit is straight-forwardly carried out. To this end we transform the Dirac equation (9.97) according to (9.236) into the Dirac representation:

$$i\gamma_D^\mu \partial_\mu \psi_D(x) - m\psi_D(x) = 0\,. \quad (9.245)$$

In this manifestly covariant formulation of the Dirac equation, we separate now explicitly the respective temporal and spatial contributions

$$i\gamma_D^0 \frac{1}{c}\frac{\partial}{\partial t}\psi_D(\mathbf{x},t) + i\gamma_D^k \partial_k \psi_D(\mathbf{x},t) - m\psi_D(\mathbf{x},t) = 0\,. \quad (9.246)$$

The Dirac equation (9.246) can then be rewritten in the form of a Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi_D(\mathbf{x},t) = H_D(\mathbf{x})\psi_D(\mathbf{x},t)\,, \quad (9.247)$$

where the Dirac Hamiltonian is given by

$$H_D(\mathbf{x}) = -ic\hbar\,\boldsymbol{\alpha}\boldsymbol{\nabla} + c\hbar m\beta\,. \quad (9.248)$$

Here we have introduced the matrices

$$\beta = \gamma_D^0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad (9.249)$$

$$\alpha^k = \gamma_D^0 \gamma_D^k = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}\begin{pmatrix} O & \sigma^k \\ -\sigma^k & O \end{pmatrix} = \begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix}, \quad (9.250)$$

where we used (9.240) and (9.241). With this we obtain the anti-commutator relations

$$\begin{aligned}
[\beta,\beta]_+ &= 2\begin{pmatrix} I & O \\ O & -I \end{pmatrix}\begin{pmatrix} I & O \\ O & -I \end{pmatrix} = 2\mathcal{I}\,, && (9.251)\\
\left[\alpha^k,\beta\right]_+ &= \begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix}\begin{pmatrix} I & O \\ O & -I \end{pmatrix} + \begin{pmatrix} I & O \\ O & -I \end{pmatrix}\begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix} = \mathcal{O}\,, && (9.252)\\
\left[\alpha^k,\alpha^l\right]_+ &= \begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix}\begin{pmatrix} O & \sigma^l \\ \sigma^l & O \end{pmatrix} + \begin{pmatrix} O & \sigma^l \\ \sigma^l & O \end{pmatrix}\begin{pmatrix} O & \sigma^k \\ \sigma^k & O \end{pmatrix}\\
&= \begin{pmatrix} \left[\sigma^k,\sigma^l\right]_+ & O \\ O & \left[\sigma^l,\sigma^k\right]_+ \end{pmatrix} = 2\delta_{kl}\,\mathcal{I}\,, && (9.253)
\end{aligned}$$

where in the latter case we applied the Clifford algebra of the Pauli matrices (9.2). Furthermore, we introduced as new abbreviations both the 4×4 unit matrix

$$\mathcal{I} = \begin{pmatrix} I & O \\ O & I \end{pmatrix} \qquad (9.254)$$

and the 4×4 zero matrix

$$\mathcal{O} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}\,. \qquad (9.255)$$

Thus, we read off from (9.251)–(9.253) that the 4×4 matrices β, α^k represent a Clifford algebra with $N=4$ generators in the sense of (9.4).

In close analogy to the Weyl representation in (9.88), we now decompose also in the Dirac representation the four-component Dirac spinor into two two-component Weyl spinors

$$\psi_D(\mathbf{x},t) = \begin{pmatrix} \xi_D(\mathbf{x},t) \\ \eta_D(\mathbf{x},t) \end{pmatrix}\,. \qquad (9.256)$$

Inserting (9.256) into (9.247) and (9.248) as well as taking into account (9.249) and (9.250) then leads to

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \xi_D(\mathbf{x},t) \\ \eta_D(\mathbf{x},t) \end{pmatrix} = -ic\hbar\begin{pmatrix} O & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & O \end{pmatrix}\boldsymbol{\nabla}\begin{pmatrix} \xi_D(\mathbf{x},t) \\ \eta_D(\mathbf{x},t) \end{pmatrix} + c\hbar m\begin{pmatrix} I & O \\ O & -I \end{pmatrix}\begin{pmatrix} \xi_D(\mathbf{x},t) \\ \eta_D(\mathbf{x},t) \end{pmatrix}\,, \qquad (9.257)$$

which reduces to two coupled equations of motion for these Weyl spinors in the Dirac representation:

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\xi_D(\mathbf{x},t) &= -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\eta_D(\mathbf{x},t) + c\hbar m\xi_D(\mathbf{x},t)\,, && (9.258)\\
i\hbar\frac{\partial}{\partial t}\eta_D(\mathbf{x},t) &= -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\xi_D(\mathbf{x},t) + c\hbar m\eta_D(\mathbf{x},t)\,. && (9.259)
\end{aligned}$$

As discussed already in Fig. 7.1 we now take into account that the relativistic and the non-relativistic energy scales are shifted against each other by the rest energy Mc^2, which leads to the ansatz

$$\psi_D(\mathbf{x},t) = \begin{pmatrix} \xi_D(\mathbf{x},t) \\ \eta_D(\mathbf{x},t) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_D(\mathbf{x},t)\,e^{-iMc^2t/\hbar} \\ \tilde{\eta}_D(\mathbf{x},t)\,e^{-iMc^2t/\hbar} \end{pmatrix}\,. \qquad (9.260)$$

Thus the coupled equations of motion (9.258), (9.259) go over into

$$i\hbar\frac{\partial}{\partial t}\tilde{\xi}_D(\mathbf{x},t) = -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\eta}_D(\mathbf{x},t) + \left(c\hbar m - Mc^2\right)\tilde{\xi}_D(\mathbf{x},t)\,, \tag{9.261}$$

$$i\hbar\frac{\partial}{\partial t}\tilde{\eta}_D(\mathbf{x},t) = -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\xi}_D(\mathbf{x},t) + \left(-c\hbar m - Mc^2\right)\tilde{\eta}_D(\mathbf{x},t)\,. \tag{9.262}$$

Fixing the yet undetermined parameter m as being inversely proportional to the Compton wave length (7.21) according to (9.87), the rest energy Mc^2 turns out to appear only in the second equation of motion:

$$i\hbar\frac{\partial}{\partial t}\tilde{\xi}_D(\mathbf{x},t) = -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\eta}_D(\mathbf{x},t)\,, \tag{9.263}$$

$$i\hbar\frac{\partial}{\partial t}\tilde{\eta}_D(\mathbf{x},t) = -ic\hbar\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\xi}_D(\mathbf{x},t) - 2Mc^2\tilde{\eta}_D(\mathbf{x},t)\,. \tag{9.264}$$

Performing now the non-relativistic limes $c \to \infty$ the kinetic energy of the Weyl spinor $\tilde{\eta}_D$ is negligible in comparison with its rest energy, i.e.

$$\left|i\hbar\frac{\partial}{\partial t}\tilde{\eta}_D(\mathbf{x},t)\right| \ll \left|Mc^2\tilde{\eta}_D(\mathbf{x},t)\right|\,, \tag{9.265}$$

so that the Weyl spinor $\tilde{\eta}_D$ can approximately be expressed by the Weyl spinor $\tilde{\xi}_D$:

$$\tilde{\eta}_D(\mathbf{x},t) = \frac{-i\hbar}{2Mc}\,\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\xi}_D(\mathbf{x},t)\,. \tag{9.266}$$

Neglecting the temporal derivative in (9.264) thus leads to an adiabatic elimination of the Weyl spinor $\tilde{\eta}_D(\mathbf{x},t)$, i.e. it now longer has an independent dynamics but its temporal evolution follows quasi-instantaneously the corresponding one of the Weyl spinor $\tilde{\xi}_D(\mathbf{x},t)$. Note that similar applications of an adiabatic elimination of degrees of freedom are ubiquitous in theoretical physics:

- One prominent example is provided by the Born-Oppenheimer approximation in molecular physics. It is based on recognizing the large difference between the electron mass and the masses of atomic nuclei, and correspondingly the respective time scales of their motion. Given the same amount of kinetic energy, the nuclei move much more slowly than the electrons. Therefore, it is a valid assumption that the wave functions of atomic nuclei and electrons in a molecule can be treated separately. This enables a separation of the Hamiltonian operator into electronic and nuclear terms, where cross-terms between electrons and nuclei are neglected, so that the two smaller and decoupled systems can be solved more efficiently. As a result an effective electronic Hamilton operator for the electronic degrees of freedom is solved, where the positions of the nuclei are fixed quantities. In the second step of the Born-Oppenheimer approximation the Schrödinger equation for the nuclear motion is treated.

- Another important example is the semi-classical laser theory, where the electric field described by the Maxwell theory couples to the matter degrees of freedom, which are dealt with quantum mechanically. For the laser it turns out that the electric field evolves on a much larger time scale than the matter degrees of freedom. This allows to adiabatically eliminate the matter degrees of freedom from the dynamics and obtain an effective evolution equation for the electric field, which describes the spontaneous emergence of coherent laser light from an originally incoherent lamp light by increasing the pump power. This adiabatic elimination of fast (stable) degrees of freedom in favour of obtaining a resulting order parameter equation for slow (unstable) degrees of freedom was recognized by Hermann Haken in the realm of synergetics, which is a theory of self-organization. This fundamental discovery leads to many fascinating applications in natural and, partially, also in social sciences.

After this excursion we return to working out the non-relativistic limit of the Dirac equation. Substituting (9.266) into (9.263) leads to a Schrödinger equation for the Weyl spinor $\tilde{\xi}_D(\mathbf{x},t)$:

$$i\hbar\frac{\partial}{\partial t}\tilde{\xi}_D(\mathbf{x},t) = -\frac{\hbar^2}{2M}\,\sigma^k\partial_k\sigma^l\partial_l\tilde{\xi}_D(\mathbf{x},t) = -\frac{\hbar^2}{4M}\,\left[\sigma^k,\sigma^l\right]_+\partial_l\partial_k\tilde{\xi}_D(\mathbf{x},t) = -\frac{\hbar^2}{2M}\,\Delta\tilde{\xi}_D(\mathbf{x},t) \tag{9.267}$$

with applying the Clifford algebra of the Pauli matrices (9.2). In the exercises we work out the non-relativistic limit of the Dirac equation in the presence of a minimal coupling to the electromagnetic field in a more systematic way by performing the so-called Foldy-Wouthuysen transformation. This leads then not to the Schrödinger equation (9.267) but to the Pauli equation for the Weyl spinor $\tilde{\xi}_D(\mathbf{x},t)$ containing automatically the correct Landé factor $g_s = 2$ for a point-like massive spin 1/2 particle. Note that both the proton and the neutron are also massive spin 1/2 particles but measurements show that their respective Landé factors 2.79 and - 1.91 deviate significantly from 2.0 which indicates that they are not point-like but composite particles. Indeed, according to the standard model of elementary particle physics, each of these nucleons consists of three quarks, which are point-like massive spin 1/2 particles according to the present day knowledge.

Let us consider now the non-relativistic limit of the Dirac action (9.92), (9.96) in the Dirac representation

$$\mathcal{A} = \frac{A}{c}\int d^4x\,\overline{\psi}_D(x)\,(i\gamma_D^\mu\partial_\mu - m)\,\psi_D(x)\,. \tag{9.268}$$

As a first preparatory step we separate explicitly the respective temporal and spatial contributions:

$$\mathcal{A} = \frac{A}{c}\int d^4x\left\{i\overline{\psi}_D(\mathbf{x},t)\gamma^0\frac{1}{c}\frac{\partial}{\partial t}\psi_D(\mathbf{x},t) + i\overline{\psi}_D(\mathbf{x},t)\boldsymbol{\gamma}_D\boldsymbol{\nabla}\psi_D(\mathbf{x},t) - m\overline{\psi}_D(\mathbf{x},t)\psi_D(\mathbf{x},t)\right\}. \tag{9.269}$$

Then we take into account how the Dirac spinor decomposes into the Weyl spinors according to (9.260) and the corresponding expression for the Dirac adjoint Dirac spinor following from (9.102) and (9.240):

$$\overline{\psi}_D(\mathbf{x},t) = \psi_D^\dagger(\mathbf{x},t)\gamma_D^0 = \left(\tilde{\xi}_D^\dagger(\mathbf{x},t)e^{iMc^2t/\hbar}, -\tilde{\eta}_D^\dagger(\mathbf{x},t)e^{iMc^2t/\hbar}\right)\,. \tag{9.270}$$

Using in addition (9.87), (9.240), and (9.241) as well as (9.260) and (9.270), the Dirac action (9.268) reduces to

$$\begin{aligned} \mathcal{A} &= A\int dt \int d^3x \left\{ \frac{i}{c}\left[\tilde{\xi}_D^\dagger(\mathbf{x},t)\,\frac{\partial\tilde{\xi}_D(\mathbf{x},t)}{\partial t} + \tilde{\eta}_D^\dagger(\mathbf{x},t)\,\frac{\partial\tilde{\eta}_D(\mathbf{x},t)}{\partial t}\right]\right. \\ &\left. +i\left[\tilde{\xi}_D^\dagger(\mathbf{x},t)\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\eta}_D(\mathbf{x},t) + \tilde{\eta}_D^\dagger(\mathbf{x},t)\boldsymbol{\sigma}\boldsymbol{\nabla}\tilde{\xi}_D(\mathbf{x},t)\right] + \frac{2Mc}{\hbar}\,\tilde{\eta}_D^\dagger(\mathbf{x},t)\tilde{\eta}_D(\mathbf{x},t)\right\}. \end{aligned} \tag{9.271}$$

If one now expresses the Weyl spinor $\tilde{\eta}_D$ according to (9.266) by the Weyl spinor $\tilde{\xi}_D$ and takes into account the calculation rule (9.7), then (9.271) goes over in the non-relativistic limes $c \to \infty$ into

$$\mathcal{A} = A\int dt \int d^3x \left\{\frac{i}{c}\,\tilde{\xi}_D^\dagger(\mathbf{x},t)\frac{\partial\tilde{\xi}(\mathbf{x},t)}{\partial t} + \frac{\hbar}{2Mc}\,\tilde{\xi}_D^\dagger(\mathbf{x},t)\Delta\tilde{\xi}_D(\mathbf{x},t)\right\}. \tag{9.272}$$

Fixing the yet undetermined parameter A according to

$$\alpha = c\hbar\,, \tag{9.273}$$

then (9.272) reduces to the Schrödinger action for the Weyl spinor $\tilde{\xi}_D$:

$$\mathcal{A} = \int dt \int d^3x \left\{i\hbar\,\tilde{\xi}_D^\dagger(\mathbf{x},t)\,\frac{\partial\tilde{\xi}_D(\mathbf{x},t)}{\partial t} + \frac{\hbar^2}{2M}\,\tilde{\xi}_D^\dagger(\mathbf{x},t)\Delta\tilde{\xi}_D(\mathbf{x},t)\right\}. \tag{9.274}$$

Furthermore, according to (9.87) and (9.273), we then conclude that the Dirac Lagrange density in the Weyl representation (9.93) reads

$$\mathcal{L} = \overline{\psi}(x)\left(i\hbar c\gamma^\mu\partial_\mu - Mc^2\right)\psi(x)\,. \tag{9.275}$$

And finally, inserting (9.260) and (9.266) into the conserved charge (9.201), we read off in the non-relativistic limit $c \to \infty$ that the yet undetermined parameter K has to be identified with

$$K = 1\,, \tag{9.276}$$

so that we obtain in the Dirac representation the conserved quantity expected for a Schrödinger theory:

$$Q = \int d^3x\,\tilde{\xi}_D^\dagger(\mathbf{x},t)\tilde{\xi}_D(\mathbf{x},t)\,. \tag{9.277}$$

Thus, we conclude that the conserved charge (9.201) of the Dirac theory reads

$$Q = \int d^3x\,\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)\,. \tag{9.278}$$

9.14 Plane Waves

We now determine the fundamental solutions of the Dirac equation in the Weyl representation (9.97), which reads by taking into account (9.87):

$$\left(i\gamma^\mu \partial_\mu - \frac{Mc}{\hbar}\right)\psi(x) = 0\,. \tag{9.279}$$

One solution method relies on performing a plane wave ansatz for the Dirac spinor $\psi(x)$, which converts the differential equation (9.279) into an algebraic equation for the corresponding spinor amplitudes. The latter would then have to be solved on the basis of the concrete form of the Dirac matrices in the Weyl representation. In this section, however, we work out a different solution method, which is group theoretically inspired. To this end we determine at first the trivial plane wave solutions in the rest frame of the massive spin 1/2 particle and then we boost them to a uniformly moving reference frame.

9.14.1 Rest Frame

In the rest frame of the massive spin 1/2 particle, the Dirac spinor can only depend on time t:

$$\psi_R(x) = \psi(t)\,. \tag{9.280}$$

Inserting (9.280) in (9.279) leads to

$$\left(i\gamma^0 \frac{\partial}{\partial t} - \frac{Mc^2}{\hbar}\right)\psi(t) = 0\,. \tag{9.281}$$

Multiplying (9.281) with the operator $(-i\gamma^0\partial/\partial t - Mc^2/\hbar)$ and taking into account $(\gamma^0)^2 = \mathcal{I}$ due to (9.94) then yields

$$\left(-i\gamma^0 \frac{\partial}{\partial t} - \frac{Mc^2}{\hbar}\right)\left(i\gamma^0 \frac{\partial}{\partial t} - \frac{Mc^2}{\hbar}\right)\psi(t) = \left\{\frac{\partial^2}{\partial t^2} + \left(\frac{Mc^2}{\hbar}\right)^2\right\}\psi(t) = 0\,. \tag{9.282}$$

Thus, we obtain the two solutions

$$\psi(t) = \psi\, e^{\mp iMc^2 t/\hbar}\,, \tag{9.283}$$

where the spinor amplitude ψ satisfies due to (9.281) and (9.283) the algebraic equation

$$\left(\pm\gamma^0 - \mathcal{I}\right)\psi = 0\,. \tag{9.284}$$

Taking into account the explicit form of the Dirac matrix γ^0 in the Weyl representation (9.94), then (9.284) reduces to

$$(\gamma^0 - \mathcal{I})\psi = \left\{\begin{pmatrix} O & I \\ I & O \end{pmatrix} - \begin{pmatrix} I & O \\ O & I \end{pmatrix}\right\}\psi = \begin{pmatrix} -I & I \\ I & -I \end{pmatrix}\psi = 0\,, \tag{9.285}$$

$$(-\gamma^0 - \mathcal{I})\psi = \left\{\begin{pmatrix} O & -I \\ -I & O \end{pmatrix} - \begin{pmatrix} I & O \\ O & I \end{pmatrix}\right\}\psi = \begin{pmatrix} -I & -I \\ -I & -I \end{pmatrix}\psi = 0\,. \tag{9.286}$$

Assuming that $\chi(+1/2)$ and $\chi(-1/2)$ are two orthonormal bi-spinors, i.e.

$$\chi^\dagger(\lambda)\chi(\lambda') = \delta_{\lambda\lambda'}\,, \tag{9.287}$$

the two solutions of (9.285) are given by

$$\psi^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi(1/2) \\ \chi(1/2) \end{pmatrix}, \qquad \psi^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi(-1/2) \\ \chi(-1/2) \end{pmatrix}. \tag{9.288}$$

Then we construct bi-spinors $\chi^c(\pm 1/2)$, which are charge conjugated with respect to $\chi(\pm 1/2)$, by defining analogous to (9.172) and (9.182)

$$\chi^c\left(\pm\frac{1}{2}\right) = c\,\chi^*\left(\pm\frac{1}{2}\right). \tag{9.289}$$

They turn out to be orthonormal as well due to (9.189), (9.287), and (9.289):

$$\chi^{c\,\dagger}(\lambda)\chi^c(\lambda') = \left(\chi^{c\,\dagger}(\lambda)\chi^c(\lambda')\right)^T = \left(\chi^T(\lambda)\,c^\dagger c\,\chi^*(\lambda')\right)^T = \chi^\dagger(\lambda')\chi(\lambda) = \delta_{\lambda\lambda'}\,. \tag{9.290}$$

With this we obtain also the two solutions of (9.286) according to

$$\psi^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi^c(1/2) \\ -\chi^c(1/2) \end{pmatrix}, \qquad \psi^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \chi^c(-1/2) \\ -\chi^c(-1/2) \end{pmatrix}. \tag{9.291}$$

We note that $\psi^{(3)}$ and $\psi^{(4)}$ just represent the charge conjugated Dirac spinors of $\psi^{(1)}$ and $\psi^{(2)}$. Namely the Dirac adjoint Dirac spinors

$$\overline{\psi}^{(1,2)} = \psi^{(1,2)\dagger}\gamma^0 \tag{9.292}$$

read explicitly with (9.288)

$$\overline{\psi}^{(1,2)} = \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\pm\frac{1}{2}\right), \chi^\dagger\left(\pm\frac{1}{2}\right)\right)\begin{pmatrix} O & I \\ I & O \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\pm\frac{1}{2}\right), \chi^\dagger\left(\pm\frac{1}{2}\right)\right), \tag{9.293}$$

so the charge conjugation yields due to (9.172), (9.182), (9.289), (9.291) and (9.293)

$$\psi_C'^{(1,2)} = C\overline{\psi}^{(1,2)T} = \begin{pmatrix} c & O \\ O & -c \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \chi^*(\pm 1/2) \\ \chi^*(\pm 1/2) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} c\chi^*(\pm 1/2) \\ -c\chi^*(\pm 1/2) \end{pmatrix} = \psi^{(3,4)}\,. \tag{9.294}$$

The finding (9.294) justifies a posteriori to define the charge conjugation of bi-spinors according to (9.289).

9.14.2 Boost to Uniformly Moving Reference Frame

Now we boost the fundamental solutions (9.283), (9.288), and (9.291) of the Dirac equation in the rest frame to a uniformly moving reference frame:

$$\psi^{(1,2)}\,e^{-iMc^2t/\hbar} \quad\longrightarrow\quad \psi_{\mathbf{p}}^{(1,2)}(x) = \psi_{\mathbf{p}}^{(1,2)}\,e^{-ipx/\hbar}\,, \tag{9.295}$$

$$\psi^{(3,4)}\,e^{+iMc^2t/\hbar} \quad\longrightarrow\quad \psi_{\mathbf{p}}^{(3,4)}(x) = \psi_{\mathbf{p}}^{(3,4)}e^{+ipx/\hbar}\,, \tag{9.296}$$

where the momentum four-vector is transferred from the rest frame (9.27) to the uniformly moving reference frame (9.28). Despite of such a boost transformation a scalar product remains invariant, so the time-like component of the boosted momentum four-vector (6.16) is fixed by its spatial components according to

$$p_R^\mu p_{R\mu} = p^\mu p_\mu \quad \Longrightarrow \quad M^2c^2 = (p^0)^2 - \mathbf{p}^2 \quad \Longrightarrow \quad E_\mathbf{p} = p^{(0)}c = \sqrt{\mathbf{p}^2c^2 + M^2c^4}\,. \tag{9.297}$$

Note that this represents precisely the relativistic energy-momentum dispersion relation. Furthermore, the corresponding spinor amplitudes $\psi_\mathbf{p}^{(\nu)}$ for $\nu = 1, 2, 3, 4$ in the uniformly moving reference frame emerge from boosting the spinor amplitudes $\psi^{(\nu)}$ in the rest frame:

$$\psi_\mathbf{p}^{(\nu)} = D(B)\psi^{(\nu)}\,. \tag{9.298}$$

Here the boost representation in the space of the Dirac spinors from (9.13), (9.14), (9.46), (9.47), and (9.101) reads in the Weyl representation:

$$D(B) = \begin{pmatrix} D^{(1/2,0)}(B) & O \\ O & D^{(0,1/2)}(B) \end{pmatrix} = \begin{pmatrix} e^{-\boldsymbol{\sigma\xi}/2} & O \\ O & e^{\boldsymbol{\sigma\eta}/2} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} & O \\ O & \sqrt{\frac{p\tilde{\sigma}}{Mc}} \end{pmatrix}. \tag{9.299}$$

Note that the spinor representations for boosts (9.46) and (9.47) represent here efficient shortcut notations for the more involved concrete expressions (9.39) and (9.43). Thus, applying (9.299) to both (9.288) and (9.291) yields

$$\psi_\mathbf{p}^{(1,2)} = D(B)\psi^{(1,2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi\left(\pm\frac{1}{2}\right) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi\left(\pm\frac{1}{2}\right) \end{pmatrix}, \tag{9.300}$$

$$\psi_\mathbf{p}^{(3,4)} = D(B)\psi^{(3,4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c\left(\pm\frac{1}{2}\right) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c\left(\pm\frac{1}{2}\right) \end{pmatrix}. \tag{9.301}$$

With the side calculation following from (6.21) and (9.80)

$$(p\sigma)(p\tilde{\sigma}) = p_\mu\sigma^\mu p_\nu\tilde{\sigma}^\nu = \frac{1}{2}p_\mu p_\nu(\sigma^\mu\tilde{\sigma}^\nu + \sigma^\nu\tilde{\sigma}^\mu) = p_\mu p_\nu g^{\mu\nu} I = p^2 I = (Mc)^2 I\,, \tag{9.302}$$

we see then explicitly that we have thus constructed solutions of the Dirac equation (9.279). At first we conclude from (9.295)

$$\left(i\gamma^\mu\partial_\mu - \frac{Mc}{\hbar}\right)\psi_\mathbf{p}^{(1,2)}(x) = 0 \qquad \Longrightarrow \qquad (\gamma^\mu p_\mu - Mc)\,\psi_\mathbf{p}^{(1,2)} = 0 \tag{9.303}$$

From (9.94), (9.300), and (9.302) follows then indeed:

$$\begin{pmatrix} O & p\sigma \\ p\tilde{\sigma} & O \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi(\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi(\pm\frac{1}{2}) \end{pmatrix} = \frac{Mc}{\sqrt{2}}\begin{pmatrix} \frac{p\sigma}{Mc}\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi(\pm\frac{1}{2}) \\ \frac{p\tilde{\sigma}}{Mc}\sqrt{\frac{p\sigma}{Mc}}\,\chi(\pm\frac{1}{2}) \end{pmatrix}$$

$$= \frac{Mc}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\sqrt{\frac{(p\sigma)(p\tilde{\sigma})}{(Mc)^2}}\,\chi(\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\sqrt{\frac{(p\tilde{\sigma})(p\sigma)}{(Mc)^2}}\,\chi(\pm\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} Mc\,I & O \\ O & Mc\,I \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi(\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi(\pm\frac{1}{2}) \end{pmatrix}. \tag{9.304}$$

In a similar way we read off from (9.296)

$$\left(i\gamma^\mu\partial_\mu - \frac{Mc}{\hbar}\right)\psi_{\mathbf{p}}^{(3,4)}(x) = 0 \qquad \Longrightarrow \qquad \left(\gamma^\mu P_\mu + Mc\right)\psi_{\mathbf{p}}^{(3,4)} = 0\,. \tag{9.305}$$

And from (9.94), (9.301), and (9.302) we get then indeed:

$$\begin{aligned}
&\begin{pmatrix} O & p\sigma \\ p\tilde{\sigma} & O \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \frac{Mc}{\sqrt{2}}\begin{pmatrix} -\frac{p\sigma}{Mc}\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\pm\frac{1}{2}) \\ +\frac{p\tilde{\sigma}}{Mc}\sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\pm\frac{1}{2}) \end{pmatrix} \\
&= \frac{Mc}{\sqrt{2}}\begin{pmatrix} -\sqrt{\frac{p\sigma}{Mc}}\sqrt{\frac{(p\sigma)(p\tilde{\sigma})}{(Mc)^2}}\,\chi^c(\pm\frac{1}{2}) \\ +\sqrt{\frac{p\tilde{\sigma}}{Mc}}\sqrt{\frac{(p\tilde{\sigma})(p\sigma)}{(Mc)^2}}\,\chi^c(\pm\frac{1}{2}) \end{pmatrix} = -\begin{pmatrix} McI & O \\ O & McI \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\pm\frac{1}{2}) \end{pmatrix}.
\end{aligned} \tag{9.306}$$

We note that $\psi_{\mathbf{p}}^{(3)}$ and $\psi_{\mathbf{p}}^{(4)}$ just represent the charge-conjugate of the Dirac spinors $\psi_{\mathbf{p}}^{(1)}$ and $\psi_{\mathbf{p}}^{(2)}$. At first, we determine the Dirac adjoint Dirac spinors

$$\begin{aligned}
\overline{\psi}_{\mathbf{p}}^{(1,2)} &= \psi_{\mathbf{p}}^{(1,2)\dagger}\gamma^0 = \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\pm\frac{1}{2}\right)\sqrt{\frac{p\sigma}{Mc}}, \chi^\dagger\left(\pm\frac{1}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}\right)\begin{pmatrix} O & I \\ I & O \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\pm\frac{1}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \chi^\dagger\left(\pm\frac{1}{2}\right)\sqrt{\frac{p\sigma}{Mc}}\right).
\end{aligned} \tag{9.307}$$

In addition, we summarize (9.185), (9.186) and conclude from (9.189)

$$c^{-1}\sigma^\mu c = (\tilde{\sigma}^\mu)^T \qquad \Longrightarrow \qquad c(\sigma^\mu)^T c^{-1} = \tilde{\sigma}^\mu\,, \qquad c(\tilde{\sigma}^\mu)^T c^{-1} = \sigma^\mu\,. \tag{9.308}$$

The latter two relations can be generalized to any function of Pauli matrices $f(\sigma^\mu)$ or $f(\tilde{\sigma}^\mu)$, which has a Taylor series:

$$cf(\sigma^\mu)^T c^{-1} = f(\tilde{\sigma}^\mu)\,, \qquad cf(\tilde{\sigma}^\mu)^T c^{-1} = f(\sigma^\mu)\,. \tag{9.309}$$

The charge conjugation of the Dirac spinors $\psi_{\mathbf{P}}^{(1)}$ and $\psi_{\mathbf{P}}^{(2)}$ then leads to due to (9.172), (9.182), (9.287), (9.307), and (9.309):

$$\begin{aligned}
\psi_C^{(1,2)} &= C\overline{\psi}^{(1,2)T} = \begin{pmatrix} c & O \\ O & -c \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\tilde{\sigma}}{Mc}}^T\chi^*(\pm\frac{1}{2}) \\ \sqrt{\frac{p\sigma}{Mc}}^T\chi^*(\pm\frac{1}{2}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} c\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,c^{-1}\,c\chi^*(\pm\frac{1}{2}) \\ -c\sqrt{\frac{p\sigma}{Mc}}\,c^{-1}\,c\,\chi^*(\pm\frac{1}{2}) \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \psi_{\mathbf{p}}^{(3,4)}\,.
\end{aligned} \tag{9.310}$$

The spinor amplitudes (9.300) and (9.301) can now be written as

$$\psi_{\mathbf{p}}^{(\nu)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{M}}\,\chi\left(\frac{(-1)^{\nu+1}}{2}\right) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi\left(\frac{(-1)^{\nu+1}}{2}\right) \end{pmatrix}; \quad \nu = 1,2 \tag{9.311}$$

$$\psi_{\mathbf{p}}^{(\nu)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c\left(\frac{(-1)^{\nu+1}}{2}\right) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c\left(\frac{(-1)^{\nu+1}}{2}\right) \end{pmatrix}; \quad \nu = 3,4\,. \tag{9.312}$$

9.14.3 Orthonormality Relations

After having obtained the plane wave solutions, we embark now upon determining their respective orthonormality relations. To this end we start with mentioning the adjoint of the spinor amplitudes (9.311):

$$\psi_{\mathbf{p}}^{(\nu)\dagger} = \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}, \chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}\right); \quad \nu = 1,2\,, \tag{9.313}$$

$$\psi_{\mathbf{p}}^{(\nu)\dagger} = \frac{1}{\sqrt{2}}\left(\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}, -\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}\right); \quad \nu = 3,4\,. \tag{9.314}$$

Furthermore, we remark that the spinor amplitudes $\psi_{\mathbf{p}}^{(\nu)}$ for $\nu = 1,2$ and $\psi_{-\mathbf{p}}^{(\nu)}$ for $\nu = 3,4$ satisfy the following orthonormality relations:

1. case: $\nu = 1,2$; $\nu' = 1,2$:

$$\psi_{\mathbf{p}}^{(\nu)\dagger}\psi_{\mathbf{p}}^{(\nu')} = \frac{1}{2}\left(\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}, \chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}\right)\begin{pmatrix}\sqrt{\frac{p\sigma}{Mc}}\,\chi(\frac{(-1)^{\nu'+1}}{2})\\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi(\frac{(-1)^{\nu'+1}}{2})\end{pmatrix} \tag{9.315}$$

$$= \chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\frac{p\sigma + p\tilde{\sigma}}{2Mc}\,\chi\left(\frac{(-1)^{\nu'+1}}{2}\right) = \frac{E_{\mathbf{p}}}{Mc^2}\,\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\chi\left(\frac{(-1)^{\nu'+1}}{2}\right) = \frac{E_{\mathbf{p}}}{Mc^2}\,\delta_{\nu\nu'}\,,$$

2. case: $\nu = 3,4$; $\nu' = 3,4$:

$$\psi_{-\mathbf{p}}^{(\nu)\dagger}\psi_{-\mathbf{p}}^{(\nu')} = \frac{1}{2}\left(\chi^c\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}, -\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}\right)\begin{pmatrix}\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\frac{(-1)^{\nu'+1}}{2})\\ -\sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\frac{(-1)^{\nu'+1}}{2})\end{pmatrix} \tag{9.316}$$

$$= \chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\frac{p\tilde{\sigma} + p\sigma}{2Mc}\,\chi^c\left(\frac{(-1)^{\nu'+1}}{2}\right) = \frac{E_{\mathbf{p}}}{Mc^2}\,\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^c\left(\frac{(-1)^{\nu'+1}}{2}\right) = \frac{E_{\mathbf{p}}}{Mc^2}\,\delta_{\nu\nu'}\,,$$

3. case: $\nu = 1,2$; $\nu' = 3,4$:

$$\begin{aligned}\psi_{\mathbf{p}}^{(\nu)\dagger}\psi_{-\mathbf{p}}^{(\nu')} &= \frac{1}{2}\left(\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}, \chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}\right)\begin{pmatrix}\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi^c(\frac{(-1)^{\nu'+1}}{2})\\ -\sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\frac{(-1)^{\nu'+1}}{2})\end{pmatrix}\\ &= \frac{1}{2}\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\left(\sqrt{\frac{(p\sigma)(p\tilde{\sigma})}{(Mc)^2}} - \sqrt{\frac{(p\tilde{\sigma})(p\sigma)}{(Mc)^2}}\right)\chi^c\left(\frac{(-1)^{\nu'+1}}{2}\right) = 0\,,\end{aligned} \tag{9.317}$$

4. case $\nu = 3,4$; $\nu' = 1,2$:

$$\begin{aligned}\psi_{-\mathbf{p}}^{(\nu)\dagger}\psi_{\mathbf{p}}^{(\nu')} &= \frac{1}{2}\left(\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}, -\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}\right)\begin{pmatrix}\sqrt{\frac{p\sigma}{Mc}}\,\chi(\frac{(-1)^{\nu'+1}}{2})\\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi(\frac{(-1)^{\nu'+1}}{2})\end{pmatrix}\\ &= \frac{1}{2}\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\left(\sqrt{\frac{(p\tilde{\sigma})(p\sigma)}{(Mc)^2}} - \sqrt{\frac{(p\sigma)(p\tilde{\sigma})}{(Mc)^2}}\right)\chi\left(\frac{(-1)^{\nu'+1}}{2}\right) = 0\,.\end{aligned} \tag{9.318}$$

The orthonormality relations (9.315)–(9.318) can be summarized as follows

$$\psi_{\varepsilon_\nu \mathbf{p}}^{(\nu)\dagger} \, \psi_{\varepsilon_{\nu'} \mathbf{p}}^{(\nu')} = \frac{E_{\mathbf{p}}}{Mc^2} \, \delta_{\nu\nu'} \, , \tag{9.319}$$

where have introduced the abbreviation

$$\varepsilon_\nu = \begin{cases} +1 \, ; & \nu = 1,2 \\ -1 \, ; & \nu = 3,4 \end{cases} \, . \tag{9.320}$$

With this, we can check whether the fundamental solutions

$$\psi_{\mathbf{p}}^{(\nu)}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(\nu)} e^{-\frac{i}{\hbar}\varepsilon_\nu (E_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} \tag{9.321}$$

fulfill orthonormality relations. Taking into account (9.297) and (9.319) we obtain from (9.321)

$$\int d^3x \, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t) \psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(\nu)\dagger} \, \psi_{\mathbf{p}'}^{(\nu')} \, e^{\frac{i}{\hbar}(\varepsilon_{\nu'} E_{\mathbf{p}'} - \varepsilon_\nu E_{\mathbf{p}})t} (2\pi\hbar)^3 \delta(\varepsilon_\nu \mathbf{p} - \varepsilon_{\nu'} \mathbf{p}') \tag{9.322}$$

$$= (2\pi\hbar)^3 \psi_{\mathbf{p}}^{(\nu)\dagger} \, \psi_{\mathbf{p}\varepsilon_\nu \varepsilon_{\nu'}}^{(\nu')} \, e^{\frac{i}{\hbar}(\varepsilon_{\nu'} E_{\varepsilon_{\nu'}\varepsilon_\nu \mathbf{p}} - \varepsilon_\nu E_{\mathbf{p}})t} \delta(\mathbf{p}' - \varepsilon_\nu \varepsilon_{\nu'} \mathbf{p}) = \frac{(2\pi\hbar)^3 E_{\mathbf{p}}}{Mc^2} \, \delta_{\nu\nu'} \delta(\mathbf{p}' - \mathbf{p}) \, . \tag{9.323}$$

If we now replace (9.321) with

$$\psi_{\mathbf{p}}^{(\nu)}(\mathbf{x},t) = \sqrt{\frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}}}} \, \psi_{\mathbf{p}}^{(\nu)} \, e^{-\frac{i}{\hbar}\epsilon_\nu (E_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} \, , \tag{9.324}$$

then the fundamental solutions of the Dirac equation satisfy the orthonormality relations

$$\int d^3x \, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t) \psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \delta_{\nu\nu'} \, \delta(\mathbf{p} - \mathbf{p}') \, . \tag{9.325}$$

9.14.4 Dirac Representation

For the sake of completeness, we finally determine the fundamental solutions (9.325) in the Dirac representation. To this end we have to calculate at first the spinor amplitudes (9.288) and (9.291) in the rest system in the Dirac representation:

$$\psi_D^{(1,2)} = S_D \psi^{(1,2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi(\pm\frac{1}{2}) \\ \chi(\pm\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \chi(\pm\frac{1}{2}) \\ 0 \end{pmatrix} , \tag{9.326}$$

$$\psi_D^{(3,4)} = S_D \psi^{(3,4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(\pm\frac{1}{2}) \\ -\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi^c(\pm\frac{1}{2}) \end{pmatrix} . \tag{9.327}$$

By boosting from the rest frame into the uniformly moving reference frame we then get

$$\psi_{\mathbf{p}D}^{(1,2)} = S_D \psi_{\mathbf{p}}^{(1,2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} \chi(\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}} \chi(\pm\frac{1}{2}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \left(\sqrt{\frac{p\sigma}{Mc}} + \sqrt{\frac{p\tilde{\sigma}}{Mc}} \right) \chi(\pm\frac{1}{2}) \\ \left(-\sqrt{\frac{p\tilde{\sigma}}{Mc}} + \sqrt{\frac{p\sigma}{Mc}} \right) \chi(\pm\frac{1}{2}) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \left(\frac{p\sigma + Mc}{\sqrt{2Mc(p^0+Mc)}} + \frac{p\tilde{\sigma} + Mc}{\sqrt{2Mc(p^0+Mc)}} \right) \chi(\pm\frac{1}{2}) \\ \left(-\frac{p\sigma + Mc}{\sqrt{2mc(p^0+Mc)}} + \frac{p\tilde{\sigma} + Mc}{\sqrt{2Mc(p^0+Mc)}} \right) \chi(\pm\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E_{\mathbf{p}} + Mc^2}{2Mc^2}} \, \chi(\pm\frac{1}{2}) \\ \frac{\boldsymbol{\sigma}\mathbf{p}c}{\sqrt{2Mc^2(E_{\mathbf{p}} + Mc^2)}} \, \chi(\pm\frac{1}{2}) \end{pmatrix} \tag{9.328}$$

and, correspondingly,

$$\psi_{\mathbf{p}D}^{(3,4)} = S_D \psi_{\mathbf{p}}^{(3,4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi^c(\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \left(\sqrt{\frac{p\sigma}{Mc}} - \sqrt{\frac{p\tilde{\sigma}}{Mc}}\right)\chi^c(\pm\frac{1}{2}) \\ \left(-\sqrt{\frac{p\sigma}{Mc}} - \sqrt{\frac{p\tilde{\sigma}}{Mc}}\right)\chi^c(\pm\frac{1}{2}) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \left(\frac{p\sigma+Mc}{\sqrt{2Mc(p^0+Mc)}} - \frac{p\tilde{\sigma}+Mc}{\sqrt{2Mc(p^0+Mc)}}\right)\chi^c(\pm\frac{1}{2}) \\ \left(-\frac{p\sigma+Mc}{\sqrt{2Mc(p^0+Mc)}} - \frac{p\tilde{\sigma}+Mc}{\sqrt{2Mc(p^0+Mc)}}\right)\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \begin{pmatrix} \frac{-\boldsymbol{\sigma}\mathbf{p}c}{\sqrt{2Mc^2(E_{\mathbf{p}}+Mc^2)}}\chi^c(\pm\frac{1}{2}) \\ -\sqrt{\frac{E_{\mathbf{p}}+Mc^2}{2M^2}}\chi^c(\pm\frac{1}{2}) \end{pmatrix} . \tag{9.329}$$

Note that the results (9.328) and (9.329) are obtained in the exercises in a different way by invoking a Foldy-Wouthuysen transformation. Furthermore, we recognize in the non-relativistic limes $c \to \infty$ that the lower or upper components of the Dirac spinor are small at $\psi_{\mathbf{P}D}^{(1,2)}$ or $\psi_{\mathbf{P}D}^{(3,4)}$ in (9.328) or (9.329), respectively.

9.15 Helicity Spinors

In the considerations of the previous section, the two orthonormal bi-spinors $\chi(+1/2)$ and $\chi(-1/2)$ have not yet been specified. It is now time to catch up with this deficiency and to make a particular choice for those orthonormal bi-spinors. In the following we introduce even two possible choices, which depend on the quantization axis for the spin 1/2.

9.15.1 Rest Frame

At first, we consider spin 1/2 particles in the rest frame, where the spin is quantized with respect to the z-axis. In this case we define the orthonormal bi-spinors according to

$$\chi\left(+\frac{1}{2}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \chi\left(-\frac{1}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{9.330}$$

as they represent the orthonormal eigenvectors of the generator $D(L_3) = \sigma^3/2$ for a rotation around the z-axis:

$$\frac{1}{2}\sigma^3\,\chi\left(\pm\frac{1}{2}\right) = \pm\frac{1}{2}\,\chi\left(\pm\frac{1}{2}\right) . \tag{9.331}$$

From (9.188), (9.289), and (9.330) we then get the explicit form of the charge conjugated bi-spinors:

$$\chi^c\left(+\frac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{9.332}$$

$$\chi^c\left(-\frac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} . \tag{9.333}$$

Accordingly, the charge conjugated bi-spinors satisfy the eigenvalue problem

$$\frac{1}{2}\sigma^3\,\chi^c\left(\pm\frac{1}{2}\right) = \mp\frac{1}{2}\,\chi^c\left(\pm\frac{1}{2}\right)\,. \tag{9.334}$$

A comparison of (9.331) with (9.334) shows that the eigenvalues of $\chi(\pm 1/2)$ and $\chi^c(\pm 1/2)$ are just exchanged.

The Dirac spinors (9.288) and (9.291) formed with the bi-spinors $\chi(\pm 1/2)$ and $\chi^c(\pm 1/2)$ in the rest system of the particle turn out to represent eigenvectors of the generator $D(L_3)$ of the rotation about the z-axis:

$$D(L_3)\psi^{(\nu)} = \frac{(-1)^{\nu+1}}{2}\,\psi^{(\nu)}\,; \quad \nu = 1,2\,, \qquad D(L_3)\psi^{(\nu)} = \frac{(-1)^{\nu}}{2}\,\psi^{(\nu)}\,; \quad \nu = 3,4\,. \tag{9.335}$$

Namely, taking into account (9.120), the following holds:

$$\begin{pmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi(\pm\frac{1}{2}) \\ \chi(\pm\frac{1}{2}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2}\sigma^3\,\chi(\pm\frac{1}{2}) \\ \frac{1}{2}\sigma^3\,\chi(\pm\frac{1}{2}) \end{pmatrix} = \pm\frac{1}{2}\,\frac{1}{\sqrt{2}} \begin{pmatrix} \chi(\pm\frac{1}{2}) \\ \chi(\pm\frac{1}{2}) \end{pmatrix}, \tag{9.336}$$

$$\begin{pmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(\pm\frac{1}{2}) \\ -\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2}\sigma^3\,\chi^c(\pm\frac{1}{2}) \\ -\frac{1}{2}\sigma^3\,\chi^c(\pm\frac{1}{2}) \end{pmatrix} = \mp\frac{1}{2}\,\frac{1}{\sqrt{2}} \begin{pmatrix} \chi^c(\pm\frac{1}{2}) \\ -\chi^c(\pm\frac{1}{2}) \end{pmatrix}. \tag{9.337}$$

9.15.2 Helicity Operator

In the following we embark on considering spin 1/2 particles, whose spin is quantized with respect to the direction of the respective particle momentum $\mathbf{p}$. To this end we construct the corresponding helicity spinors analogous to Section 8.10, where the polarisation vectors of circularly polarised plane waves were determined in the realm of electrodynamics.

To this end we determine at first the helicity operator (6.185) in the space of bi-spinors, where the spin vector is given by $D(\mathbf{L}) = \boldsymbol{\sigma}/2$ due to (9.120):

$$h(\mathbf{p}) = \frac{D(\mathbf{L})\mathbf{p}}{p} = \frac{\boldsymbol{\sigma}\mathbf{p}}{2p} \tag{9.338}$$

Taking into account the explicit form of the Pauli matrices (9.8) this yields

$$h(\mathbf{p}) = \frac{1}{2p}\left\{ p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \frac{1}{2p} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}. \tag{9.339}$$

Now we define the helicity spinors $\chi_h(\mathbf{p}, \pm 1/2)$ as eigenvectors of the helicity operator (9.338) with the eigenvalues $\pm 1/2$:

$$h(\mathbf{p})\;\chi_h\left(\mathbf{p}, \pm\frac{1}{2}\right) = \pm\frac{1}{2}\,\chi_h\left(\mathbf{p}, \pm\frac{1}{2}\right)\,. \tag{9.340}$$

From (9.330) and (9.338) follows then that the bi-spinors $\chi(\pm 1/2)$ are eigenvectors of the helicity operator $h(p\,\mathbf{e}_z)$ to the eigenvalue $\pm 1/2$:

$$h(p\,\mathbf{e}_z)\chi\left(+\frac{1}{2}\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\chi\left(+\frac{1}{2}\right), \tag{9.341}$$

$$h(p\,\mathbf{e}_z)\chi\left(-\frac{1}{2}\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\chi\left(-\frac{1}{2}\right). \tag{9.342}$$

Thus, due to (9.340), we then conclude

$$\chi_h\left(p\,\mathbf{e}_z, \pm\frac{1}{2}\right) = \chi\left(\pm\frac{1}{2}\right). \tag{9.343}$$

9.15.3 Uniformly Moving Rest Frame

Now we consider a uniformly moving rest frame, where the spin is quantized with respect to the momentum vector, where $\mathbf{p}$ is described with the help of spherical coordinates p, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi)$:

$$\mathbf{p} = p\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{9.344}$$

Then we know that the rotation matrix (8.128) determined in (8.131) yields (9.344) analogous to (8.132):

$$R(\theta, \phi)p\mathbf{e}_z = \mathbf{p}. \tag{9.345}$$

Therefore, we determine the rotation matrix $D(R(\theta, \phi)$ in the space of bi-spinors, where first the rotation $D(R_y(\theta))$ and then the rotation $D(R_z(\phi))$ is performed:

$$D(R(\theta, \phi)) = D(R_z(\phi))\ D(R_y(\theta)). \tag{9.346}$$

The individual rotation matrices follow from (9.8), (9.10), (9.11), and (9.21):

$$\begin{aligned} D(R_z(\phi)) &= e^{-iD(L_3)\phi} = e^{-\frac{i}{2}\sigma^3\phi} = \cos\left(\frac{\phi}{2}\right)I - i\sin\left(\frac{\phi}{2}\right)\sigma^3 \\ &= \cos\left(\frac{\phi}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\phi}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \end{aligned} \tag{9.347}$$

$$\begin{aligned} D(R_y(\theta)) &= e^{-iD(L_2)\theta} = e^{-\frac{i}{2}\sigma^2\theta} = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\sigma^2 \\ &= \cos\left(\frac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\theta}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \end{aligned} \tag{9.348}$$

Thus, the resulting rotation matrix (9.346) is given by:

$$D(R(\theta,\phi)) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} & -\sin\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right)\, e^{i\phi/2} & \cos\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} . \tag{9.349}$$

Now we map the bi-spinors $\chi(\pm 1/2)$, which describe a quantization of the spin 1/2 with respect to the z-axis, with the rotation matrix $D(R(\theta,\phi)$ and obtain the helicity bi-spinors, which describe a spin quantization with respect to the direction of the momentum $\mathbf{p}$:

$$\chi_h\left(\mathbf{p}, \pm\frac{1}{2}\right) = D(R(\theta,\phi))\, \chi\left(\pm\frac{1}{2}\right) . \tag{9.350}$$

With the explicit form of the dual spinors (9.330) and the rotation matrix (9.349), the helicity dual spinors are then:

$$\chi_h\left(\mathbf{p}, +\frac{1}{2}\right) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} , \qquad \chi_h\left(\mathbf{p}, -\frac{1}{2}\right) = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \cos\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} . \tag{9.351}$$

In special case $\mathbf{p} = p\,\mathbf{e}_z$, i.e. $\theta = \phi = 0$, the result (9.351) reduces to (9.330) according to (9.343). Furthermore, the charge conjugation of the helicity spinors $\chi_h(\mathbf{p}, \pm 1/2)$ leads to:

$$\chi_h^c\left(\mathbf{p}, +\frac{1}{2}\right) = c\,\chi_h^*\left(\mathbf{p}, +\frac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \\ \sin\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \end{pmatrix} = \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \cos\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} , \tag{9.352}$$

$$\chi_h^c\left(\mathbf{p}, -\frac{1}{2}\right) = c\,\chi_h^*\left(\mathbf{p}, -\frac{1}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \\ \cos\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \end{pmatrix} = \begin{pmatrix} -\cos\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ -\sin\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} . \tag{9.353}$$

In case $\mathbf{p} = p\,\mathbf{e}_z$, i.e. $\theta = \phi = 0$, (9.352) and (9.353) reduce to (9.330):

$$\chi_h^c\left(p\,\mathbf{e}_z, \pm\frac{1}{2}\right) = \chi^c\left(\pm\frac{1}{2}\right) . \tag{9.354}$$

Furthermore, we remark that the mapping of the charge conjugated bi-spinors (9.332) with the rotation matrix (9.349) leads to the charge conjugated helicity spinors (9.352) and (9.353):

$$\chi_h^c\left(\mathbf{p}, \pm\frac{1}{2}\right) = D(R(\theta,\phi))\, \chi^c\left(\pm\frac{1}{2}\right) . \tag{9.355}$$

As a crosscheck we also verify that the constructed helicity spinors $\chi_h(\mathbf{p}, \pm 1/2)$ are, indeed, eigenvectors of the helicity operator $h(\mathbf{p})$ from (9.339) with the eigenvalue $\pm\frac{1}{2}$:

$$\begin{aligned} h(\mathbf{p})\chi_h\left(\mathbf{p}, +\frac{1}{2}\right) &= \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right)\, e^{i\phi/2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi/2} \end{pmatrix} = \frac{1}{2}\,\chi_h\left(\mathbf{p}, +\frac{1}{2}\right) , \end{aligned} \tag{9.356}$$

$$\begin{aligned} h(\mathbf{p})\chi_h\left(\mathbf{p}, -\frac{1}{2}\right) &= \frac{1}{2} \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)\, e^{-i\phi/2} \\ \cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ -\cos\left(\frac{\theta}{2}\right) e^{i\phi/2} \end{pmatrix} = -\frac{1}{2}\,\chi_h\left(\mathbf{p}, -\frac{1}{2}\right) . \end{aligned} \tag{9.357}$$

Furthermore, we show that the constructed charge conjugated helicity spinors $\chi_h^c(\mathbf{p},\pm 1/2)$ are eigenvectors of the helicity operator $h(\mathbf{p})$ with the eigenvalue $\mp 1/2$:

$$\begin{aligned} h(\mathbf{p})\chi_h^c\left(\mathbf{p},+\frac{1}{2}\right) &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ -\cos\left(\frac{\theta}{2}\right) e^{i\phi/2} \end{pmatrix} = -\frac{1}{2}\chi_h^c\left(\mathbf{p},+\frac{1}{2}\right), \end{aligned} \tag{9.358}$$

$$\begin{aligned} h(\mathbf{p})\chi_h^c\left(\mathbf{p},-\frac{1}{2}\right) &= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} -\cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ -\sin\left(\frac{\theta}{2}\right) e^{i\phi/2} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -\cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ -\sin\left(\frac{\theta}{2}\right) e^{+i\phi/2} \end{pmatrix} = \frac{1}{2}\,\chi_h^c\left(\mathbf{p},-\frac{1}{2}\right). \end{aligned} \tag{9.359}$$

Now we come back to the Dirac spinors (9.300) and (9.301) in the uniformly moving reference frame, where $\chi(\pm 1/2)$ and $\chi^c(\pm 1/2)$ denoted two sets of orthonormal bi-spinors, which are charge conjugated with respect to each other. Whereas we have discussed in the two previous subsections the case of choosing the z-axis as the quantization axis, we come now to another appropriate physical choice to identify $\chi(\pm 1/2)$ and $\chi^c(\pm 1/2)$. Namely we assume that the spin is quantized with respect to the direction of motion $\mathbf{p}/p$, which amounts to identifying $\chi(\pm 1/2)$ and $\chi^c(\pm 1/2)$ with the helicity spinors $\chi_h(\mathbf{p},\pm 1/2)$ and $\chi_h^c(\mathbf{p},\pm 1/2)$, yielding concretely

$$\psi_{\mathbf{p}}^{(1,2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h\left(\mathbf{p},\pm\frac{1}{2}\right) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h\left(\mathbf{p},\pm\frac{1}{2}\right) \end{pmatrix}, \tag{9.360}$$

$$\psi_{\mathbf{p}}^{(3,4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h^c\left(\mathbf{p},\pm\frac{1}{2}\right) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h^c\left(\mathbf{p},\pm\frac{1}{2}\right) \end{pmatrix}. \tag{9.361}$$

In order to justify this choice we define the helicity operator in the space of Dirac spinors due to (6.185) and (9.120):

$$H(\mathbf{p}) = \frac{D(\mathbf{L})\mathbf{p}}{p} = \frac{1}{2p}\begin{pmatrix} \boldsymbol{\sigma}\mathbf{p} & O \\ O & \boldsymbol{\sigma}\mathbf{p} \end{pmatrix} = \begin{pmatrix} h(\mathbf{p}) & O \\ O & h(\mathbf{p}) \end{pmatrix}. \tag{9.362}$$

According to (9.36), (9.45), and (9.338) as well as the Lie algebra of the Pauli matrices (9.5), the helicity operator $h(\mathbf{p})$ in the space of bi-spinors commutates with the boost representation in the space of bi-spinors:

$$\left[\sqrt{\frac{p\sigma}{Mc}}, h(\mathbf{p})\right]_- = \left[\sqrt{\frac{p\tilde{\sigma}}{Mc}}, h(\mathbf{p})\right]_- = 0\,. \tag{9.363}$$

Therefore, the Dirac spinors (9.360) and (9.361) are eigenstates of the helicity operator

$$H(\mathbf{p})\psi_{\mathbf{p}}^{(\nu)} = \eta_\nu\,\psi_{\mathbf{p}}^{(\nu)} \tag{9.364}$$

with the eigenvalues

$$\eta_\nu = \frac{(-1)^{\nu+1}}{2}\,; \quad \nu = 1,2\,, \qquad \eta_\nu = \frac{(-1)^{\nu}}{2}\,; \quad \nu = 3,4\,. \tag{9.365}$$

In detail, due to (9.283), (9.284), (9.339), (9.362), and (9.363) the following applies for $\nu = 1,2$

$$H(\mathbf{p})\psi_{\mathbf{p}}^{(\nu)} = \begin{pmatrix} h(\mathbf{p}) & O \\ O & h(\mathbf{p}) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h(\mathbf{p},\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h(\mathbf{p},\pm\frac{1}{2}) \end{pmatrix} = \frac{\pm 1}{2\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h(\mathbf{p},\pm\frac{1}{2}) \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h(\mathbf{p},\pm\frac{1}{2}) \end{pmatrix} = \pm\frac{1}{2}\,\psi_{\mathbf{p}}^{(\nu)}\,,$$

and, correspondingly, we have for $\nu = 3,4$

$$H(\mathbf{p})\psi_{\mathbf{p}}^{(\nu)} = \begin{pmatrix} h(\mathbf{p}) & O \\ O & h(\mathbf{p}) \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h^c(\mathbf{p},\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h^c(\mathbf{p},\pm\frac{1}{2}) \end{pmatrix} = \frac{\mp 1}{2\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}}\,\chi_h^c(\mathbf{p},\pm\frac{1}{2}) \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}}\,\chi_h^c(\mathbf{p},\pm\frac{1}{2}) \end{pmatrix} = \mp\frac{1}{2}\,\psi_{\mathbf{p}}^{(\nu)}\,.$$

Thus, in conclusion, we have determined in a group theoretically inspired approach the plane wave solutions of the Dirac equation (9.324), where the corresponding Dirac spinor amplitudes are given by (9.360) and (9.361). This result will turn out to be indispensable for the subsequent canonical field quantization of the Dirac theory.

9.16 Canonical Field Quantisation

In order to determine the Hamiltonian formulation of classical field theory from the Lagrangian formulation, one has to find at first the momentum fields, which are canonically conjugated to the independent field degrees of freedom. In case of the Dirac field, the following canonically conjugated momentum fields are obtained for the Dirac spinor $\psi(\mathbf{x},t)$ and the Dirac adjoint Dirac spinor $\overline{\psi}(\mathbf{x},t)$, respectively:

$$\pi(\mathbf{x},t) = \frac{\delta\mathcal{A}}{\delta\left(\frac{\partial\psi(\mathbf{x},t)}{\partial t}\right)} = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi(\mathbf{x},t)}{\partial t}\right)} = i\hbar\overline{\psi}(\mathbf{x},t)\gamma^0 = i\hbar\psi^\dagger(\mathbf{x},t)\,, \tag{9.366}$$

$$\overline{\pi}(\mathbf{x},t) = \frac{\delta\mathcal{A}}{\delta\left(\frac{\partial\overline{\psi}(\mathbf{x},t)}{\partial t}\right)} = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\overline{\psi}(\mathbf{x},t)}{\partial t}\right)} = 0\,. \tag{9.367}$$

Note that the last equality in (9.366) follows from taking into account (9.102). Thus, in the Hamiltonian formulation of the Dirac theory, one can consider $\psi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ or, equivalently, $\psi(\mathbf{x},t)$ and $\psi^\dagger(\mathbf{x},t)$ as the independent fields.

And, according to the Noether theorem being explored and applied to the Dirac field in the Appendix, any conserved physical quantity of the Dirac theory turns out to be bilinear in these independent fields. Namely, due to the sandwich principle, each conserved quantity follows from a spatial integral over the respective first-quantized operator, which is multiplied with $\psi^\dagger(\mathbf{x},t)$ from the left and $\psi(\mathbf{x},t)$ from the right. Indeed, the charge of the Dirac field is given by (9.278) and analogous expressions also hold for the energy, the momentum, and the helicity of the Dirac field:

$$E = \int d^3x\, \psi^\dagger(\mathbf{x},t) H_D(\mathbf{x}) \psi(\mathbf{x},t)\,, \tag{9.368}$$

$$\mathbf{P} = \int d^3x\, \psi^\dagger(\mathbf{x},t)\, \frac{\hbar}{i} \boldsymbol{\nabla}\, \psi(\mathbf{x},t)\,, \tag{9.369}$$

$$h = \int d^3x\, \psi^\dagger(\mathbf{x},t) \begin{pmatrix} \boldsymbol{\sigma}/2 & O \\ O & \boldsymbol{\sigma}/2 \end{pmatrix} \frac{\hbar\boldsymbol{\nabla}/i}{|\hbar\boldsymbol{\nabla}/i|}\, \psi(\mathbf{x},t)\,. \tag{9.370}$$

Note that the Dirac Hamiltonian $H_D(\mathbf{x})$ was already defined in (9.248) and reduces due to (9.87) to

$$H_D(\mathbf{x}) = -ic\hbar\, \boldsymbol{\alpha}\boldsymbol{\nabla} + Mc^2\beta\,. \tag{9.371}$$

Furthermore, we have used in (9.370) that the helicity (6.185) stems from the generators of the rotations (9.120) in the space of Dirac spinors.

In a canonical quantization of the Dirac field the independent fields $\psi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ or $\psi(\mathbf{x},t)$ and $\psi^\dagger(\mathbf{x},t)$ of the Hamilton field theory become field operators $\hat{\psi}(\mathbf{x},t)$ and $\hat{\pi}(\mathbf{x},t)$ or $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^\dagger(\mathbf{x},t)$. Since a bosonic quantisation of the Dirac field turns out to violate microcausality and, thus, leads inevitably to contradictions, one has to perform a fermionic quantisation. Therefore, the following equal-time anti-commutator algebra is required:

$$\left[\hat{\psi}_\alpha(\mathbf{x},t), \hat{\psi}_\beta(\mathbf{x}',t)\right]_+ = \left[\hat{\pi}_\alpha(\mathbf{x},t), \hat{\pi}_\beta(\mathbf{x}',t)\right]_+ = 0\,,\ \left[\hat{\psi}_\alpha(\mathbf{x},t), \hat{\pi}_\beta(\mathbf{x}',t)\right]_+ = i\hbar\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')\,, \tag{9.372}$$

where α, β denote the spinorial components. Due to the definition of the momentum field in (9.366) the anti-commutator algebra (9.372) reduces to

$$\left[\hat{\psi}_\alpha(\mathbf{x},t), \hat{\psi}_\beta(\mathbf{x}',t)\right]_+ = \left[\hat{\psi}^\dagger_\alpha(\mathbf{x},t), \hat{\psi}^\dagger_\beta(\mathbf{x}',t)\right]_+ = 0\,,\ \left[\hat{\psi}_\alpha(\mathbf{x},t), \hat{\psi}^\dagger_\beta(\mathbf{x}',t)\right]_+ = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}')\,. \tag{9.373}$$

Thus, the conserved quantities of the first quantized Dirac theory, i.e. the charge (9.278), the energy (9.368), the momentum (9.369), and the helicity (9.370), become second quantized operators due to the canonical field quantisation:

$$\hat{Q} = \int d^3x\, \hat{\psi}^\dagger(\mathbf{x},t)\hat{\psi}(\mathbf{x},t)\,, \tag{9.374}$$

$$\hat{H} = \int d^3x\, \hat{\psi}^\dagger(\mathbf{x},t) H_D(\mathbf{x}) \hat{\psi}(\mathbf{x},t)\,, \tag{9.375}$$

$$\hat{\mathbf{P}} = \int d^3x\, \hat{\psi}^\dagger(\mathbf{x},t)\, \frac{\hbar}{i} \nabla\, \hat{\psi}(\mathbf{x},t)\,, \tag{9.376}$$

$$\hat{h} = \int d^3x\, \hat{\psi}^\dagger(\mathbf{x},t) \begin{pmatrix} \boldsymbol{\sigma}/2 & O \\ O & \boldsymbol{\sigma}/2 \end{pmatrix} \frac{\hbar\boldsymbol{\nabla}/i}{|\hbar\boldsymbol{\nabla}/i|}\, \hat{\psi}(\mathbf{x},t)\,. \tag{9.377}$$

In order to determine the Heisenberg equations of motion (3.62), one needs to take into account both the first and the second quantized Hamilton operator (9.371) and (9.376) as well as to apply the calculation rule (3.94). With this the Heisenberg equations of motion of the field operators $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^\dagger(\mathbf{x},t)$ result in

$$i\hbar \frac{\partial\hat{\psi}(\mathbf{x},t)}{\partial t} = \left[\hat{\psi}(\mathbf{x},t),\hat{H}\right]_- = H_D(\mathbf{x})\hat{\psi}(\mathbf{x},t) = \left(-ic\hbar\boldsymbol{\alpha}\nabla + Mc^2\beta\right)\hat{\psi}(\mathbf{x},t)\,, \tag{9.378}$$

$$i\hbar \frac{\partial\hat{\psi}^\dagger(\mathbf{x},t)}{\partial t} = \left[\hat{\psi}^\dagger(\mathbf{x},t),\hat{H}\right]_- = -\left\{H_D(\mathbf{x})\hat{\psi}(\mathbf{x},t)\right\}^\dagger = \left(-ic\hbar\boldsymbol{\alpha}\nabla - Mc^2\beta\right)\hat{\psi}^\dagger(\mathbf{x},t)\,. \tag{9.379}$$

Thus, the field operators $\hat{\psi}(\mathbf{x},t)$ and $\hat{\psi}^\dagger(\mathbf{x},t)$ satisfy the Dirac equation (9.247) and the adjoint Dirac equation, respectively.

9.17 Decomposition Into Plane Waves

The field operator $\hat{\psi}(\mathbf{x},t)$ is now decomposed with respect to the fundamental plane wave solutions $\psi_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)$ of the Dirac equation defined in (9.324). The expansion coefficients in this decomposition are then operators of second quantisation:

$$\hat{\psi}(\mathbf{x},t) = \sum_{\nu=1}^{4}\int d^3p\,\psi_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)\hat{a}_{\mathbf{p}}^{(\nu)}\,. \tag{9.380}$$

Correspondingly, one obtains for the adjoint field operator:

$$\hat{\psi}^\dagger(\mathbf{x},t) = \sum_{\nu=1}^{4}\int d^3p\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\,. \tag{9.381}$$

With the help of the orthonormality relation (9.325) of the fundamental plane wave solutions, the expansions (9.380) and (9.381) can be inverted, yielding:

$$\int d^3x\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\hat{\psi}(\mathbf{x},t) = \hat{a}_{\mathbf{p}}^{(\nu)}\,, \tag{9.382}$$

$$\int d^3x\,\hat{\psi}^\dagger(\mathbf{x},t)\psi_{\mathbf{p}}^{(\nu)}(\mathbf{x},t) = \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\,. \tag{9.383}$$

From the equal-time anti-commutator algebra (9.373) of the field operator $\hat{\psi}(\mathbf{x},t)$ and its adjoint $\hat{\psi}^\dagger(\mathbf{x},t)$, a corresponding anti-commutator algebra can then be determined for the expansion coefficients $\hat{a}_{\mathbf{p}}^{(\nu)}$ and $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$:

$$\left[\hat{a}_{\mathbf{p}}^{(\nu)},\hat{a}_{\mathbf{p}'}^{(\nu')}\right]_+ = \int d^3x\int d^3x' \sum_{\alpha,\alpha'=1}^{4}\psi_{\mathbf{p},\alpha}^{(\nu)}(\mathbf{x},t)\psi_{\mathbf{p}',\alpha'}^{(\nu')}(\mathbf{x}',t)\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}_{\alpha'}(\mathbf{x}',t)\right]_+ = 0\,, \tag{9.384}$$

$$\left[\hat{a}_{\mathbf{p}}^{(\nu)\dagger},\hat{a}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \int d^3x\int d^3x' \sum_{\alpha,\alpha'=1}^{4}\psi_{\mathbf{p},\alpha}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}',\alpha'}^{(\nu')\dagger}(\mathbf{x}',t)\left[\hat{\psi}_\alpha^\dagger(\mathbf{x},t),\hat{\psi}_{\alpha'}^\dagger(\mathbf{x}',t)\right]_+ = 0\,, \tag{9.385}$$

$$\left[\hat{a}_{\mathbf{p}}^{(\nu)},\hat{a}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \int d^3x\int d^3x' \sum_{\alpha,\alpha'=1}^{4}\psi_{\mathbf{p},\alpha}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}',\alpha'}^{(\nu')}(\mathbf{x}',t)\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}_{\alpha'}^\dagger(\mathbf{x}',t)\right]_+$$

$$= \int d^3x\sum_{\alpha=1}^{4}\psi_{\mathbf{p},\alpha}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}',\alpha}^{(\nu')}(\mathbf{x},t) = \int d^3x\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \delta_{\nu\nu'}\delta(\mathbf{p}-\mathbf{p}')\,. \tag{9.386}$$

Note that in (9.386) the orthonormality relation (9.325) is applied. As the operators $\hat{a}_{\mathbf{p}}^{(\nu)}$, $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ fulfill according to (9.384)–(9.386) the canonical anti-commutator algebra, they are interpreted for the time being as annihilation and creation operators of fermionic particles.

9.18 Second Quantizied Operators

Inserting (9.380) and (9.381) into (9.374) and taking into account the orthonormality relation (9.325) the charge operator $\hat{Q}$ in second quantisation can be expressed in terms of the creation and annihilation operators $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ and $\hat{a}_{\mathbf{p}}^{(\nu)}$:

$$\begin{aligned}\hat{Q} &= \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p\int d^3p'\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\int d^3x\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\\ &= \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p\int d^3p'\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\delta_{\nu\nu'}\delta(\mathbf{p}-\mathbf{p}') = \sum_{\nu=1}^{4}\int d^3p\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)}\,. \qquad (9.387)\end{aligned}$$

Since the particle number operator $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)}$ is positive definite, also the charge operator $\hat{Q}$ is positive definite due to (9.387). Thus, it looks like as if the fermionic particles seem to have only a positive charge.

Accordingly, inserting (9.380) and (9.381) into (9.375) one obtains for the Hamilton operator $\hat{H}$ of second quantisation at first

$$\hat{H} = \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p\int d^3p'\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\int d^3x\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)H_D(\mathbf{x})\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \qquad (9.388)$$

Here we can take into account that the plane waves $\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)$ from (9.324) are eigenfunctions of the Dirac Hamiltonian operator of the first quantisation (9.371) as they were determined in Section 9.14 to solve the Dirac equation (9.279):

$$H_D(\mathbf{x})\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = i\hbar\frac{\partial}{\partial t}\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \varepsilon_{\nu'}E_{\mathbf{p}'}\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \qquad (9.389)$$

With the help of the orthonormality relation (9.325) the Hamilton operator of second quantisation (9.388) then results in

$$\begin{aligned}\hat{H} &= \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p\int d^3p'\,\varepsilon_{\nu'}E_{\mathbf{p}'}\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\int d^3x\,\psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\\ &= \sum_{\nu=1}^{4}\int d^3p\,\varepsilon_\nu E_{\mathbf{p}}\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)} = \int d^3p\left(\sum_{\nu=1}^{2}E_{\mathbf{p}}\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)} - \sum_{\nu=3}^{4}E_{\mathbf{p}}\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)}\right), \qquad (9.390)\end{aligned}$$

where we have used the abbreviation (9.320) in the last step. Thus, the fermionic particles with $\nu = 1, 2$ appear to have positive energies $E_{\mathbf{p}}$, while those with $\nu = 3, 4$ seem to have correspondingly negative energies $-E_{\mathbf{p}}$.

Subsequently, we insert (9.380) and (9.381) into (9.376), so the momentum operator $\hat{\mathbf{P}}$ of second quantisation results at first in

$$\hat{\mathbf{P}} = \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p \int d^3p'\, \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')} \int d^3x\, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\, \frac{\hbar}{i}\boldsymbol{\nabla}\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \tag{9.391}$$

Here we use the fact that the plane waves $\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)$ from (9.324) are eigenfunctions of the momentum operator of first quantisation:

$$\frac{\hbar}{i}\boldsymbol{\nabla}\,\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \varepsilon_{\nu'}\mathbf{p}'\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \tag{9.392}$$

Thus, with the orthonormality relation (9.325) the momentum operator of second quantisation (9.377) reduces to

$$\begin{aligned}\hat{\mathbf{P}} &= \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p \int d^3p'\, \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\varepsilon_{\nu'}\mathbf{p}' \int d^3x\, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\\ &= \sum_{\nu=1}^{4}\int d^3p\,\varepsilon_\nu\mathbf{p}\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)} = \int d^3p\left(\sum_{\nu=1}^{2}\mathbf{p}\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\,\hat{a}_{\mathbf{p}}^{(\nu)} - \sum_{\nu=3}^{4}\mathbf{p}\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)}\right).\end{aligned} \tag{9.393}$$

We conclude that the fermionic particles with $\nu = 1, 2$ seem to have the momentum $\mathbf{p}$ and, correspondingly, those with $\nu = 3, 4$ the momentum $-\mathbf{p}$.

In a similar way we also proceed for the helicity operator (9.377), where we insert (9.380) and (9.381), yielding

$$\hat{h} = \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p \int d^3p'\, \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')} \int d^3x\, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\begin{pmatrix}\boldsymbol{\sigma}/2 & O\\ O & \boldsymbol{\sigma}/2\end{pmatrix}\frac{\hbar\boldsymbol{\nabla}/i}{|\hbar\boldsymbol{\nabla}/i|}\,\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \tag{9.394}$$

Applying the eigenvalue problem (9.392) and the first quantized helicity operator (9.362) this reduces to

$$\hat{h} = \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p \int d^3p'\, \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\varepsilon_{\nu'} \int d^3x\, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\, H(\mathbf{p}')\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \tag{9.395}$$

Here we use the fact that the plane waves $\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)$ from (9.324) are eigenfunctions of the helicity operator of first quantisation according to (9.364):

$$H(\mathbf{p}')\,\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) = \eta_{\nu'}\,\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t)\,. \tag{9.396}$$

Thus, with this and the orthonormality relation (9.325) the helicity operator of second quantisation (9.395) reads

$$\begin{aligned}\hat{h} &= \sum_{\nu=1}^{4}\sum_{\nu'=1}^{4}\int d^3p \int d^3p'\, \hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}'}^{(\nu')}\varepsilon_{\nu'}\eta_{\nu'} \int d^3x\, \psi_{\mathbf{p}}^{(\nu)\dagger}(\mathbf{x},t)\psi_{\mathbf{p}'}^{(\nu')}(\mathbf{x},t) \qquad (9.397)\\ &= \sum_{\nu=1}^{4}\int d^3p\,\varepsilon_\nu\eta_\nu\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)} = \int d^3p\left(\sum_{\nu=1}^{2}\frac{(-1)^{\nu+1}}{2}\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\,\hat{a}_{\mathbf{p}}^{(\nu)} + \sum_{\nu=3}^{4}\frac{(-1)^{\nu+1}}{2}\,\hat{a}_{\mathbf{p}}^{(\nu)\dagger}\hat{a}_{\mathbf{p}}^{(\nu)}\right).\end{aligned}$$

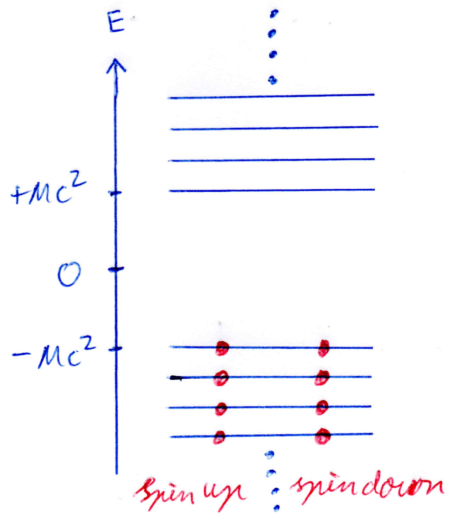

Figure 9.2: Schematic sketch of the Dirac sea, which models the physical vacuum as an infinite sea of particles with negative energy.

Note that we have used in the last step the abbreviations (9.320) and (9.365). The result (9.397) means that the fermionic particles with $\nu = 1, 3$ ($\nu = 2, 4$) have supposedly the helicity $+1/2$ ($-1/2$).

Finally, we conclude this section by summarizing that, indeed, the second quantized operators for the charge (9.387), the energy (9.390), the momentum (9.393), and the helicity (9.397) have turned out to not explicitly depend on time. This reflects that these second quantized operators correspond to conserved quantities.

9.19 Dirac Sea

Within the framework of the canonical field quantisation, the vacuum state $|0\rangle_V$ is usually defined by the fact that it does not contain any particle. This is guaranteed provided that all annihilation operators $\hat{a}_{\mathbf{p}}^{(\nu)}$ annul the vacuum state $|0\rangle_V$:

$$\hat{a}_{\mathbf{p}}^{(\nu)} |0\rangle_V = 0 \quad \text{for all } \nu, \mathbf{p} \,. \tag{9.398}$$

On the other, in the second quantized Dirac theory we are confronted with the fact that particles with both positive and negative energies appear, see Eq. (9.390). In order to provide a physical interpretation for the latter observation, Paul Dirac assumed in 1930 that instead of the vacuum state $|0\rangle_V$ a physical vacuum state $|0\rangle_P$ is realised in nature. It is defined by the condition that all states with negative energies, i.e. those with $\nu = 3, 4$, are occupied, forming the so-called Fermi sea, see Fig. 9.2:

$$|0\rangle_P = \prod_{\nu=3,4} \prod_{\mathbf{p}} \hat{a}_{\mathbf{p}}^{(\nu)\dagger} |0\rangle_V \,. \tag{9.399}$$

Here, a continuous product is formed with respect to all momenta $\mathbf{p}$. Dirac justifies this transition from the vacuum state $|0\rangle_V$ to the physical vacuum state $|0\rangle_P$ by the argument that the Dirac sea is always present and can, therefore, not be measured in any experiment, Thus, the infinitely large energy or charge of the Dirac sea can be renormalised.

An immediate consequence of the definition of the physical vacuum state $|0\rangle_P$ in (9.399) is that it is annulled by the annihilation operators $\hat{a}_{\mathbf{p}}^{(\nu)}$ for $\nu = 1, 2$ because of (9.398) and by the creation operators $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ for $\nu = 3, 4$ due to the anti-commutator algebra (9.385):

$$\hat{a}_{\mathbf{p}}^{(\nu)} |0\rangle_P = 0 \quad \text{for all } \nu = 1, 2 \text{ and } \mathbf{p}\,; \qquad \hat{a}_{\mathbf{p}}^{(\nu)\dagger} |0\rangle_P = 0 \quad \text{for all } \nu = 3, 4 \text{ and } \mathbf{p}\,. \quad (9.400)$$

If one takes into account the anti-commutator algebra (9.384)–(9.386) and the property (9.400) of the physical vacuum $|0\rangle_P$, a reinterpretation of the annihilation and creation operators becomes possible. While $\hat{a}_{\mathbf{p}}^{(\nu)}$ and $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ for $\nu = 1, 2$ continue to be understood as annihilation and creation operators of particles, $\hat{a}_{\mathbf{p}}^{(\nu)}$ and $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ for $\nu = 3, 4$ can now be interpreted inversely as the creation and annihilation operators of particles. For instance, applying $\hat{a}_{\mathbf{p}}^{(\nu)}$ for $\nu = 3, 4$ to the physical vacuum state (9.399) annihilates a particle in the Dirac sea of Fig. 9.2, which corresponds to the creation of a hole.

Consequently, by convention we consider in the Dirac hole theory that the indices $\nu = 1, 2$ ($\nu = 3, 4$) describe particles (antiparticles), for instance electrons (positrons) with spin up and down. The double role of the expansion operators $\hat{a}_{\mathbf{p}}^{(\nu)}$ and $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ as creation and annihilation operators, respectively, makes the theory at a first glance confusing. Therefore, it is suggestive to introduce different symbols in order to discriminate already from the notation between the operators of particles and antiparticles. For the particles we use from now on the following definition for the creation operators

$$\hat{a}_{\mathbf{p}}^{(1)\dagger} = \hat{b}_{\mathbf{p}}^{(1)\dagger}\,, \qquad \hat{a}_{\mathbf{p}}^{(2)\dagger} = \hat{b}_{\mathbf{p}}^{(2)\dagger} \tag{9.401}$$

and for the annihilation operators

$$\hat{a}_{\mathbf{p}}^{(1)} = \hat{b}_{\mathbf{p}}^{(1)}\,, \qquad \hat{a}_{\mathbf{p}}^{(2)} = \hat{b}_{\mathbf{p}}^{(2)}\,. \tag{9.402}$$

Correspondingly, we introduce for the antiparticles the creation operators

$$\hat{a}_{\mathbf{p}}^{(3)} = \hat{d}_{\mathbf{p}}^{(1)\dagger}\,, \qquad \hat{a}_{\mathbf{p}}^{(4)} = \hat{d}_{\mathbf{p}}^{(2)\dagger} \tag{9.403}$$

and the annihilation operators

$$\hat{a}_{\mathbf{p}}^{(3)\dagger} = \hat{d}_{\mathbf{p}}^{(1)}\,, \qquad \hat{a}_{\mathbf{p}}^{(4)\dagger} = \hat{d}_{\mathbf{p}}^{(2)}\,. \tag{9.404}$$

For $\nu = 1, 2$ this redefinition just corresponds to a simple renaming. But for $\nu = 3, 4$ the creation and annihilation operators exchange their roles. Note that the anti-commutator algebra (9.384)–(9.386) remains invariant due to this redefinition, since creation and annihilation operators appear there on equal footing:

$$\left[\hat{b}_{\mathbf{p}}^{(\nu)}, \hat{b}_{\mathbf{p}'}^{(\nu')}\right]_+ = \left[\hat{b}_{\mathbf{p}}^{(\nu)}, \hat{d}_{\mathbf{p}'}^{(\nu')}\right]_+ = \left[\hat{d}_{\mathbf{p}}^{(\nu)}, \hat{d}_{\mathbf{p}'}^{(\nu')}\right]_+ = \left[\hat{b}_{\mathbf{p}}^{(\nu)}, \hat{d}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = 0\,, \tag{9.405}$$

$$\left[\hat{b}_{\mathbf{p}}^{(\nu)\dagger}, \hat{b}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \left[\hat{b}_{\mathbf{p}}^{(\nu)\dagger}, \hat{d}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \left[\hat{d}_{\mathbf{p}}^{(\nu)\dagger}, \hat{d}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \left[\hat{b}_{\mathbf{p}}^{(\nu)\dagger}, \hat{d}_{\mathbf{p}'}^{(\nu')}\right]_+ = 0\,, \tag{9.406}$$

$$\left[\hat{b}_{\mathbf{p}}^{(\nu)}, \hat{b}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \left[\hat{d}_{\mathbf{p}}^{(\nu)}, \hat{d}_{\mathbf{p}'}^{(\nu')\dagger}\right]_+ = \delta_{\nu\nu'}\delta(\mathbf{p}-\mathbf{p}')\,. \tag{9.407}$$

The physical vacuum state $|0\rangle_P$ is now determined by the fact that it is annulled by the annihilation operators $\hat{b}_{\mathbf{p}}^{(\nu)}$, $\hat{d}_{\mathbf{p}}^{(\nu)}$ of both the particles and the antiparticles:

$$\hat{b}_{\mathbf{p}}^{(\nu)}\,|0\rangle_P = 0\,, \tag{9.408}$$

$$\hat{d}_{\mathbf{p}}^{(\nu)}\,|0\rangle_P = 0\,. \tag{9.409}$$

The Hamilton operator (9.390) of the second quantisation has both positive and negative energy values. Due to the redefinition of second quantized operators (9.401)–(9.404) it changes into

$$\hat{H} = \sum_{\nu=1}^{2} \int d^3p\, E_{\mathbf{p}} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger}\hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)}\hat{d}_{\mathbf{p}}^{(\nu)\dagger}\right)\,. \tag{9.410}$$

But, taking into account the anti-commutator algebra (9.407), the expression (9.410) for the Hamilton operator is transformed into:

$$\hat{H} = \sum_{\nu=1}^{2} \int d^3p\, E_{\mathbf{p}} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger}\hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)\dagger}\hat{d}_{\mathbf{p}}^{(\nu)}\right) - \sum_{\nu=1}^{2} \int d^3p\, E_{\mathbf{p}}\, \delta(\mathbf{0})\,. \tag{9.411}$$

The expectation value of this Hamilton operator with respect to the physical vacuum state $|0\rangle_P$ reads due to (9.408) and (9.409)

$${}_P\langle 0|\,\hat{H}\,|0\rangle_P = -\sum_{\nu=1}^{2} \int d^3p\, E_{\mathbf{p}}\, \delta(\mathbf{0})\,. \tag{9.412}$$

First of all we note that the vacuum energy for the fermions of the Dirac theory turns out to be negative in contrast to the bosonic cases of the Klein-Gordon theory in (7.119) and the Maxwell theory in (8.158). This is an immediate consequence of having an underlying anti-commutator algebra instead of a commutator algebra. But also in the fermionic case the vacuum energy (9.412) is divergent due to two reasons. On the one hand the respective momentum integral over the relativistic energy-momentum dispersion (9.297) is divergent and on the other hand the factor $\delta(\mathbf{0})$ is divergent as well. The renormalisation of the Hamilton operator (9.411) is performed by simply subtracting this infinitely large expectation value (9.412), yielding the normal-ordered Hamilton operator

$$:\hat{H}: = \hat{H} - {}_P\langle 0|\,\hat{H}\,|0\rangle_P = \sum_{\nu=1}^{2} \int d^3p\, E_{\mathbf{p}} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger}\hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)\dagger}\hat{d}_{\mathbf{p}}^{(\nu)}\right)\,. \tag{9.413}$$

This physical Hamilton operator is positive definite as both particles and antiparticles have the same energy $E_{\mathbf{p}} > 0$.

Quite correspondingly, the charge operator $\hat{Q}$, the momentum operator $\hat{P}$, and the helicity operator from (9.387), (9.393), and (9.397) change due to the redefinition of second quantized operators (9.401)–(9.404) to

$$\hat{Q} = \sum_{\nu=1}^{2} \int d^3p \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)} \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \right) , \tag{9.414}$$

$$\hat{\mathbf{P}} = \sum_{\nu=1}^{2} \int d^3p \, \mathbf{p} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)} \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \right) , \tag{9.415}$$

$$\hat{h} = \sum_{\nu=1}^{2} \int d^3p \, \frac{(-1)^{\nu+1}}{2} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)} \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \right) . \tag{9.416}$$

Applying the anti-commutator algebra (9.407) yields

$$\hat{Q} = \sum_{\nu=1}^{2} \int d^3p \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) + \sum_{\nu=1}^{2} \int d^3p \, \delta(\mathbf{0}) , \tag{9.417}$$

$$\hat{\mathbf{P}} = \sum_{\nu=1}^{2} \int d^3p \, \mathbf{p} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) - \sum_{\nu=1}^{2} \int d^3p \, \mathbf{p} \, \delta(\mathbf{0}) , \tag{9.418}$$

$$\hat{h} = \sum_{\nu=1}^{2} \int d^3p \, \frac{(-1)^{\nu+1}}{2} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) + \sum_{\nu=1}^{2} \frac{(-1)^{\nu+1}}{2} \int d^3p \, \delta(\mathbf{0}) . \tag{9.419}$$

The charge operator $\hat{Q}$ can be renormalised by subtracting its divergency, which amount to going over to the normal ordered charge operator

$$: \hat{Q} : = \hat{Q} - {}_P \langle 0 | \, \hat{Q} \, | 0 \rangle_P = \sum_{\nu=1}^{2} \int d^3p \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) . \tag{9.420}$$

In contrast to that a renormalisation of the momentum operator $\hat{\mathbf{P}}$ is not necessary, since the expectation value of (9.418) with respect to the physical vacuum state $|0\rangle_P$ vanishes due to symmetry reasons in the momentum integral. Thus, the momentum operator (9.418) is already normal ordered:

$$: \hat{\mathbf{P}} : = \hat{\mathbf{P}} = \sum_{\nu=1}^{2} \int d^3p \, \mathbf{p} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} + \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) . \tag{9.421}$$

We conclude that particles carry the charge $+1$ and possess the momentum $\mathbf{p}$, while antiparticles have the negative charge -1 and also possess the momentum $\mathbf{p}$. And, finally, we recognize that also a renormalization of the helicity operator $\hat{h}$ is superfluous as the expectation value of (9.419) with respect to the physical vacuum state $|0\rangle_P$ vanishes due to symmetry reasons in the discrete sum. Thus, the helicity operator (9.419) is already normal ordered:

$$: \hat{h} : = \hat{h} = \sum_{\nu=1}^{2} \int d^3p \, \frac{(-1)^{\nu+1}}{2} \left(\hat{b}_{\mathbf{p}}^{(\nu)\dagger} \hat{b}_{\mathbf{p}}^{(\nu)} - \hat{d}_{\mathbf{p}}^{(\nu)\dagger} \hat{d}_{\mathbf{p}}^{(\nu)} \right) . \tag{9.422}$$

This means that particles with $\nu = 1$ ($\nu = 2$) and antiparticles with $\nu = 2$ ($\nu = 1$) have a positive (negative) helicity.

9.20 Propagator as Green Function

Analogous to the Klein-Gordon or the Maxwell propagator, also the Dirac propagator is defined as the expectation value of the time-ordered product of the field operators $\hat{\psi}_\alpha(\mathbf{x},t)$ and $\hat{\overline{\psi}}_\beta(\mathbf{x}',t')$ with respect to the physical vacuum $|0\rangle_P$:

$$S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = {}_P\left\langle 0 \left| \hat{T}\left(\hat{\psi}_\alpha(\mathbf{x},t)\hat{\overline{\psi}}_\beta(\mathbf{x}',t')\right)\right| 0\right\rangle_P . \tag{9.423}$$

Note that the definition (7.124) of two time-dependent operators $\hat{A}(t)$ and $\hat{B}(t')$ in the context of bosonic operators is not valid for fermionic operators and is given instead by

$$\hat{T}\left(\hat{A}(t)\,\hat{B}(t')\right) = \Theta(t-t')\,\hat{A}(t)\,\hat{B}(t') - \Theta(t'-t)\,\hat{B}(t')\,\hat{A}(t) \tag{9.424}$$

with the Heaviside function (7.125). Note the appearance of the minus in (9.424), which reflects the anti-commutativity of fermionic operators. Due to (9.424) the Dirac propagator (9.423) reads explicitly

$$S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = \Theta(t-t')_P\left\langle 0\left|\hat{\psi}_\alpha(\mathbf{x},t)\hat{\overline{\psi}}_\beta(\mathbf{x}',t')\right|0\right\rangle_P - \Theta(t'-t)_P\left\langle 0\left|\hat{\overline{\psi}}_\beta(\mathbf{x}',t')\hat{\psi}_\alpha(\mathbf{x},t)\right|0\right\rangle_P . \tag{9.425}$$

At first we derive the equation of motion for the Dirac propagator by performing the time derivative of (9.425) and by taking into account (7.128):

$$i\hbar\frac{\partial}{\partial t}S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = i\hbar\delta(t-t')_P\left\langle 0\left|\left[\hat{\psi}_\alpha(\mathbf{x},t),\hat{\overline{\psi}}_\beta(\mathbf{x}',t')\right]_+\right|0\right\rangle_P \tag{9.426}$$
$$+\Theta(t-t')_P\left\langle 0\left|i\hbar\frac{\partial\hat{\psi}_\alpha(\mathbf{x},t)}{\partial t}\hat{\overline{\psi}}_\beta(\mathbf{x}',t')\right|0\right\rangle_P - \Theta(t'-t)_P\left\langle 0\left|\hat{\overline{\psi}}_\beta(\mathbf{x}',t')i\hbar\frac{\partial\hat{\psi}_\alpha(\mathbf{x},t)}{\partial t}\right|0\right\rangle_P .$$

With the definition of the Dirac adjoint Dirac spinor (9.91), the equal-time anti-commutator algebra (9.373), the Heisenberg equation of the Dirac spinor (9.378), and (9.423) we then yield

$$i\hbar\frac{\partial}{\partial t}S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = \sum_{\gamma=1}^{4}\left(-i\hbar c\,\boldsymbol{\alpha}_{\alpha\gamma}\boldsymbol{\nabla} + Mc^2\beta_{\alpha\gamma}\right)S_{\gamma\beta}(\mathbf{x},t;\mathbf{x}',t') + i\hbar\gamma^0_{\alpha\beta}\delta(t-t')\delta(\mathbf{x}-\mathbf{x}') . \tag{9.427}$$

Thus, the Dirac propagator is just the Green function of the Dirac equation, which follows from (9.87), (9.247), and (9.248). Multiplying (9.427) from the left by γ^0/c and taking into account (9.249), (9.250) the equation of motion of the Dirac propagator can also be rewritten in a manifestly covariant form:

$$\left(i\hbar\gamma^\mu\partial_\mu - Mc\right)S(x;x') = i\hbar\delta(x-x') . \tag{9.428}$$

9.21 Propagator Calculation

In order to derive a Fourier representation for the Dirac propagator, we must first transfer the Dirac reinterpretation for the creation and annihilation operators $\hat{a}_{\mathbf{p}}^{(\nu)\dagger}$ and $\hat{a}_{\mathbf{p}}^{(\nu)}$ to the plane wave expansions (9.380) and (9.381) of the field operator $\hat{\psi}(\mathbf{x},t)$ and its adjoint $\hat{\psi}^\dagger(\mathbf{x},t)$. To this end we introduce the following notation for the plane waves of the particles

$$u_{\mathbf{p}}^{(1)}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(1)}(\mathbf{x},t)\,, \qquad u_{\mathbf{p}}^{(2)}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(2)}(\mathbf{x},t) \tag{9.429}$$

and, correspondingly, for the antiparticles

$$v_{\mathbf{p}}^{(1)}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(3)}(\mathbf{x},t)\,, \qquad v_{\mathbf{p}}^{(2)}(\mathbf{x},t) = \psi_{\mathbf{p}}^{(4)}(\mathbf{x},t)\,. \tag{9.430}$$

Taking into account (9.401)–(9.404) the expansions (9.380), (9.381) then merge into

$$\hat{\psi}(\mathbf{x},t) = \sum_{\nu=1}^{2} \int d^3p \left\{ u_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)\hat{b}_{\mathbf{p}}^{(\nu)} + v_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)\hat{d}_{\mathbf{p}}^{(\nu)\dagger} \right\}\,, \tag{9.431}$$

$$\hat{\overline{\psi}}(\mathbf{x},t) = \sum_{\nu=1}^{2} \int d^3p \left\{ \overline{u}_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)\hat{b}_{\mathbf{p}}^{(\nu)\dagger} + \overline{v}_{\mathbf{p}}^{(\nu)}(\mathbf{x},t)\hat{d}_{\mathbf{p}}^{(\nu)} \right\}\,. \tag{9.432}$$

Now we can insert (9.431) and (9.432) into (9.425). As the annihilation operators $\hat{b}_{\mathbf{p}}^{(\nu)}$, $\hat{d}_{\mathbf{p}}^{(\nu)}$ annul the ket vacuum state $|0\rangle_P$ according to (9.408), (9.409) and, correspondingly, the creation operators $\hat{b}_{\mathbf{p}}^{(\nu)\dagger}$, $\hat{d}_{\mathbf{p}}^{(\nu)\dagger}$ annul the bra vacuum state $_P\langle 0|$, we get

$$\begin{aligned} S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') \;&=\; \Theta(t-t')\sum_{\nu=1}^{2}\sum_{\nu'=1}^{2}\int d^3p \int d^3p'\, u_{\mathbf{p}\alpha}^{(\nu)}(\mathbf{x},t)\overline{u}_{\mathbf{p}'\beta}^{(\nu')}(\mathbf{x}',t')_P\langle 0|\,\hat{b}_{\mathbf{p}}^{(\nu)}\hat{b}_{\mathbf{p}'}^{(\nu')\dagger}\,|0\rangle_P \\ &\quad -\Theta(t'-t)\sum_{\nu=1}^{2}\sum_{\nu'=1}^{2}\int d^3p \int d^3p'\, \overline{v}_{\mathbf{p}'\beta}^{(\nu')}(\mathbf{x}',t')v_{\mathbf{p}\alpha}^{(\nu)}(\mathbf{x},t)_P\langle 0|\,\hat{d}_{\mathbf{p}'}^{(\nu')}\hat{d}_{\mathbf{p}}^{(\nu)\dagger}\,|0\rangle_P\,. \end{aligned} \tag{9.433}$$

Due to the anti-commutator algebra (9.407) this reduces to

$$S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = \sum_{\nu=1}^{2}\int d^3p\,\left\{\Theta(t-t')u_{\mathbf{p}\alpha}^{(\nu)}(\mathbf{x},t)\overline{u}_{\mathbf{p}\beta}^{(\nu)}(\mathbf{x}',t') - \Theta(t'-t)v_{\mathbf{p}\alpha}^{(\nu)}(\mathbf{x},t)\overline{v}_{\mathbf{p}\beta}^{(\nu)}(\mathbf{x}',t')\right\} \tag{9.434}$$

Inserting the plane waves (9.324) into (9.434) and considering (9.429) as well as (9.430) one obtains for the Fourier representation of the Dirac propagator

$$\begin{aligned} S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') &= \int d^3p\,\frac{Mc^2}{(2\pi\hbar)^3E_{\mathbf{p}}} \\ \times &\left\{\Theta(t-t')e^{-i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]}P_{\alpha\beta}^{u}(p) - \Theta(t'-t)e^{+i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]}P_{\alpha\beta}^{v}(p)\right\}\,, \end{aligned} \tag{9.435}$$

where the following polarisation sums for both particles and antiparticles are introduced:

$$P_{\alpha\beta}^{u}(p) \;=\; \sum_{\nu=1}^{2} u_{\mathbf{p}\alpha}^{(\nu)}\,\overline{u}_{\mathbf{p}\beta}^{(\nu)} = \sum_{\nu=1}^{2}\psi_{\mathbf{p}\alpha}^{(\nu)}\,\overline{\psi}_{\mathbf{p}\beta}^{(\nu)}\,, \tag{9.436}$$

$$P_{\alpha\beta}^{v}(p) \;=\; \sum_{\nu=1}^{2} v_{\mathbf{p}\alpha}^{(\nu)}\,\overline{v}_{\mathbf{p}\beta}^{(\nu)} = \sum_{\nu=3}^{4}\psi_{\mathbf{p}\alpha}^{(\nu)}\,\overline{\psi}_{\mathbf{p}\beta}^{(\nu)}\,. \tag{9.437}$$

In order to evaluate these polarisation sums we have to perform several auxiliary calculations. To this end we start with the Dirac adjoint spinor amplitudes resulting from (9.314) with the help of (9.94) and (9.102):

$$\overline{\psi}_{\mathbf{p}}^{(\nu)} = \psi_{\mathbf{p}}^{(\nu)\dagger}\gamma^0 = \frac{1}{\sqrt{2}}\left(\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}\right) \quad \text{for } \nu = 1, 2\,, \tag{9.438}$$

$$\overline{\psi}_{\mathbf{p}}^{(\nu)} = \psi_{\mathbf{p}}^{(\nu)\dagger}\gamma^0 = \frac{1}{\sqrt{2}}\left(-\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right)\sqrt{\frac{p\sigma}{Mc}}\right) \quad \text{for } \nu = 3, 4\,. \tag{9.439}$$

We also note that the bi-spinors $\chi(\pm 1/2)$ are complete:

$$\sum_{\nu=1}^{2}\chi\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right) = \chi\left(\frac{1}{2}\right)\chi^\dagger\left(\frac{1}{2}\right) + \chi\left(-\frac{1}{2}\right)\chi^\dagger\left(-\frac{1}{2}\right) = I\,. \tag{9.440}$$

In fact, for the quantisation of the spin 1/2 with respect to the direction of the momentum $\mathbf{p}$ we obtain according to (9.351)

$$\begin{aligned}
&\begin{pmatrix} \cos(\frac{\theta}{2})e^{-i\phi/2} \\ \sin(\frac{\theta}{2})e^{+i\phi/2} \end{pmatrix}\left(\cos\left(\frac{\theta}{2}\right)e^{+i\phi/2}, \sin\left(\frac{\theta}{2}\right)e^{-i\phi/2}\right) \\
&+\begin{pmatrix} -\sin(\frac{\theta}{2})e^{-i\phi/2} \\ \cos(\frac{\theta}{2})e^{+i\phi/2} \end{pmatrix}\left(-\sin\left(\frac{\theta}{2}\right)e^{+i\phi/2}, \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I\,.
\end{aligned} \tag{9.441}$$

Furthermore, from the completeness of the bi-spinors $\chi(\pm 1/2)$ we then conclude the completeness of the charge-conjugated bi-spinors $\chi^c(\pm 1/2)$:

$$\begin{aligned}
&\sum_{\nu=1}^{2}\chi^c\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^{c\dagger}\left(\frac{(-1)^{\nu+1}}{2}\right) = \sum_{\nu=1}^{2}c\chi^*\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^T\left(\frac{(-1)^{\nu+1}}{2}\right)c^\dagger \\
&= c\left\{\sum_{\nu=1}^{2}\chi\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\right\}^T c^\dagger = cIc^\dagger = cc^\dagger = I\,.
\end{aligned} \tag{9.442}$$

After these preparations, the polarisation sum of the particles is calculated as follows. At first, we insert (9.313) and (9.438) in (9.436):

$$P^u(p) = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}} \end{pmatrix}\sum_{\nu=1}^{2}\chi\left(\frac{(-1)^{\nu+1}}{2}\right)\chi^\dagger\left(\frac{(-1)^{\nu+1}}{2}\right)\left(\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \sqrt{\frac{p\sigma}{Mc}}\right)\,. \tag{9.443}$$

Due to the completeness relation (9.440) this reduces to

$$P^u(p) = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} \\ \sqrt{\frac{p\tilde{\sigma}}{Mc}} \end{pmatrix}\left(\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \sqrt{\frac{p\sigma}{Mc}}\right) = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{p\sigma\, p\tilde{\sigma}}{(Mc)^2}} & \frac{p\sigma}{Mc} \\ \frac{p\tilde{\sigma}}{Mc} & \sqrt{\frac{p\tilde{\sigma}\, p\sigma}{(Mc)^2}} \end{pmatrix}\,. \tag{9.444}$$

And, finally, using the side calculation (9.302) we yield with the Dirac matrices (9.94)

$$P^u(p) = \frac{1}{2}\left\{\frac{p_\mu}{Mc}\begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} + \begin{pmatrix} I & O \\ O & I \end{pmatrix}\right\} = \frac{p_\mu\gamma^\mu + Mc}{2Mc}\,. \tag{9.445}$$

The polarisation sum of the antiparticles is calculated along similar lines. Inserting (9.314) and (9.439) in (9.437) we get

$$P^v(p) = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}} \end{pmatrix} \sum_{\nu=1}^{2} \chi^c \left(\frac{(-1)^{\nu+1}}{2} \right) \chi^{c\dagger} \left(\frac{(-1)^{\nu+1}}{2} \right) \left(-\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \sqrt{\frac{p\sigma}{Mc}} \right) , \quad (9.446)$$

which reduces according to the completeness relation (9.442)

$$P^v(p) = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{p\sigma}{Mc}} \\ -\sqrt{\frac{p\tilde{\sigma}}{Mc}} \end{pmatrix} \left(-\sqrt{\frac{p\tilde{\sigma}}{Mc}}, \sqrt{\frac{p\sigma}{Mc}} \right) = \frac{1}{2} \begin{pmatrix} -\sqrt{\frac{p\sigma\, p\tilde{\sigma}}{(Mc)^2}} & \frac{p\sigma}{Mc} \\ \frac{p\tilde{\sigma}}{Mc} & -\sqrt{\frac{p\tilde{\sigma}\, p\sigma}{(Mc)^2}} \end{pmatrix} . \quad (9.447)$$

With the side calculation (9.302) and the Dirac matrices (9.94) we finally obtain

$$P^v(p) = \frac{1}{2} \left\{ \frac{p_\mu}{Mc} \begin{pmatrix} O & \sigma^\mu \\ \tilde{\sigma}^\mu & O \end{pmatrix} - \begin{pmatrix} I & O \\ O & I \end{pmatrix} \right\} = \frac{p_\mu \gamma^\mu - Mc}{2Mc} . \quad (9.448)$$

A comparison of (9.445) and (9.448) reveals that there is a simple relationship between the polarisation sums of the particles and the antiparticles:

$$P^v(p) = -P^u(-p) . \quad (9.449)$$

Using (9.449) in (9.435), the minus sign between the polarisation sums of the particles and antiparticles compensates the minus sign, which originally stems from the definition of the time-ordered product of fermionic operators in (9.424), yielding

$$S_{\alpha\beta}(\mathbf{x}, t; \mathbf{x}', t') = \int d^3p \, \frac{Mc^2}{(2\pi\hbar)^3 E_\mathbf{p}} \times \left\{ \Theta(t-t') e^{-i[E_\mathbf{p}(t-t') - \mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} P^u_{\alpha\beta}(p) + \Theta(t'-t) e^{+i[E_\mathbf{p}(t-t') - \mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} P^u_{\alpha\beta}(-p) \right\} . \quad (9.450)$$

It turns out that this form of the Dirac propagator is universally valid for massive particles with arbitrary spin. The respective spin dependencies are hidden in the polarisation sum of the particles. For example, the result (9.450) agrees with the Klein-Gordon propagator (7.139) with the plane waves (7.113) provided that the polarisation sum is identified according to $P^u_{\alpha\beta}(p) = 1$.

9.22 Four-Dimensional Fourier Representation

Substituting the explicit form of the polarisation sum of the particles (9.445) into (9.450), one obtains

$$S_{\alpha\beta}(\mathbf{x}, t; \mathbf{x}', t') = \int d^3p \, \frac{Mc^2}{(2\pi\hbar)^3 E_\mathbf{p}} \left\{ \Theta(t-t') e^{-i[E_\mathbf{p}(t-t') - \mathbf{p}(\mathbf{x}-\mathbf{x}')]/\hbar} \, \frac{p_\mu \gamma^\mu_{\alpha\beta} + Mc\, \delta_{\alpha\beta}}{2Mc} + \Theta(t'-t) e^{+i[E_\mathbf{p}(t-t') - \mathbf{p}(\mathbf{x}-\mathbf{x}')]/\hbar} \, \frac{-p_\mu \gamma^\mu_{\alpha\beta} + Mc\, \delta_{\alpha\beta}}{2Mc} \right\} . \quad (9.451)$$

The four-momentum vector in the polarisation sum of the particles can now be understood as the effect of applying the four-momentum operator on the plane waves, see (6.96):

$$\begin{aligned} S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') &= \int d^3p \, \frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}}} \left\{ \Theta(t-t') \, \frac{i\hbar\partial_\mu \gamma^\mu_{\alpha\beta} + Mc\,\delta_{\alpha\beta}}{2Mc} \, e^{-i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} \right. \\ & \left. +\Theta(t'-t) \, \frac{i\hbar\partial_\mu \gamma^\mu_{\alpha\beta} + Mc\,\delta_{\alpha\beta}}{2Mc} \, e^{+i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} \right\} . \end{aligned} \tag{9.452}$$

As now both terms involve the same differential operator it is suggestive to bring it in front of the momentum integral. Note that this manipulation leads to an additional term due to applying the time derivative upon the Heavside functions. But one can convince oneself that this additional term vanishes due to the odd symmetry of the respective momentum integral. With this we yield

$$\begin{aligned} S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') &= \frac{i\hbar\partial_\mu \gamma^\mu_{\alpha\beta} + Mc\,\delta_{\alpha\beta}}{2Mc} \int d^3p \, \frac{Mc^2}{(2\pi\hbar)^3 E_{\mathbf{p}}} \\ & \times \left\{ \Theta(t-t') e^{-i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} + \Theta(t'-t) e^{+i[E_{\mathbf{p}}(t-t')-\mathbf{p}(\mathbf{x}-\mathbf{x}')/\hbar]} \right\} . \end{aligned} \tag{9.453}$$

The remaining momentum integral just represents the Klein-Gordon propagator as discussed below Eq. (9.450). Thus, the Dirac propagator can be obtained directly from the Klein-Gordon propagator by applying the following differential rule:

$$S_{\alpha\beta}(\mathbf{x},t;\mathbf{x}',t') = \frac{i\hbar\partial_\mu \gamma^\mu_{\alpha\beta} + Mc\,\delta_{\alpha\beta}}{2Mc} \, G(\mathbf{x},t;\mathbf{x}',t') \,. \tag{9.454}$$

Since we have already found a covariant formulation for the Klein-Gordon propagator in Section 7.12, also the Dirac propagator can be formulated covariantly according to (9.454):

$$S(x;x') = \frac{i\hbar\partial_\mu \gamma^\mu + Mc}{2Mc} \, G(x;x') \,. \tag{9.455}$$

Note that (9.455) can be generalized to any massive particles with arbitrary spin according to the remarks below (9.450):

$$S(x;x') = P^u(i\hbar\partial) \, G(x;x') \,. \tag{9.456}$$

Indeed, inserting the explicit form of the polarisation sum of the particles (9.445) for the Dirac theory in (9.456) yields back (9.455). Substituting the four-dimensional Fourier representation of the Klein-Gordon propagator (7.169) into (9.455), we obtain a corresponding four-dimensional Fourier representation of the Dirac propagator:

$$\begin{aligned} S(x;x') &= \frac{i\hbar\partial_\mu \gamma^\mu + Mc}{2Mc} \, i\hbar 2Mc \lim_{\eta\downarrow 0} \int \frac{d^4p}{(2\pi\hbar)^4} \, \frac{1}{p^2 - M^2c^2 + i\eta} \, e^{-ip(x-x')/\hbar} \\ &= i\hbar \lim_{\eta\downarrow 0} \int \frac{d^4p}{(2\pi\hbar)^4} \, \frac{p_\mu \gamma^\mu + Mc}{p^2 - M^2c^2 + i\eta} \, e^{-ip(x-x')/\hbar} . \end{aligned} \tag{9.457}$$

With the help of the Clifford algebra (9.95) of the Dirac matrices, the denominator of (9.457) can be transformed as follows:

$$\begin{aligned} p^2 - M^2c^2 &= p_\mu p_\nu g^{\mu\nu} - M^2c^2 = \frac{1}{2} \, p_\mu p_\nu \left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) - M^2c^2 \\ &= \left(p_\mu\gamma^\mu\right)\left(p_\nu\gamma^\nu\right) - (Mc)^2 = \left(p_\mu\gamma^\mu - Mc\right)\left(p_\nu\gamma^\nu + Mc\right) . \end{aligned} \tag{9.458}$$

In the $\eta \downarrow 0$ limit, the numerator in (9.457) can be cancelled by a factor of the denominator in (9.458). With this the Dirac propagator has the following compact four-dimensional Fourier representation:

$$S(x;x') = \lim_{\eta \downarrow 0} \int \frac{d^4p}{(2\pi\hbar)^4} \frac{i\hbar}{p_\mu \gamma^\mu - Mc + i\eta} \, e^{-ip(x-x')/\hbar} \,. \tag{9.459}$$

In this form, the Dirac propagator obviously satisfies the equation of motion (9.428):

$$(i\hbar\gamma^\mu \partial_\mu - Mc)\, S(x;x') \quad = \quad i\hbar \int \frac{d^4p}{(2\pi\hbar)^4} \, e^{-ip(x-x')/\hbar} = i\hbar\, \delta(x-x') \,. \tag{9.460}$$