4 Space and Time Transformations:

So far we have discussed the propagators for both the free particle and the harmonic oscillator separately. Now we will show that both propagators can be mapped into each other by a suitable space-time transformation. In particular, applying a time transformation is unusual as non-relativistic classical and quantum mechanics are based on the Newtonian concept that time is absolute and moves uniformly: "Die absolute, wahre und mathematische Zeit verfließt an sich und vermöge ihrer Natur gleichförmig, und ohne Beziehung auf einen äußeren Gegenstand." (Newton, 1643–1727) But if one follows Dirac extends the phase space of coordinates and momenta by time and energy, then space-time transformations can be shown to be canonical and unitary in classical and quantum mechanics, respectively.

4.1 Space Transformation:

Starting point of our considerations is the path integral

$$G(x_b, t_b; x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \; e^{\frac{i}{\hbar} A[x]} \tag{235}$$

for the harmonic oscillator

$$A[x] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2} \dot{x}^2(t) - \frac{M}{2} \omega^2 x^2(t) \right\} \tag{236}$$

At first we apply the space transformation

$$x(t) = y(t)\, \xi(t) \tag{237}$$

with a yet undetermined function $\xi(t)$ and investigate its consequences for the action (236):

$$A[x] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2} \left[\dot{y}^2(t)\, \xi^2(t) + 2 \dot{y}(t)\, y(t)\, \dot{\xi}(t)\, \xi(t) + y^2(t)\, \dot{\xi}^2(t) \right] - \frac{M}{2} \omega^2 y^2(t)\, \xi^2(t) \right\} \tag{238}$$

A partial integration of the second term yields

$$\int_{t_a}^{t_b} dt\, \dot{y}(t) y(t)\, \dot{\xi}(t)\xi(t) = \frac{1}{2}\int_{t_a}^{t_b} dt \left\{ \frac{d}{dt} y^2(t) \right\} \dot{\xi}(t)\xi(t) = \frac{1}{2}\int_{t_a}^{t_b} dt\, \frac{d}{dt}\left\{ y^2(t)\,\dot{\xi}(t)\xi(t)\right\}$$

$$- \frac{1}{2}\int_{t_a}^{t_b} dt\, y^2(t) \frac{d}{dt}\left\{ \dot{\xi}(t)\xi(t)\right\} = \frac{1}{2}\left[y^2(t)\,\dot{\xi}(t)\xi(t)\right]_{t_a}^{t_b} - \frac{1}{2}\int_{t_a}^{t_b} dt\, y^2(t)\left\{ \ddot{\xi}(t)\xi(t) + \dot{\xi}^2(t)\right\} \quad (239)$$

The boundary term can be simplified with the help of (237) and the last term in (239) cancels the third term in (238), yielding the action decomposition

$$A[x] = \frac{M}{2}\left\{ x_b^2\, \frac{\dot{\xi}(t_b)}{\xi(t_b)} - x_a^2\, \frac{\dot{\xi}(t_a)}{\xi(t_a)} \right\} + \tilde{A}[y] \quad (240)$$

where the action of the new path reads

$$\tilde{A}[y] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2}\, \xi^2(t)\, \dot{y}^2(t) - \frac{M}{2}\, \omega^2 \left[\ddot{\xi}(t) + \omega^2 \xi(t)\right] \xi(t) \right\} \quad (241)$$

In order to eliminate the potential term in (241) we have to choose $\xi(t)$ such that the condition

$$\ddot{\xi}(t) + \omega^2\, \xi(t) = 0 \quad (242)$$

is fulfilled. Thus, we conclude that a linear space transformation (237) just allows to eliminate a quadratic potential term. Then (241) reduces to the action of a free particle with an effective time-dependent mass:

$$\tilde{A}[y] = \int_{t_a}^{t_b} dt\, \frac{M}{2}\, \xi^2(t)\, \dot{y}^2(t) \quad (243)$$

4.2 Time Transformation:

In view of eliminating the explicit time dependence of the mass in (243) we apply now a subsequent time transformation according to

$$\frac{dt}{ds} = g(t) \quad (244)$$

with a yet undetermined function $g(t)$. Transforming from the old time t to the new time s yields for derivatives via the chain rule

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} \overset{(244)}{=\!=\!=} \frac{1}{g(t)}\frac{d}{ds} \tag{245}$$

With the velocity definitions

$$\dot{y} = \frac{dy}{dt}, \quad \mathring{y} = \frac{dy}{ds} \tag{246}$$

(245) has to be interpreted as

$$\dot{y} = \frac{1}{g(t)}\mathring{y} \tag{247}$$

Thus the action (243) is mapped due to the time transformation (244) to

$$\mathcal{A}[y] = \frac{M}{2}\int_{s_a}^{s_b} ds\, g(t)\, f^2(t)\, \frac{\mathring{y}^2(s)}{g(t)^2} = \frac{M}{2}\int_{s_a}^{s_b} ds\, \frac{f^2(t)}{g(t)}\, \mathring{y}^2(s) \tag{248}$$

Provided we fix the function $g(t)$ according to

$$g(t) = f^2(t) \tag{249}$$

we finally end up with the action of a free particle with respect to the new time s

$$\mathcal{A}[y] = \int_{s_a}^{s_b} ds\, \frac{M}{2}\, \mathring{y}^2(s) \tag{250}$$

Now we have to clarify the relation between old time t and new time s. Inserting (249) in (244) yields at first

$$\frac{dt}{ds} = f^2(t) \tag{251}$$

which we interpret as follows:

$$\frac{dt}{f^2(t)} = ds \Rightarrow s_b - s_a = \int_{s_a}^{s_b} ds = \int_{t_a}^{t_b} dt\, \frac{1}{f^2(t)} \tag{252}$$

Here $f(t)$ is any solution of the second-order differential equation (242)

$$f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \tag{253}$$

By choosing the coefficients according to

$$c_1 = c\cos\gamma \qquad\Longleftrightarrow\qquad c = \sqrt{c_1^2 + c_2^2} \tag{254}$$
$$c_2 = c\sin\gamma \qquad\qquad\qquad \gamma = \arctan\frac{c_2}{c_1}$$

(253) is converted to

$$f(t) = c\cos(\omega t - \gamma) \tag{255}$$

Inserting (255) in (252) then yields

$$s_b - s_a = \int_{t_a}^{t_b} dt \, \frac{1}{c^2\cos^2(\omega t - \gamma)} = \frac{1}{c^2\omega}\Big[\tan(\omega t - \gamma)\Big]_{t_a}^{t_b} = \frac{\tan(\omega t_b - \gamma) - \tan(\omega t_a - \gamma)}{\omega c^2}$$

$$= \frac{1}{\omega c^2\cos(\omega t_b - \gamma)\cos(\omega t_a - \gamma)}\Big\{\sin(\omega t_b - \gamma)\cos(\omega t_a - \gamma) - \sin(\omega t_a - \gamma)\cos(\omega t_b - \gamma)\Big\}$$

$$= \frac{\sin\omega(t_b - t_a)}{\omega c^2\cos(\omega t_b - \gamma)\cos(\omega t_a - \gamma)} \overset{(255)}{=\!=} \frac{\sin\omega(t_b - t_a)}{\omega}\cdot\frac{1}{f(t_b)f(t_a)} \tag{256}$$

This relation describes the mapping between the initial and the final old and new times t and s.

4.3 Path Integral Measure:

In the last two sections we have focused on investigating the consequences of the consecutive space and time transformation (237) and (251) for the action. In order to transform correspondingly the whole path integral (235), we have now to consider also the fate of the path integral measure, which is defined by its discretized version

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x = \lim_{\substack{\varepsilon\to 0\\ N\to\infty\\ \varepsilon N = t_b - t_a}} \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dx_j\right\}\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}} \tag{257}$$

The space transformation (237) yields immediately

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x = \lim_{\substack{\epsilon\to 0\\ N\to\infty\\ \epsilon N = t_b - t_a}} \left\{ \prod_{j=1}^{N-1} \int_{-\infty}^{+\infty} dx_j \, f(t_j) \right\} \left(\frac{M}{2\pi i \hbar \epsilon} \right)^{\frac{N}{2}} \tag{258}$$

In order to implement subsequently the time transformation (251), we recall that ϵ denotes the equidistant slicing of the old time t:

$$\epsilon = t_{j+1} - t_j \tag{259}$$

so that (258) has to be interpreted as

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x = \lim_{N\to\infty} \left\{ \prod_{j=1}^{N-1} \int_{-\infty}^{+\infty} dx_j \, f(t_j) \right\} \prod_{j=0}^{N-1} \sqrt{\frac{M}{2\pi i \hbar (t_{j+1} - t_j)}} \tag{260}$$

We now consider (251) as a differential equation for the relation $t(s)$ between old and new time

$$\frac{dt(s)}{ds} = f^2(t(s)) \tag{261}$$

which is integrated between two time steps

$$t_{j+1} - t_j = \int_{t_j}^{t_{j+1}} dt = \int_{s_j}^{s_{j+1}} f^2(t(s))\, ds \tag{262}$$

For an infinitesimal small difference $s_{j+1} - s_j$ the integral (262) can be approximated in different ways up to first order in $s_{j+1} - s_j$:

$$t_{j+1} - t_j \approx f^2(t(s_{j+1}))(s_{j+1} - s_j) + \mathcal{O}((s_{j+1} - s_j)^2) \tag{263}$$

$$\approx f^2(t(s_j))(s_{j+1} - s_j) + \mathcal{O}((s_{j+1} - s_j)^2) \tag{264}$$

$$\approx f(t(s_{j+1}))f(t(s_j))(s_{j+1} - s_j) + \mathcal{O}((s_{j+1} - s_j)^2) \tag{265}$$

A priori we can not decide which discretization (263)–(265) should be chosen. But the discretization, which corresponds to the transformation of the continuous actions in sections 4.1 and 4.2 can be identified with (265). To this end one has to apply the space and time transformation (237) and (251) in the corresponding discrete action, which we will not reveal explicitly here. With this we get

$$t_{j+1}-t_j = g(t_{j+1})\,g(t_j)\,(s_{j+1}-s_j) + \mathcal{O}\left((s_{j+1}-s_j)^2\right) \qquad (266)$$

as we identify

$$t_j = f(s_j)\,, \quad t_{j+1} = f(s_{j+1}) \qquad (267)$$

Thus, (266) destroys the equidistance of the slicing of time axis and we get from (260)

$$\int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x = \lim_{N\to\infty} \left\{\prod_{j=1}^{N-1} \int_{-\infty}^{+\infty} dy_j\, g(t_j)\right\} \prod_{j=0}^{N-1} \sqrt{\frac{M}{2\pi i\hbar\,(s_{j+1}-s_j)\,g(t_{j+1})\,g(t_j)}}$$

$$= \frac{\cancel{g(t_1)}\cdot\ldots\cdot\cancel{g(t_{N-1})}}{\sqrt{g(t_a)\,g(t_1)\cancel{\cdot g(t_1)}\,g(\cancel{t_2})\cdots g(t_{N-2})\,g(\cancel{t_{N-1}})\cdot\cancel{g(t_{N-1})}\,g(t_b)}} \lim_{N\to\infty} \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dy_j\right\} \prod_{j=0}^{N-1} \sqrt{\frac{M}{2\pi i\hbar(s_{j+1}-s_j)}}$$

$$= \frac{1}{\sqrt{g(t_a)\,g(t_b)}} \lim_{\substack{N\to\infty \\ \varepsilon\to 0 \\ N\varepsilon = s_b-s_a}} \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dy_j\right\} \prod_{j=0}^{N-1}\sqrt{\frac{M}{2\pi i\hbar\varepsilon}} = \frac{1}{\sqrt{g(t_b)\,g(t_a)}} \int_{y(t_a)=y_a}^{y(t_b)=y_b} \mathcal{D}y \qquad (268)$$

4.4 Propagator Mapping:

Now we combine all previous results for the space-time transformation of the path integral to a corresponding mapping of propagator:

$$G(x_b,t_b;x_a,t_a) \underset{(235),(236)}{=} \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{M}{2}\dot{x}^2(t) - \frac{M}{2}\omega^2 x^2(t)\right]\right\}$$

$$\underset{(240),(250),(268)}{=} \frac{e^{\frac{iM}{2\hbar}\left\{x_b^2 \frac{\dot{g}(t_b)}{g(t_b)} - x_a^2 \frac{\dot{g}(t_a)}{g(t_a)}\right\}}}{\sqrt{g(t_b)\,g(t_a)}} \int_{y(t_a)=y_a}^{y(t_b)=y_b} \mathcal{D}y \exp\left\{\frac{i}{\hbar}\int_{s_a}^{s_b} ds\, \frac{M}{2}\,\dot{y}^2(t)\right\} \qquad (269)$$

The remaining path integral in (269) represents that of the free particle. Thus, by taking into account the space transformation (237) and the time transformation (250) we finally obtain a mapping between the propagators of the harmonic oscillator and the free particle:

$$G(x_b,t_b;x_a,t_a) = \frac{e^{\frac{iM}{2\hbar}\left\{x_b^2 \frac{\dot\varrho(t_b)}{\varrho(t_b)} - x_a^2 \frac{\dot\varrho(t_a)}{\varrho(t_a)}\right\}}}{\sqrt{\varrho(t_b)\,\varrho(t_a)}}\, G_{FP}\left(\frac{x_b}{\varrho(t_b)}, \frac{\sin\omega(t_b-t_a)}{\omega\,\varrho(t_b)\,\varrho(t_a)}; \frac{x_a}{\varrho(t_a)}, 0\right) \qquad (270)$$

In order to verify (270) we insert the free particle propagator determined in Chapter 2

$$G_{FP}(y_b, s_b-s_a; y_a, 0) = \sqrt{\frac{M}{2\pi i\hbar(s_b-s_a)}}\, \exp\left\{\frac{iM}{2\hbar}\,\frac{(y_b-y_a)^2}{s_b-s_a}\right\} \qquad (271)$$

and obtain

$$G(x_b,t_b;x_a,t_a) = \sqrt{\frac{M\omega\,\cancel{\varrho(t_b)\varrho(t_a)}}{2\pi i\hbar\,\sin\omega(t_b-t_a)}}\,\frac{1}{\cancel{\sqrt{\varrho(t_b)\,\varrho(t_a)}}}\,\exp\left\{\frac{iM}{2\hbar}\left[x_b^2\,\frac{\dot\varrho(t_b)}{\varrho(t_b)} - x_a^2\,\frac{\dot\varrho(t_a)}{\varrho(t_a)}\right]\right\}$$

$$\exp\left\{\frac{iM\omega\,\varrho(t_b)\varrho(t_a)}{2\hbar\,\sin\omega(t_b-t_a)}\left[\frac{x_b}{\varrho(t_b)} - \frac{x_a}{\varrho(t_a)}\right]^2\right\} = \sqrt{\frac{M\omega}{2\pi i\hbar\,\sin\omega(t_b-t_a)}}$$

$$\exp\left\{\frac{iM\omega}{2\hbar\,\sin\omega(t_b-t_a)}\left[-2x_a x_b + x_b^2\left[\frac{\varrho(t_a)}{\varrho(t_b)} + \frac{\sin\omega(t_b-t_a)}{\omega}\,\frac{\dot\varrho(t_b)}{\varrho(t_b)}\right] + x_a^2\left[\frac{\varrho(t_b)}{\varrho(t_a)} - \frac{\sin\omega(t_b-t_a)}{\omega}\,\frac{\dot\varrho(t_a)}{\varrho(t_a)}\right]\right]\right\}$$

The prefactor of x_b^2 yields with (255)

$$B = \frac{C}{\varrho(t_b)}\left\{\cos\omega(t_a-\tau) - \sin\omega(t_b-t_a)\,\sin\omega(t_b-\tau)\right\}$$

$$= \frac{C}{\varrho(t_b)}\left\{\cos\omega(t_a-\tau) - \sin\omega(t_b-t_a)\left[\sin\omega(t_b-t_a)\cos\omega(t_a-\tau) - \cos\omega(t_b-t_a)\sin\omega(t_a-\tau)\right]\right\}$$

$$= \frac{C}{\varrho(t_b)}\left\{\cos\omega(t_a-\tau)\cos^2\omega(t_b-t_a) + \sin\omega(t_b-t_a)\cos\omega(t_b-t_a)\sin\omega(t_a-\tau)\right\}$$

$$= \frac{C}{\varrho(t_b)}\left\{\ \right.$$

$$= \frac{\cancel{\cos\omega(t_b-t_a)}}{\cancel{\cos\omega(t_b-\tau)}}\cdot\left\{\underbrace{\cos\omega(t_b-t_a)\cos\omega(t_a-\tau) + \sin\omega(t_b-t_a)\sin\omega(t_a-\tau)}_{=\cos\omega(t_b-t_a)}\right\} = \cos\omega(t_b-t_a) \qquad (273)$$

and correspondingly the prefactor of x_a^2 gives

$$A = \frac{C}{\varrho(t_a)}\left\{\cos\omega(t_b-\tau) + \sin\omega(t_b-t_a)\,\sin\omega(t_a-\tau)\right\}$$

$$= \frac{C}{\varrho(t_a)}\left\{\cos\omega(t_b-\tau) + \sin\omega(t_b-t_a)\left[\sin\omega(t_b-\tau)\cos\omega(t_b-t_a) - \cos\omega(t_b-\tau)\sin\omega(t_b-t_a)\right]\right\}$$

$$= \frac{\ell \cos\omega(t_b-t_a)}{\ell \cos\omega(t_a-\tau)} \Big\{ \underbrace{\cos\omega(t_b-\tau)\cos\omega(t_b-t_a) + \sin\omega(t_b-\tau)\sin\omega(t_b-t_a)}_{=\cos\omega(t_a-\tau)} \Big\} = \cos\omega(t_b-t_a) \qquad (274)$$

Thus, we conclude that (272) reduces together with (273) and (274) to the propagator of the harmonic oscillator (270).

4.5 Comments:

The propagator mapping (270) between the harmonic oscillator and the free particle is quite peculiar as the former system consists of a discrete spectrum, whereas the latter system possesses a continuous spectrum. Thus, one has to understand (270) such that the spectral data as a whole are transformed via the space-time transformation (237) and (251).

Furthermore, note that (237) and (251) just represents a particular example for a more general class of space-time transformations which have been applied to the path integral. Most prominently, it is also possible to map the three-dimensional hydrogen atom to the four-dimensional harmonic oscillator. This mapping of the corresponding path integrals was found in 1979 by Duru and Kleinert and is based on the Kustaanheimo-Stiefel transformation from 1965, which maps the three-dimensional Kepler problem to the four-dimensional harmonic oscillator. The latter transformation was developed within the realm of celestial mechanics in order to eliminate the singularity of the gravitational potential of the Kepler problem.