scattering problem at basis of Schrödinger $\longrightarrow$ integral equation

$$(\Delta_{\vec x} + k^2)\,\psi(\vec x) = \frac{2\mu}{\hbar^2} V^{(int)}(\vec x)\,\psi(\vec x)\,, \quad E = \frac{\hbar^2 k^2}{2\mu}\,, \quad k = |\vec k|$$

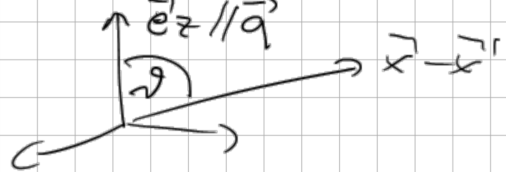
inhomogeneous Helmholtz equation:

$$\boxed{\psi(\vec x) = \underbrace{e^{i\vec k\cdot\vec x}}_{\text{homogeneous solution}} + \underbrace{\int d^3x'\, G(\vec x,\vec x')\, V^{(int)}(\vec x')\,\psi(\vec x')\,\frac{2\mu}{\hbar^2}}_{\text{particular solution}}} \qquad (*)$$

$\hat=$ incoming plane wave $\qquad \hat=$ scattered wave (outgoing spherical wave)

$$G(\vec x,\vec x') \underset{\uparrow}{=} G(\vec x-\vec x') = \int \frac{d^3q}{(2\pi)^3}\, \underbrace{G(\vec q)}_{= \frac{-1}{\vec q^{\,2}-k^2}}\, e^{i\vec q(\vec x-\vec x')}\,, \quad (\Delta_{\vec x}+k^2)\,G(\vec x-\vec x') = \delta(\vec x-\vec x')$$

homogeneity of Helmholtz equation



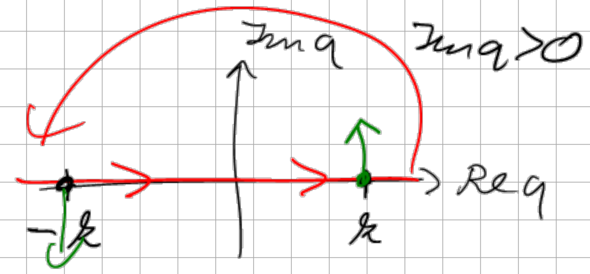
$$G(\vec x-\vec x') = \frac{-1}{8\pi^3}\int_0^\infty dq\, q^2 \int_0^\pi d\vartheta \sin\vartheta\, \frac{1}{q^2-k^2}\, e^{iq|\vec x-\vec x'|\cos\vartheta}$$

$$= \frac{-2\pi}{8\pi^3}\int_0^\infty dq\, q^2 \underbrace{\int_{-1}^{+1} du\, e^{iq|\vec x-\vec x'|u}}_{= \frac{e^{iq|\vec x-\vec x'|} - e^{-iq|\vec x-\vec x'|}}{i|\vec x-\vec x'|\,q}} \frac{1}{q^2-k^2}$$

$\uparrow\ u(\vartheta) = \cos\vartheta$

$$= \frac{1}{|\vec x-\vec x'|}\,\frac{i}{4\pi^2}\left\{\int_0^\infty dq\, \frac{q\, e^{iq|\vec x-\vec x'|}}{q^2-k^2} - \underbrace{\int_0^{-\infty} dq}_{-\int_{-\infty}^{0} dq} \underbrace{\frac{q\, e^{+iq|\vec x-\vec x'|}}{q^2-k^2}}_{q\,\to\,-q}\right\} = \frac{i}{4\pi^2|\vec x-\vec x'|}\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} dq\, \frac{q}{q^2-k^2-i\varepsilon}\, e^{iq|\vec x-\vec x'|}$$

ambiguity in performing q-integral resolved by the Feynman prescription $\hat{=}$ causality



zero of denominator: $q^2 = k^2 + i\varepsilon = \sqrt{k^4+\varepsilon^2}\, e^{i \arctan\frac{\varepsilon}{k^2}}$

square root $\hookrightarrow$ $\underline{q_\pm} = \pm \underbrace{\sqrt[4]{k^4+\varepsilon^2}}\, e^{\frac{i}{2}\arctan\frac{\varepsilon}{k^2}}$

$$= \pm k\left(1+\frac{1}{4}\cancel{\frac{\varepsilon^2}{k^4}}\right)\left(1+\frac{i}{2}\frac{\varepsilon}{k^2}\right) = \pm k \underline{\pm} \frac{i}{2}\frac{\varepsilon}{k} \quad (\varepsilon > 0)$$

$$\left| e^{iq|\vec{x}-\vec{x}'|} \right| = e^{-\underbrace{\mathrm{Im}\, q}_{>0}\underbrace{|\vec{x}-\vec{x}'|}_{>0}} \longrightarrow \text{convergence}$$

$$G(\vec{x}-\vec{x}') = \frac{i}{4\pi\cancel{\pi}|\vec{x}-\vec{x}'|}\cancel{2}\cancel{\pi}\, i \underbrace{\operatorname*{Res}_{q=k} \frac{q}{q^2-k^2}\, e^{iq|\vec{x}-\vec{x}'|}}$$

$$= \lim_{q\to k} \cancel{(q-k)}\frac{q}{\cancel{(q-k)}(q+k)}\, e^{iq|\vec{x}-\vec{x}'|}$$

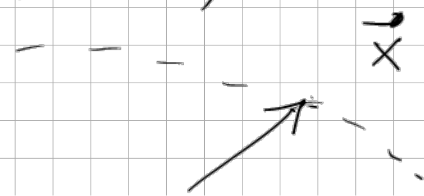
$$= \frac{k}{\cancel{2k}}\, e^{ik|\vec{x}-\vec{x}'|}$$

$$= -\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}$$

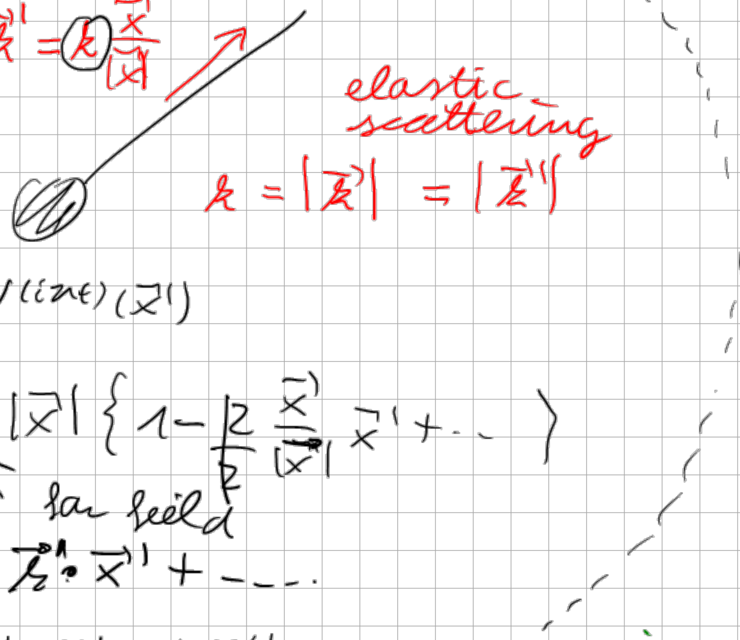
Implication for scattering solution (*):

$$\psi(\vec{x}) = e^{i\vec{k}\vec{x}} - \frac{\mu}{2\pi\hbar^2}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} V^{(int)}(\vec{x}')\,\psi(\vec{x}')$$

How to interpret this in terms of scattering theory



according to sketch:
detector far away
from scattering potential



$\vec{k}' = k \frac{\vec{x}}{|\vec{x}|}$

elastic scattering

$k = |\vec{k}| = |\vec{k}'|$

$V^{(int)}(\vec{x}')$

$$|\vec{x} - \vec{x}'| = \sqrt{(\vec{x} - \vec{x}')^2}$$
$$= \sqrt{\vec{x}^2 - 2\vec{x}\cdot\vec{x}' + \vec{x}'^2} = |\vec{x}| \sqrt{1 - 2\frac{\vec{x}}{|\vec{x}|^2}\vec{x}' + \frac{\vec{x}'^2}{|\vec{x}|^2}} \approx |\vec{x}|\left\{1 - \frac{2}{2}\frac{\vec{x}}{|\vec{x}|^2}\vec{x}' + \dots\right\}$$

far field

$$k|\vec{x} - \vec{x}'| = kr - \underbrace{k\frac{\vec{x}}{|\vec{x}|}}_{=\vec{k}'}\cdot\vec{x}' + \dots =$$
$$= kr - \vec{k}'\cdot\vec{x}' + \dots$$

$$\psi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} - \frac{\mu}{2\pi\hbar^2}\int d^3x' \frac{e^{ikr}e^{-i\vec{k}'\cdot\vec{x}'}}{r} V^{(int)}(\vec{x}')\,\psi(\vec{x}')$$
$$= e^{i\vec{k}\vec{x}} + \frac{e^{ikr}}{r} f(\vartheta, \varphi)$$

$$f(\vec{k}') = \int d^3x' e^{-i\vec{k}'\vec{x}'} V^{(int)}(\vec{x}')\,\boxed{\psi(\vec{x}')}$$

not yet known

incoming plane wave

not yet normalized

outgoing spherical wave

scattering amplitude

$\Rightarrow [f] = 1\,m$

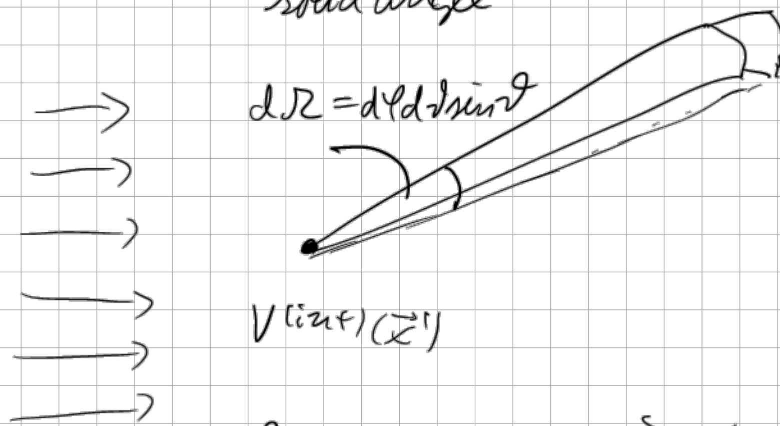
$$\vec{k}' = k\begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}$$

7.8 Scattering Cross-Section:

differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{\text{count scattered particles per solid angle } d\Omega \text{ per time}}{\text{number of incoming particles per time and area}} = \frac{dN(\Omega)}{d\Omega\, dt\, j_{in}}$$

$$\left[\frac{d\sigma}{d\Omega}\right] = 1\,m^2$$



solid angle

$d\Omega = d\varphi\, d\vartheta \sin\vartheta$

$V^{(int)}(\vec{x}')$

current of particles coming in

Schrödinger equation: $\hbar i \frac{\partial}{\partial t}\psi(\vec{x},t) = \left\{-\frac{\hbar^2}{2\mu}\Delta + V^{(int)}(\vec{x})\right\}\psi(\vec{x},t)$

continuity equation: $\frac{\partial}{\partial t}\underbrace{\rho}_{=\psi(\vec{x},t)\psi^*(\vec{x},t)} + \mathrm{div}\,\underbrace{\vec{j}(\vec{x},t)}_{=\frac{\hbar}{2\mu i}\left\{\psi^*(\vec{x},t)\vec{\nabla}\psi(\vec{x},t) - \psi(\vec{x},t)\vec{\nabla}\psi^*(\vec{x},t)\right\}} = 0$

$\psi(\vec{x}) = e^{i\vec{k}\vec{x}}$ $\qquad \vec{\nabla}\psi(\vec{x}) = i\vec{k}\, e^{i\vec{k}\vec{x}}$

$\vec{j}(\vec{x}) = \frac{\hbar}{2\mu i}\left\{\cancel{e^{-i\vec{k}\vec{x}}}\, i\vec{k}\, \cancel{e^{i\vec{k}\vec{x}}} - \cancel{e^{i\vec{k}\vec{x}}}(-i)\vec{k}\,\cancel{e^{-i\vec{k}\vec{x}}}\right\} = \frac{\hbar\vec{k}}{\mu}$

$$j_{in} = |\vec{j}| = \frac{\hbar k}{\mu}$$

$$\psi(\vec{x}) = \frac{e^{ikr}}{r} f(\vartheta,\varphi)$$

$$j_{out} = j_{radial} = \frac{\hbar}{2\mu i}\left\{\psi^*\frac{\partial}{\partial r}\psi - \psi\frac{\partial}{\partial r}\psi^*\right\}$$

$$\frac{\partial\psi}{\partial r} = \left(-\frac{1}{r^2} + \frac{ik}{r}\right)e^{ikr} f$$

$$j_{out} = \frac{\hbar}{2\mu i}2\left\{\frac{e^{-ikr}}{r}f^*\left(-\cancel{\frac{1}{r^2}} + \frac{ik}{r}\right)e^{ikr} f - \cancel{f e^{ikr}\left(-\frac{1}{r^2} - \frac{ik}{r}\right)e^{-ikr}f^*}\right\}$$

$$= \frac{\hbar}{\mu i}\frac{ik}{r^2}|f|^2 = \frac{\hbar k}{\mu r^2}|f|^2$$

$$dN = j_{out}\, dt\, r^2\, d\Omega = \frac{\hbar k}{\mu}|f|^2\, d\Omega\, dt$$

$$\frac{d\sigma}{d\Omega} = \frac{\frac{\hbar k}{\mu}|f|^2 d\Omega\, dt}{d\Omega\, dt\, \frac{\hbar k}{\mu}} = |f|^2, \quad \left[\frac{d\sigma}{d\Omega}\right] = m^2$$

generic unit

$1\ \underbrace{\text{barn}}_{\text{"Scheune"}} = 10^{-24}\,\text{cm}^2$

$\boxed{\sigma = \int d\Omega\left(\frac{d\sigma}{d\Omega}\right)}$ total cross-section

⇒ General structure is here available

How to determine $\psi(\vec{x})$?

- Exact solution for $\psi(\vec{x})$ available, e.g. Fermi pseudopotential Problem 17, Problem set 16 ⇒ $f(\vartheta, \varphi) = f(\vartheta, \varphi; k)$ $\qquad E = \frac{\hbar^2 k^2}{2\mu}$, $\quad k = \sqrt{\frac{2E\mu}{\hbar^2}}$
- Born approximation: perturbation of first order

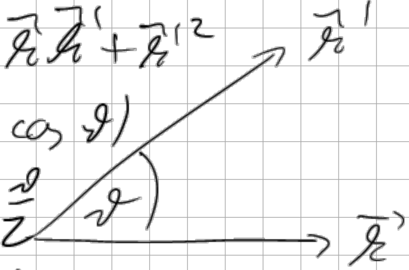
7.9 Born approximation:

$$\psi^{(n+1)}(\vec{x}) = e^{i\vec{k}\vec{x}} \underbrace{- \frac{\mu}{2\pi\hbar^2} \int d^3x' \, e^{-i\vec{k}'\vec{x}'} V^{(int)}(\vec{x}') \psi^{(n)}(\vec{x}')}_{= f(\vartheta, \varphi)} \frac{e^{ikr}}{r}$$

↑ far field result

$$f(\vartheta, \varphi) = \left(-\frac{\mu}{2\pi\hbar^2}\right) \underbrace{\int d^3x' \underbrace{e^{-i\vec{k}'\vec{x}'} V^{(int)}(\vec{x}') e^{i\vec{k}\vec{x}'}}_{= e^{i(\vec{k}-\vec{k}')\vec{x}'} V^{(int)}(\vec{x}')}}_{V^{(int)}(\vec{k}-\vec{k}')}$$

$$(\vec{k}-\vec{k}')^2 = k^2 - 2\vec{k}\vec{k}' + \vec{k}'^2 = 2k^2(1-\cos\vartheta) = 4k^2 \sin^2\frac{\vartheta}{2} \qquad \vec{k}'^2 = k^2$$



result: In Born approximation the scattering amplitude is given by the Fourier transformed of interaction potential

$$V^{(int)}(\vec{x}) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |\vec{x}|} \qquad V^{(int)}(\vec{k}) = \frac{Q_1 Q_2}{\cancel{4\pi}\varepsilon_0} \frac{\cancel{4\pi}}{k^2}$$

$$f(\vartheta, \varphi) = -\frac{\mu}{2\pi\hbar^2} \frac{Q_1 Q_2}{\varepsilon_0} \frac{1}{(\vec{k}-\vec{k}')^2}$$

↑ repulsive

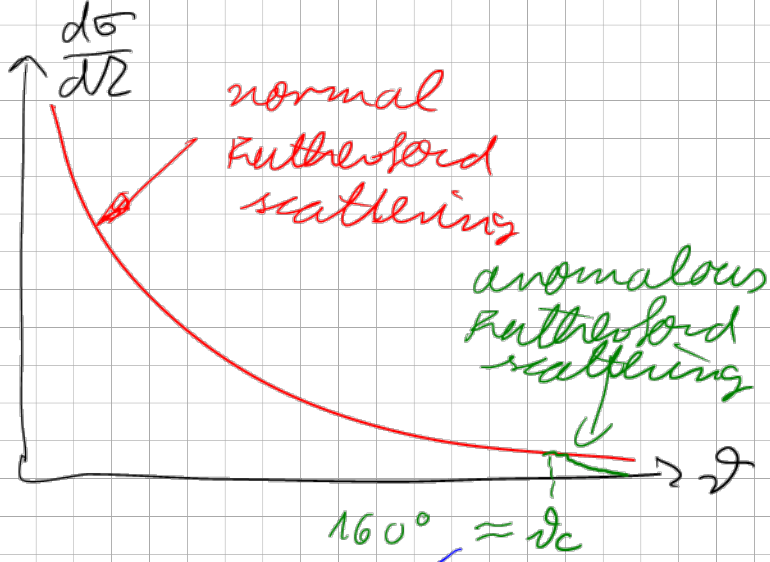
Rutherford scattering
α-particles, gold
$Q_1 = 2e > 0$, gold: $Q_2 = \underbrace{Z}_{=79} e$

$$f(\vartheta,\varphi) = -\frac{\cancel{\mu}}{\cancel{2\pi\hbar^2}}\,\frac{Q_1 Q_2}{\varepsilon_0}\cdot\underbrace{\frac{1}{4\,k^2 \sin^2\vartheta/2}}_{\frac{1}{4\cdot\frac{2\mu}{\cancel{\hbar^2}} E \sin^2\vartheta/2}} \quad , \quad E = \frac{\hbar^2 k^2}{2\mu} \Rightarrow k^2 = \frac{2\mu}{\hbar^2}E$$

$\hbar$ gone
≙ classical

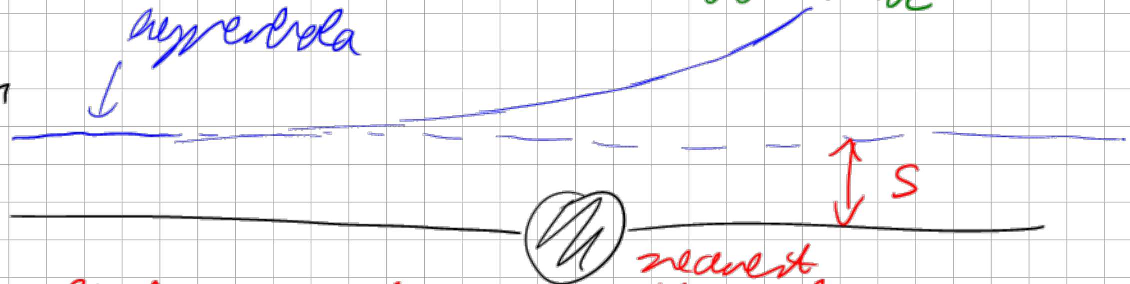
$$= -\frac{Q_1 Q_2}{16\pi\varepsilon_0}\,\frac{1}{E\sin^2\vartheta/2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Q_1 Q_2}{16\pi\varepsilon_0 E}\right)^2\cdot\frac{1}{\sin^4\vartheta/2}$$



$$s_c = \frac{Q_1 Q_2 e^2}{8\pi\varepsilon_0 E_c}\,\mathrm{ctg}\,\frac{\vartheta_c}{2} = 5\cdot 10^{-15}\,\mathrm{fm}$$

$$1\,\mathrm{fm} = 10^{-15}\,\mathrm{fm}$$



Stoßparameter = length scale
impact parameter

Note: $\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega}$ diverges for Rutherford scattering
≙ reflects long-range character of Coulomb potential

7.10 Partial Waves:

Now: specialize to $V^{(int)}(\vec{x}) = v^{(int)}(|\vec{x}|)$

Plane Wave:

divergent at origin

$$\underbrace{e^{i\vec{k}\cdot\vec{x}}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left\{A_{\ell m}\, j_\ell(kr) + \cancel{B_{\ell m}\, n_\ell(kr)}\right\} Y_{\ell m}(\vartheta,\varphi) \qquad \vec{k} = k\,\vec{e}_z$$

homogeneous solution of Helmholtz equation

$$e^{i\vec{k}\vec{x}} = e^{i\underbrace{kz}_{=kr\cos\vartheta}} = \sum_{\ell=0}^{\infty} A_{\ell 0}\, j_\ell(kr)\, \underbrace{Y_{\ell 0}(\vartheta,\varphi)}_{=\sqrt{\frac{2\ell+1}{4\pi}}\,P_\ell(\cos\vartheta)} \qquad \text{cylinder-symmetry}$$

$$P_\ell(z) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dz^\ell}(z^2-1)^\ell$$

$$\int_{-1}^{+1} dz\, P_\ell(z)\, P_{\ell'}(z) = \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

$\ell = 0$ s-wave

$= 1$ p-wave

$= 2$ d-wave

$\vdots$

$$\boxed{e^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty} \underbrace{A_\ell}_{=?}\sqrt{\frac{2\ell+1}{4\pi}}\, j_\ell(kr)\, P_\ell(\cos\vartheta)}$$

valid for all r

$$\Big|\ \int_0^\pi d\vartheta \sin\vartheta\, P_{\ell'}(\cos\vartheta)$$

$$\int_{-1}^{+1} \underset{\uparrow\, z=\cos\vartheta}{dz}\, P_{\ell'}(z)\, e^{ikr\underbrace{\cos\vartheta}_{=z}} = \sum_{\ell=0}^{\infty} A_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, j_\ell(kr)\, \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

$$A_\ell\, j_\ell(kr) = \sqrt{\pi(2\ell+1)} \int_{-1}^{+1} dz\, P_\ell(z)\, e^{ikrz}$$

work out $\underline{r \downarrow 0}$:

$$A_\ell \frac{(kr)^\ell}{(2\ell+1)!!} = \sqrt{\pi(2\ell+1)} \int_{-1}^{+1} dz\, P_\ell(z) \left\{ \cancel{\ldots + \frac{(ikrz)^{\ell-1}}{(\ell-1)!}} + \boxed{\frac{(ikrz)^\ell}{\ell!}} + \underbrace{\frac{(ikrz)^{\ell+1}}{(\ell+1)!} + \ldots}_{\text{do not contribute for } r\to 0} \right.$$

$$z^\ell = \underline{\frac{2^\ell (\ell!)^2}{(2\ell)!} \cdot P_\ell(z)} + \underline{\sum_{m=0}^{\ell-1} c_m P_m(z)}$$

do not contribute due orthogonality

$$\Rightarrow \quad A_\ell = i^\ell \sqrt{4\pi(2\ell+1)}$$

[Integral representation for spherical Bessel function

$$j_\ell(x) = \frac{(-i)^\ell}{2} \int_{-1}^{+1} dz\, P_\ell(z)\, e^{ixz}$$]

Intermediate result: partial wave decomposition of plane wave

$$\boxed{e^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty} i^\ell \boxed{(2\ell+1)} j_\ell(kr) \, P_\ell(\cos\vartheta)}$$

Outgoing spherical wave:

$$f(\vartheta,\varphi) = f(\vartheta) = \sum_{\ell=0}^{\infty} \boxed{(2\ell+1)} \underbrace{f_\ell}_{\text{partial wave amplitude of scattering amplitude}} P_\ell(\cos\vartheta) \qquad \left(e^{i\frac{\pi}{2}}\right)^\ell = i^\ell$$

cylinder symmetry

Scattering solution

$$\psi(\vec{x}) = e^{ikr\cos\vartheta} + \frac{e^{ikr}}{r} f(\vartheta) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\vartheta) \left\{ i^\ell \underline{j_\ell(kr)} + f_\ell \frac{e^{ikr}}{r} \right\}$$

large distances: $j_\ell(\rho) = \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right) = \frac{1}{\rho}\frac{1}{2i}\left(e^{i(\rho-\frac{\ell\pi}{2})} - e^{-i(\rho-\frac{\ell\pi}{2})}\right)$

$$= \frac{1}{2i\rho}\left\{(-i)^\ell e^{i\rho} - e^{-i\rho}\, i^\ell\right\}$$

$$\psi(\vec{x}) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)P_\ell(\cos\vartheta)}{2ikr}\left\{ \underbrace{e^{ikr}}_{\text{outgoing spherical wave}} \Big[\underset{\text{plane wave}}{1} + \underset{\text{scattering}}{2ikf_\ell}\Big] - \underset{\text{plane wave}}{(-1)^\ell}\, \underbrace{e^{-ikr}}_{\text{incoming spherical wave}} \right\}$$

far field solution of Schrödinger equation ($V(r)=0$)

$$\psi(\vec{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} C_{\ell m}\, R_\ell(r)\, Y_{\ell m}(\vartheta,\varphi)$$

cylinder symmetry

$$= \sum_{\ell=0}^{\infty} c_\ell\, \underbrace{R_\ell(r)}_{\sim\ \frac{\sin(kr-\ell\pi/2+\delta_\ell)}{kr}}\, P_\ell(\cos\vartheta)$$

$$\sim\ \frac{\sin(kr-\ell\pi/2+\delta_\ell)}{kr} \qquad r\to\infty$$

previously

$$= \sum_{\ell=0}^{\infty} \frac{C_\ell\, P_\ell(\cos\vartheta)}{2ikr}\left\{(-i)^\ell e^{i\delta_\ell} e^{ikr} - i^\ell e^{-i\delta_\ell} e^{-ikr}\right\}$$

$c_\ell = (2\ell+1)\, i^\ell\, e^{i\delta_\ell}$: incoming spherical wave

$$\boxed{f_l = \frac{e^{i\delta_l}\sin\delta_l}{k}}$$: outgoing spherical wave

$$f(\vartheta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\vartheta)$$

scattering amplitude expressed in terms of phase shifts

$$\left(\frac{d\sigma}{d\Omega}\right)(\vartheta) = |f(\vartheta)|^2 = \frac{1}{k^2}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}(2l+1)(2l'+1)\,\underline{e^{i(\delta_l-\delta_{l'})}}\sin\delta_l\sin\delta_{l'}\,\underline{P_l(\cos\vartheta)}\,\underline{P_{l'}(\cos\vartheta)}$$

$$\sigma = \int d\Omega\left(\frac{d\sigma}{d\Omega}\right)(\vartheta) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

orthogonality of P_l

Example: hard spheres

$$\delta_l = -\frac{(2l+1)}{[(2l+1)!!]^2}(ka)^{2l+1}$$

^{87}Rb : $a \simeq 100\, a_B$, $a_B = 0.529\cdot 10^{-10}\,m$, $\mu = 87\cdot u$, $u \approx 1.67\cdot 10^{-27}\,kg$

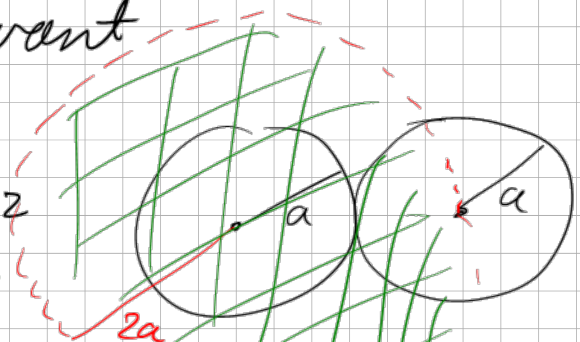
$T \simeq 300\,nK$ (ultracold)

$$E = \frac{\hbar^2 k^2}{2\mu} = k_B T \Rightarrow k = \sqrt{\frac{2\mu k_B T}{\hbar^2}}$$

$$\Rightarrow ka = \sqrt{\frac{2\mu k_B T}{\hbar^2}}\,a = 0.097 \ll 1$$ $\Rightarrow$ s-wave scattering with $l=0$ is relevant

$$\Rightarrow \delta_l = \delta_{l0}(-1)ka$$

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\underbrace{(\sin ka)^2}_{\approx (ka)^2} = 4\pi a^2 = \underline{\pi(2a)^2}$$



7.11 Indistinguishable Particles:

So far: particles, which scatter, were assumed to be distinguishable

now: indistinguishable $\Rightarrow$ same properties $m_1 = m_2 = m \Rightarrow \mu = \frac{m}{2}$

$$\Rightarrow g \underset{\substack{\uparrow \\ \text{hard spheres}}}{=} \frac{2\pi\hbar^2}{\mu} a = \frac{4\pi\hbar^2}{m} a$$

Indistinguishable particles: bosons and fermions <u>in 3D</u>

symmetric wave function (bosons) $\rightarrow$ <u>integer spin</u>

antisymmetric wave function (fermions) $\rightarrow$ <u>half-integer spin</u>

in 2D: anyons $\hat{=}$ exchange of two anyons in a many-anyon wave function yields $e^{i\Theta}$

bosons: $\Theta = 0$ $\quad\quad$ $\Theta = \pi$: fermions

---

scattering for bosons ($\varepsilon = +1$) and fermions ($\varepsilon = -1$)

$$\psi^{\varepsilon}(\vec{x}) = \frac{1}{\sqrt{2}} \left\{ \psi(\vec{x}) + \varepsilon\, \psi(-\vec{x}) \right\}$$

$\vec{x} = \vec{x}_1 - \vec{x}_2$

$\varepsilon = +1$: symmetrization

$\varepsilon = -1$: antisymmetrization

$$\psi^{\varepsilon}(\vec{x}) \underset{\text{problem}}{\overset{\text{scattering}}{=}} \frac{1}{\sqrt{2}} \left\{ e^{ikz} + \varepsilon\, e^{-ikz} \right\} + \frac{e^{ikr}}{r} \underline{f^{\varepsilon}(\vartheta)}$$

$$\vec{x} = r \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \Longleftrightarrow \quad -\vec{x} \; ;$$

$$\varphi \longrightarrow \varphi + \pi$$

$$\vartheta \longrightarrow \pi - \vartheta$$

cylinder symmetry

$$f^{\varepsilon}(\vartheta) = \frac{1}{\sqrt{2}} \left\{ f(\vartheta) + \varepsilon\, f(\pi - \vartheta) \right\}$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{\varepsilon}(\vartheta) = \frac{1}{2} \left| \underline{f(\vartheta)} + \varepsilon\, \underline{f(\pi - \vartheta)} \right|^2$$

partial wave decomposition for $\underline{f(\vartheta)}$

$$P_\ell(\cos\vartheta) \longrightarrow P_\ell(\cos(\pi - \vartheta)) = \underline{(-1)^\ell}\, P_\ell(\cos\vartheta)$$

$$P_\ell(z) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dz^\ell} (z^2 \cancel{-1})^\ell$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{\varepsilon}(\vartheta) = \frac{1}{2 k^2} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (2\ell+1)(2\ell'+1)\, e^{i(\delta_\ell - \delta_{\ell'})} \sin\delta_\ell \sin\delta_{\ell'}$$

$$\underline{\left[1 + \varepsilon(-1)^\ell\right]\left[1 + \varepsilon(-1)^{\ell'}\right]}\, P_\ell(\cos\vartheta)\, P_{\ell'}(\cos\vartheta)$$

$$\sigma^{\varepsilon} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)(\vartheta) = \frac{2\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \underbrace{\sin^2\delta_\ell}_{(ka)^2} \left[1 + \varepsilon(-1)^\ell\right]^2$$

$$\ell = 0: \quad \sigma^{\varepsilon} = \underbrace{4\pi a^2}_{\text{distinguishable particles}} \underbrace{\frac{(1+\varepsilon)^2}{2}}_{\text{statistical correction factor}}$$

$\varepsilon = +1$: 2

$\varepsilon = -1$: 0

$$\left\{ \begin{array}{l} \sigma^{+} = 8\pi a^2 \\ \sigma^{-} = 0 \end{array} \right.$$

Pauli principle