13 Dirac Equation:

Motivation:

- Klein-Gordon equation is a linear differential equation of second order in space and time describing a spin 0 relativistic quantum particle
- Instead Dirac has found a relativistic wave equation which is linear in the space-time derivatives. It turns out to describe spin 1/2 particles.
- The Dirac equation describes correctly the finestructure of hydrogen atom.

13_1 Derivation:

- Ansatz: linear differ. eq. of first order

covariant index

unknown?

$$\left(i\gamma^{\mu}\partial_{\mu}-\underbrace{\frac{mc}{\hbar}}\right)\Psi(x^{\lambda})=0$$

four space-time vector

$$\begin{cases}(x_{\lambda})=(ct,-x,-y,-z)\\(x^{\lambda})=(ct,x,y,z)\end{cases}$$

greek letter like μ, λ: $\mu,\lambda=\underbrace{0}_{\text{time}},\underbrace{1,2,3}_{\text{space}}$

"mass term" $\hat{=}$ inverse Compton wave length

object, not known?

$$=\frac{\partial}{\partial x^{\mu}}$$

contravariant index

$$=\left(\frac{1}{c}\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

Note: $\gamma^{\mu}\partial_{\mu}:=\gamma^{0}\partial_{0}+\gamma^{1}\partial_{1}+\gamma^{2}\partial_{2}+\gamma^{3}\partial_{3}$

- Application: $\left(-i\gamma^{2}\partial_{2}-\frac{mc}{\hbar}\right)\cdot$

$$\underbrace{\left(-i\gamma^\nu\partial_\nu - \frac{mc}{\hbar}\right)\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)}\Psi = 0$$

$$= \underbrace{\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu}_{\frac{1}{2}(\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu)\underbrace{\partial_\nu\partial_\mu}_{\text{symmetric in } \mu,\nu}} + \frac{m^2c^2}{\hbar^2}$$

$$\Rightarrow \left\{\frac{1}{2}(\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu + \frac{m^2c^2}{\hbar^2}\right\}\Psi = 0$$

This is a Klein-Gordon equation provided that

$$\frac{1}{2}(\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu) = \frac{1}{2}[\gamma^\nu,\gamma^\mu]_+ = \eta^{\mu\nu}, \quad (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\eta^{\mu\nu})$$

Minkowski metric

light cone: $x^\mu x^\nu \eta_{\mu\nu} = 0 = (ct)^2 - \vec{x}^2$

$$\Rightarrow \left(\underbrace{\eta^{\mu\nu}\partial_\nu\partial_\mu}_{} + \frac{m^2c^2}{\hbar^2}\right)\Psi = 0$$

$$\eta^{00}\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \eta^{11}\frac{\partial^2}{\partial x^2} + \eta^{22}\frac{\partial^2}{\partial y^2} + \eta^{33}\frac{\partial^2}{\partial z^2} = \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

- Mathematical consequence: γ^μ can not be simply complex number

$[\gamma^\mu,\gamma^\nu]_+ = 2\eta^{\mu\nu}$ Clifford algebra

$\mu = \nu = 0$: $(\gamma^0)^2 = 1$, $\mu = \nu = i = 1,2,3$: $(\gamma^i)^2 = -1$

$\mu \neq \nu$: $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$

$\Rightarrow$ Next guess: γ^μ must some matrices, then Ψ must be a "vector"

Ψ turns to be a spinor!

- Physical consequence: Each component of Ψ fulfills a Klein-Gordon equation, but due to Dirac equation their dynamics is coupled via γ^μ
- Free particle solution:

$$\Psi(x^\lambda) = \underbrace{\Psi}_{\text{fixed spinor amplitude}} e^{\frac{i}{\hbar}(\vec{p}\vec{x} - Et)}$$

contravariant, covariant } four-momentum vector $\quad (p^\lambda) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$; $p^\lambda = g^{\lambda\mu} p_\mu$

$(p_\lambda) = \begin{pmatrix} E/c \\ -\vec{p} \end{pmatrix}$; $p_\lambda = g_{\lambda\mu} p^\mu$

$$\left.\begin{aligned} i\partial_0 \Psi(x^\lambda) &= \frac{i}{c}\frac{\partial}{\partial t}\Psi(x^\lambda) = \frac{1}{\hbar}\frac{E}{c}\Psi(x^\lambda) = \frac{1}{\hbar}p^0\Psi(x^\lambda) \\ i\partial_i \Psi(x^\lambda) &= -\frac{1}{\hbar}p^i \Psi(x^\lambda) \end{aligned}\right\} \cancel{i}\partial_\mu \Psi(x^\lambda) = \frac{-i}{\hbar}p_\mu \Psi(x^\lambda)$$

$$\Rightarrow \Big(\underbrace{g^{\mu\nu}\partial_\mu\partial_\nu}_{} - \frac{m^2c^2}{\hbar^2}\Big)\Psi(x^\lambda) = 0$$

$$\underbrace{\left(g^{\mu\nu}\left(\frac{-i}{\hbar}\right)^2 p_\mu p_\nu - \frac{m^2c^2}{\hbar^2}\right)}_{=0}\Psi(x^\lambda) = 0$$

$= 0 \Rightarrow$ relativistic energy-momentum dispersion $\quad E^2 = \vec{p}^2c^2 + m^2c^2$

- Rewriting Dirac equation:

$$\left(i\gamma^0\frac{1}{c}\frac{\partial}{\partial t} + i\gamma^i\partial_i - \frac{mc}{\hbar}\right)\Psi(\vec{x},t) = 0 \quad \Big|\cdot \hbar c\gamma^0$$

$$i\hbar\underbrace{(\gamma^0)^2}_{=1}\frac{\partial}{\partial t}\Psi(\vec{x},t) = \underbrace{\Big\{c\underbrace{\gamma^0\gamma^i}_{=\alpha^i}\underbrace{\frac{\hbar}{i}\partial_i}_{\hat{p}_i} + mc^2\underbrace{\gamma^0}_{=:\beta}\Big\}}_{=\hat{H}}\Psi(\vec{x},t)$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\Psi, \quad \hat{H} = c\,\vec{\alpha}\cdot\hat{\vec{p}} + mc^2\beta$$

- Algebra of α^i, β:

$$[\beta,\beta]_+ = 2\beta^2 = 2(\gamma^0)^2 = 2$$

$$[\alpha^i,\beta]_+ = \alpha^i\beta + \beta\alpha^i = \underbrace{\gamma^0\gamma^i}_{=-\gamma^i\gamma^0}\gamma^0 + \underbrace{\gamma^0\gamma^0}_{=1}\gamma^i = 0, \qquad -\gamma^i\underbrace{\gamma^0\gamma^0}_{=1}$$

$$[\alpha^i,\alpha^j]_+ = \alpha^i\alpha^j + \alpha^j\alpha^i = \underbrace{\gamma^0\gamma^i}_{=-\gamma^i\gamma^0}\gamma^0\gamma^j + \underbrace{\gamma^0\gamma^j}_{=-\gamma^j\gamma^0}\gamma^0\gamma^i = -\gamma^i\gamma^j - \gamma^j\gamma^i = \begin{cases} 2 & i=j \\ 0 & i\neq j \end{cases}$$

$$= 2\delta^{ij}$$

- Realize α^i, β with matrices. What about 2×2 matrices?

Pauli matrices σ^i, $[\sigma^i,\sigma^j]_+ = 2\delta^{ij}$

σ^i-matrices + $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ form a basis in the space of 2×2 matrices

$$[\sigma^i, I]_+ = 2\sigma^i \neq 0$$

- 4×4 matrices:

$$\begin{pmatrix}0&0\\0&0\end{pmatrix} \qquad \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\Rightarrow [\beta,\beta]_+ = 2, \quad [\alpha^i,\beta]_+ = 0, \quad [\alpha^i,\alpha^j]_+ = 2\delta^{ij}$$

- This means that Ψ must have 4 components. It is a spinor because Ψ transforms differently under rotations and boosts as a vector.

13.2 Continuity:

Dirac equation: $i\hbar \frac{\partial}{\partial t}\Psi = i\hbar c \vec{\alpha}\vec{\nabla}\Psi + mc^2\beta\,\Psi \quad \Big| \;\Psi^+ \cdot$

$$-i\hbar\frac{\partial}{\partial t}\Psi^+ = -i\hbar c\vec{\nabla}\Psi^+ \cdot \underbrace{\vec{\alpha}^+}_{=\vec{\alpha}} + mc^2\underbrace{\beta^+}_{=\beta}\Psi^+ \quad \Big| \cdot \Psi$$

$$i\hbar\underbrace{\left(\Psi^+\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^+}{\partial t}\Psi\right)}_{=\frac{\partial}{\partial t}(\Psi^+\Psi)} = -i\hbar c\underbrace{\left[\Psi^+\vec{\alpha}\vec{\nabla}\Psi + \vec{\nabla}\Psi^+\vec{\alpha}\Psi\right]}_{=\vec{\nabla}\;\Psi^+\vec{\alpha}\Psi}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\underbrace{\rho}_{=\Psi^+\Psi} + \operatorname{div}\underbrace{\vec{j}}_{c\,\Psi^+\vec{\alpha}\,\Psi} = 0$$

positive definite

13.3 Nonrelativistic Limit:

Interaction of spin 1/2 with charge q to electromagnetic field: minimal coupling

$$\hat{\vec{p}} \Rightarrow \hat{\vec{p}} - q\vec{A}, \qquad \hat{E} \rightarrow \hat{E} - q\varphi$$

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi, \quad \hat{H} = c\vec{\alpha}\cdot(\hat{\vec{p}} - q\vec{A}) + mc^2\beta + q\varphi, \qquad \beta = \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix}$$

Note: local gauge transformations like in Schrödinger theory

Nonrelativistic limit: ansatz

$$\Psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

upper Weyl spinor (2 components)

lower Weyl spinor (2 components)

Dirac spinor (4 components)

$$i\hbar \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \left\{ c \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} (\hat{\vec{p}} - q\vec{A}) + mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + q\varphi \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right\} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$i\hbar \frac{\partial}{\partial t} u = c\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A}) v + (mc^2 + q\varphi) u$$

$$i\hbar \frac{\partial}{\partial t} v = c\vec{\sigma} (\hat{\vec{p}} - q\vec{A}) u + (-mc^2 + q\varphi) v,$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} e^{-\frac{i}{\hbar}mc^2 t}$$

nonrelativistic limit for particles

$$i\hbar \frac{\partial}{\partial t} \tilde{u} = c\vec{\sigma}(\hat{\vec{p}} - q\vec{A})\tilde{v} + q\varphi\tilde{u} \quad (1)$$

$$\underbrace{\cancel{i\hbar \frac{\partial}{\partial t} \tilde{v}}}_{\equiv 0} = c\vec{\sigma}(\hat{\vec{p}} - q\vec{A})\tilde{u} + (\underbrace{-2mc^2}_{\text{large}} + \underbrace{\cancel{q\varphi}}_{\text{scalar potential}})\tilde{v} \quad (2)$$

Adiabatic elimination of lower Weyl spinor:

$$0 \approx c\vec{\sigma}(\hat{\vec{p}} - q\vec{A})\tilde{u} - 2mc^2\tilde{v} \Rightarrow \tilde{v} = \frac{1}{2mc}\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})\tilde{u} \quad (2')$$

(2') in (1): $$i\hbar\frac{\partial\tilde{u}}{\partial t} = \frac{1}{2m}\underbrace{\boxed{[\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})][\vec{\sigma}\cdot(\hat{\vec{p}} - q\vec{A})]}}_{= ?}\tilde{u} + q\varphi\tilde{u}$$

Properties of Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Clifford algebra:

$$[\sigma^k, \sigma^l]_+ = \sigma^k\sigma^l + \sigma^l\sigma^k = 2\delta^{kl}\cdot I$$

- Lie algebra:

$$[\sigma^k, \sigma^l]_- = \sigma^k\sigma^l - \sigma^l\sigma^k = 2i\varepsilon^{klm}\sigma^m$$

Physical consequences: $\hat{S}_k = \frac{\hbar}{2}\sigma_k$

$[\hat{S}_k, \hat{S}_l]_- \hat{=} \frac{\hbar^2}{4} 2i\varepsilon_{klm}\sigma_m = i\hbar\varepsilon_{klm}\hat{S}_m$ angular momentum algebra

How can we identify which quantum number corresponds to $\hat{S}_m$?

$$\vec{S}^2 = \sum_k \hat{S}_k\hat{S}_k = \frac{\hbar^2}{4}\sum_k \underbrace{\sigma_k\sigma_k}_{\text{Clifford algebra: } = 2\delta_{kk} = 2} = \frac{\hbar^2}{4}\cdot 2\cdot 3 = \frac{3}{2}\hbar^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = s(s+1)\hbar^2$$

$\longrightarrow \boxed{s = 1/2}$

Mathematical consequence: $\sigma^k\sigma^l = \delta^{kl}I + i\varepsilon_{klm}\sigma^m$

$$\boxed{} = \sigma_k\sigma_l\left(\frac{\hbar}{i}\partial_k - qA_k\right)\left(\frac{\hbar}{i}\partial_l - qA_l\right) \hat{=}$$

$$= (\hat{\vec{p}} - q\vec{A})^2 + i\,\vec{\sigma}\cdot\underbrace{\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)\times\left(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}\right)}$$

$$= \cancel{\frac{\hbar}{i}\vec{\nabla}\times\frac{\hbar}{i}\vec{\nabla}} \cancel{- q\vec{A}\times\frac{\hbar}{i}\vec{\nabla}} - \underbrace{\frac{\hbar}{i}\vec{\nabla}\times\vec{A}q}_{\text{product rule}} \cancel{+ q\vec{A}\times q\vec{A}}$$

$$= q\frac{\hbar}{i}\,\mathrm{rot}\,\vec{A} \cancel{- \frac{\hbar}{i}\vec{A}\times\vec{\nabla}}$$

$$= (\hat{\vec{p}} - q\vec{A})^2 + i\vec{\sigma}\, q\frac{\hbar}{i}(-1)\underbrace{\mathrm{rot}\,\vec{A}}_{=\vec{B}}$$

Pauli equation for $\tilde{u}$:

$$i\hbar\frac{\partial}{\partial t}\tilde{u} = \left[\underbrace{\frac{1}{2m}(\hat{\vec{p}} - q\vec{A})^2} + \underbrace{\frac{-q\hbar}{2m}\vec{\sigma}}_{= \frac{-q}{2m}\cdot 2\cdot\underbrace{\frac{\hbar}{2}\vec{\sigma}}_{=\vec{S}}}\cdot\vec{B} + q\varphi\right]\tilde{u}$$

$\underbrace{g_s}_{=2}$

$$= \frac{\hat{\vec{p}}^2}{2m} \left(-\frac{1}{2m} 2q \vec{A}\cdot\hat{\vec{p}} \right) - \frac{q}{2m}\frac{\hbar}{i} \underbrace{\operatorname{div}\vec{A}}_{=0} + \underbrace{q^2 \vec{A}^2}_{\text{quadratic Zeeman}}/2m$$

const. magnetic field

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{x}$$

$$\operatorname{rot}\vec{A} = \vec{B}$$

$$\frac{-q}{m}\frac{1}{2}(\vec{B}\times\vec{x})\cdot\hat{\vec{p}} = \frac{-q}{2m}\underbrace{(\vec{x}\times\hat{\vec{p}})}_{=\hat{\vec{L}}}\cdot\vec{B}$$

$\underbrace{g_e}_{=1}$