Current status:

Path integral

- discrete time lattice: problem set 9

  $\rightarrow$ difference equations $\hat{=}$ differential equations

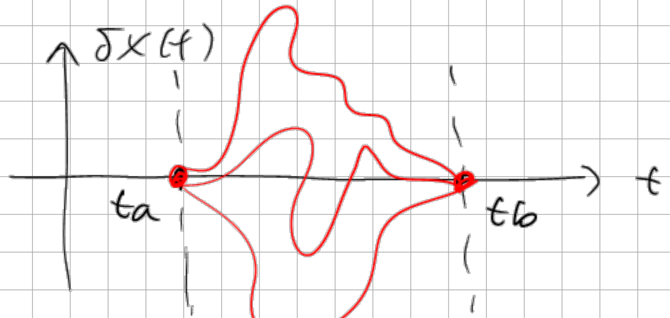
- continuum time: example of harmonic oscillator

$$G(x_b,t_b;x_a,t_a) = \underbrace{\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x}_{\text{summation over all possible paths}} \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{m}{2}\dot{x}^2(t) - \underbrace{V(x(t))}_{\Rightarrow\ \frac{m}{2}\omega^2 x^2(t)}\right]\right\}$$

Ansatz: $\boxed{x(t) = \underbrace{x_{cl}(t)}_{\text{fixed classical path: } x_{cl}(t_a)=x_a,\ x_{cl}(t_b)=x_b} + \delta x(t)}$

$$\mathcal{D}x = \mathcal{D}\,\delta x$$

$$\underbrace{\qquad\qquad}_{= S[x(\cdot)]}$$



partial integral $\curvearrowright$ $= \int_{t_a}^{t_b} dt\, \delta x(t) \underbrace{\hat{O}(t)}_{=-\frac{m}{2}\left[\frac{d^2}{dt^2}+\omega^2\right]} \delta x(t)$

$\delta x(t_a) = 0 = \delta x(t_b)$ (temporal Dirichlet boundary condition)

$$G(x_b,t_b;x_a,t_a) = \boxed{e^{\frac{i}{\hbar}\underbrace{S[x_{cl}(\cdot)]}_{=S_{cl}}}}_{=G_1(x_b,t_b;x_a,t_a)} \cdot \underbrace{\int_{\delta x(t_a)=0}^{\delta x(t_b)=0} \mathcal{D}\,\delta x\; e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left\{\frac{m}{2}\delta\dot{x}^2(t) - \frac{m}{2}\omega^2 \delta x^2(t)\right\}}}_{= G_2(x_b,t_b;x_a,t_a)}$$

How to sum practically over all fluctuations $\delta x(t)$?

Idea: basis for the function space encompassing all fluctuations

eigenvalue problem $\hat{=}$ particle in a box of "length" $t_b - t_a$

$$\begin{cases} \hat{O}(t)\, u_n(t) = \lambda_n u_n(t) & \text{differential equation} \\ u_n(t_a) = 0 = u_n(t_b) & \text{boundary conditions} \end{cases}$$

$$\boxed{u_n(t) = N_n \sin\left[\frac{n\pi (t-t_a)}{t_b - t_a}\right] \; ; \quad n = 1, 2, 3, \ldots}$$

involves no ω

$$\Rightarrow \lambda_n = \frac{M}{2}\left[\left(\frac{n\pi}{t_b - t_a}\right)^2 - \underline{\underline{\omega^2}}\right]$$

$$\int_{t_a}^{t_b} dt\, u_n(t)\, u_{n'}(t) = \delta_{nn'} \quad \Rightarrow \quad N_n = \sqrt{\frac{2}{t_b - t_a}}$$

Continue Section 9.4 Fluctuation Propagator:

$$\delta x(t) = \sum_{n=1}^{\infty} c_n \underbrace{u_n(t)}_{\text{fixed}} \quad \Rightarrow \quad \int_{\delta x(t_a)=0}^{\delta x(t_b)=0} \mathcal{D}\delta x = \underbrace{J}_{\text{= Jacobian determinant}} \left\{ \prod_{n=1}^{\infty} \int_{-\infty}^{+\infty} dc_n \right\}$$

not dependent on ω

not well defined, turns out to be infinite

Idea: Jacobian later on fixed due to $\frac{\partial}{\partial \omega} J = 0$

$$A[\delta x] = \int_{t_a}^{t_b} dt\, \underline{\delta x(t)}\, \hat{O}(t)\, \underline{\delta x(t)} = \sum_n \sum_{n'} \underbrace{c_n c_{n'} \int_{t_a}^{t_b} u_n(t) \underbrace{\hat{O}(t)\, u_{n'}(t)}_{= \lambda_{n'} u_{n'}(t)}\, dt}_{= \lambda_{n'} \delta_{nn'}}$$

$$= \sum_n \lambda_n c_n^2$$

$$G_2(x_b, t_b; x_a, t_a) = J \left\{ \prod_{n=1}^{\infty} \int_{-\infty}^{+\infty} dc_n \right\} e^{-\frac{1}{\hbar i} \sum_n \lambda_n c_n^2} \underset{\text{Fresnel}}{=} J \prod_{n=1}^{\infty} \sqrt{\frac{i\pi\hbar}{\lambda_n}}$$

$$= \mathcal{J} \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i\hbar}{m} \frac{1}{\left(\frac{n\pi}{t_b-t_a}\right)^2 - \omega^2}}$$

9.5 Jacobi Determinant:

Intermediate result obtained so far:

$$G_\omega(x_b,t_b;x_a,t_a) = \underset{\omega}{\mathcal{J}} \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i\hbar}{m} \frac{1}{\left(\frac{n\pi}{t_b-t_a}\right)^2 - \omega^2}}\; \exp\left\{\frac{im\omega}{2\hbar \sin\omega(t_b-t_a)}\left[(x_a^2+x_b^2)\cos\omega(t_b-t_a) - 2x_a x_b\right]\right\}$$

determine now $\mathcal{J}$ from the condition $\frac{\partial}{\partial\omega}\mathcal{J} = 0$

$$\lim_{\omega\downarrow 0} G_\omega(x_b,t_b;x_a,t_a) = \mathcal{J}\cdot \prod_{n=1}^{\infty}\sqrt{\frac{2\pi i\hbar}{m}\frac{1}{[n\pi/(t_b-t_a)]^2}} \cdot \exp\left\{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}\right\}$$

$$\overset{\text{free particle}}{\underset{\text{propagator}}{=}} \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}} \cdot \exp\left\{\frac{im}{2\hbar}\frac{(x_b-x_a)^2}{t_b-t_a}\right\}$$

$$\mathcal{J} = \sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}} \cdot \underbrace{\prod_{n=1}^{\infty}\sqrt{\frac{\left(\frac{n\pi}{t_b-t_a}\right)^2}{\frac{2\pi i\hbar}{m}}}}_{=\,?}$$

evaluated with

$$\boxed{\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = \frac{\sin\pi x}{\pi x}}$$

$\sin(\pi x)$ can be represented by products $(x-x_n)$ with $x_n = n,\ n\in\mathbb{Z}$ as its zeros:

$$\sin\pi x = A\prod_{n=-\infty}^{+\infty}(x-n) = \tilde{A}\overbrace{\prod_{n=-\infty}^{-1}\left(1-\frac{x}{n}\right)}^{=\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)} \times \prod_{n=1}^{\infty}\left(1-\frac{x}{n}\right)$$

$$= \tilde{A}\times\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

$\tilde{A}$ determined from $x \to 0$: $\quad \tilde{A} = \lim_{x\to 0} \frac{\sin \pi x}{x} = \pi$

Result:

$$\boxed{G(x_b,t_b;x_a,t_a) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega(t_b-t_a)}}\; \exp\left\{\frac{i m\omega}{2\hbar \sin\omega(t_b-t_a)}\left[(x_b^2+x_a^2)\cos\omega(t_b-t_a) - 2x_a x_b\right]\right\}}$$

9.6 Density of States:

$$G(x_b,t_b;x_a,t_a) \underset{\uparrow}{=} \sum_n \phi_n^*(x_b)\, e^{-\frac{i}{\hbar}E_n(t_b-t_a)}\, \phi_n(x_a)$$

spectral decomposition

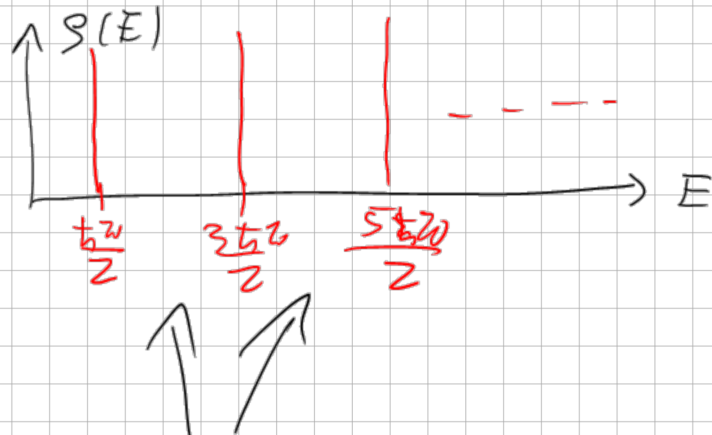
$$\int_{-\infty}^{+\infty} dx\, G(x,t_b;x,t_a) = \sum_n \underbrace{\int_{-\infty}^{+\infty} dx\, \phi_n^*(x)\phi_n(x)}_{=1}\; e^{-\frac{i}{\hbar}E_n(t_b-t_a)} = \sum_n e^{-\frac{i}{\hbar}E_n(t_b-t_a)}$$

$$\boxed{\underset{\uparrow}{\varrho(E)} = \int_{-\infty}^{+\infty} \frac{dt}{2\pi\hbar} \int_{-\infty}^{+\infty} dx\; G(x,t;x,0)\, e^{\frac{i}{\hbar}Et}}$$

density of states

$$= \sum_n \int_{-\infty}^{+\infty} \frac{dt}{2\pi\hbar}\, e^{\frac{i}{\hbar}(E-E_n)t} = \boxed{\sum_n \delta(E-E_n)}$$

$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$



$$\int_{-\infty}^{+\infty} dx\, G(x,t;x,0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega t}}$$

$$\cdot \underbrace{\int_{-\infty}^{+\infty} dx\, \exp\left\{-\frac{m\omega}{i\hbar}\frac{\cos\omega t - 1}{\sin\omega t}x^2\right\}}_{=\sqrt{\frac{i\pi\hbar}{m\omega}\cdot\frac{\sin\omega t}{\cos\omega t - 1}}} = \frac{1}{2i\sin\frac{\omega t}{2}}$$

$$\rho(E) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi\hbar}\, e^{\frac{i}{\hbar}Et}\, \underbrace{\frac{1}{2i\sin\frac{\omega t}{2}}} = \sum_{n=0}^{\infty} \delta\left(E - \hbar\omega\left(n+\tfrac{1}{2}\right)\right)$$

$$= \frac{1}{e^{\frac{i\omega}{2}t}\left(1-e^{-i\omega t}\right)} = e^{-\frac{i}{2}\omega t} \sum_{n=0}^{\infty} e^{-in\omega t}$$

9.7 Spectral decomposition: (Mehler formula)

rewrite propagator of harmonic oscillator by introducing dimensionless units: $a_{ho} = \sqrt{\hbar/m\omega}$ harmonic oscillator length

$$u = \frac{x}{a_{ho}}, \quad u_0 = \frac{x_0}{a_{ho}},$$

$$G(u,t;u_0,0) = \sqrt{\frac{m\omega}{\pi\hbar}}\, \underbrace{e^{-\frac{i}{2}\omega t}\, e^{-\frac{1}{2}(u^2+u_0^2)}}_{\text{"ground state"}} \cdot \overbrace{F(u,t;u_0,0)}^{\text{"excited states"}}$$

$$w := e^{-i\omega t}$$

$$= e^{u^2+u_0^2}\, \underbrace{\frac{1}{\sqrt{1-w^2}}\, \exp\left\{\frac{2uu_0w-(u^2+u_0^2)}{1-w^2}\right\}}_{=f(u,t;u_0,0)}$$

$$f(k,t;k_0,0) = \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} du_0\, e^{-2iku-2ik_0u_0}\, f(u,t;u_0,0)$$

$$= \frac{1}{\sqrt{1-w^2}} \int d^2x\, e^{-\vec{x}^T A \vec{x} + \vec{b}^T \vec{x}}$$

$$\vec{x} = \begin{pmatrix} u \\ u_0 \end{pmatrix}, \quad \vec{b} = 2i \begin{pmatrix} k \\ k_0 \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{1-w^2} & \frac{-w}{1-w^2} \\ \frac{-w}{1-w^2} & \frac{1}{1-w^2} \end{pmatrix}$$

$$f(k,t,k_0,0) = \frac{1}{\sqrt{1-w^2}}\sqrt{\frac{\pi^2}{\det A}}\; e^{\frac{1}{4}\vec{b}^T A^{-1}\vec{b}}$$

$$= \dots = \pi \exp\left\{-(k^2+k_0^2 + 2wkk_0)\right\} = \pi\, e^{-(k^2+k_0^2)}\sum_{n=0}^{\infty}\frac{1}{n!}(-2wkk_0)^n$$

Taylor expansion into $w = e^{-i\omega t}$

Inverse Fourier transformation

$$f(u,t;u_0,0) = \int_{-\infty}^{+\infty}\frac{dk}{\pi}\int_{-\infty}^{+\infty}\frac{dk_0}{\pi}\, e^{2i(ku+k_0u_0)}\, f(k,t;k_0,0)$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{w^n}{2^n n!}\underbrace{\left\{\int_{-\infty}^{+\infty}dk\,\underbrace{(2ik)^n}_{=\frac{\partial^n}{\partial u^n}} e^{-k^2+2iku}\right\}}_{=\frac{\partial^n}{\partial u^n}\sqrt{\pi}\,e^{-u^2}}\underbrace{\left\{\int_{-\infty}^{+\infty}dk_0\,(2ik_0)^n\, e^{-k_0^2+2ik_0u_0}\right\}}_{\text{dito}}$$

$$F(u,t;u_0,0) = \sum_{n=0}^{\infty}\frac{w^n}{2^n n!}\underbrace{\left\{e^{u^2}\frac{\partial^n}{\partial u^n}\underbrace{e^{-u^2}}_{e^{-(u-\alpha)^2}\big|_{\alpha=0}}\right\}}\left\{e^{u_0^2}\frac{\partial^n}{\partial u_0^n}e^{-u_0^2}\right\}$$

$$= (-1)^n e^{u^2}\frac{\partial^n}{\partial\alpha^n}e^{-(u-\alpha)^2}\Big|_{\alpha=0}$$

$$= (-1)^n\frac{\partial^n}{\partial\alpha^n}e^{-\alpha^2+2u\alpha}\Big|_{\alpha=0}$$

generating function for Hermite polynomials

$$\underline{S(u,\alpha)} = e^{-\alpha^2+2\alpha u} = \sum_{n=0}^{\infty} H_n(u)\frac{\alpha^n}{n!}$$

$$\left.\frac{\partial^n S(u,\alpha)}{\partial \alpha^n}\right|_{\alpha=0} = H_n(u)$$

$\downarrow$

$\vdots$

$\downarrow$

(Mehler formula)

$$G(x,t;x_0,0) = \sqrt{\frac{m\omega}{\pi\hbar}} \sum_{n=0}^{\infty} \frac{e^{-i\omega(n+\frac{1}{2})t}}{n!\, 2^n} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x_0\right) e^{-\frac{1}{2}\frac{m\omega}{\hbar}(x^2+x_0^2)}$$

Comparing with spectral decomposition

$$\psi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{n!\, 2^n}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

Note:

Evaluating the path integral via the decomposition

$$x(t) = x_{cl}(t) + \delta x(t), \quad \mathcal{D}x = \mathcal{D}\delta x \quad \Leftarrow$$

is also possible approximately for a general potential $V(x)$

$\Rightarrow$ Gelfand Yaglom formula

$\Rightarrow$ Pauli, de Witt-Morette formula (semi-classical propagator)

$\Rightarrow$ Gutzwiller trace formula for quantum chaos