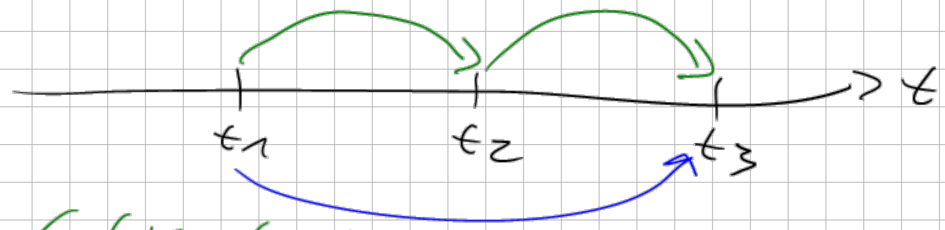
$$\psi(x,t) = \int_{-\infty}^{+\infty} dx_0\, \underbrace{G(x,t;x_0,t_0)}_{\text{propagator}}\, \psi(x_0,t_0)$$

group property

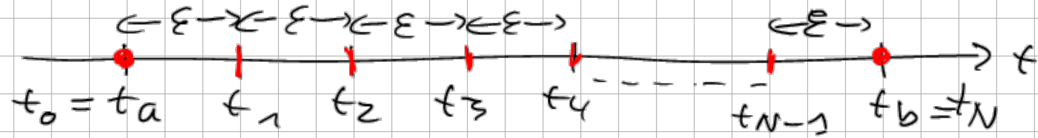


$$G(x_3,t_3;x_1,t_1) = \int_{-\infty}^{+\infty} dx_2\; G(x_3,t_3;x_2,t_2)\, G(x_2,t_2;x_1,t_1) \qquad (*)$$

8.3 Long- and Short-Time Propagator:

equidistant slices of time

$$\varepsilon = t_{j+1}-t_j = \frac{t_b-t_a}{N}$$

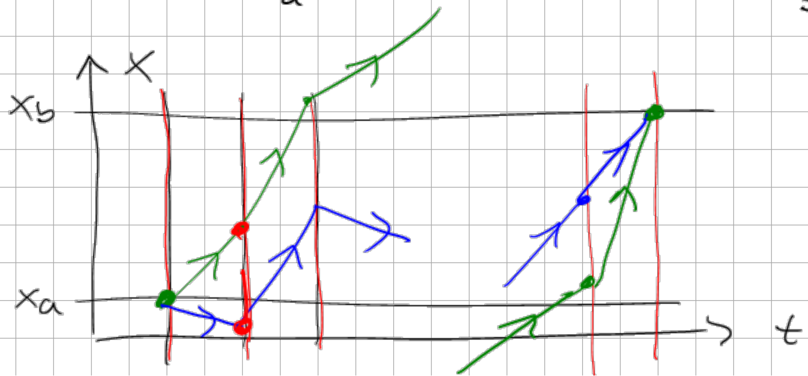


Apply $(*)$ $(N-1)$-times:

$$G_N(x_b,t_b;x_a,t_a) = \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dx_j\right\}\left\{\prod_{j=0}^{N-1} G(x_{j+1},t_{j+1};x_j,t_j)\right\} \text{ with } x_0 = x_a,\ x_N = x_b$$

Idea: $N\to\infty$, $\varepsilon\to 0$, $\varepsilon N = t_b - t_a$

$$\underbrace{G(x_b,t_b;x_a,t_a)}_{\text{long-time propagator}} = \lim_{\substack{N\to\infty\\ \varepsilon\to 0\\ \varepsilon N = t_b-t_a}} \left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dx_j\right\}\left\{\prod_{j=0}^{N-1} \underbrace{G(x_{j+1},t_{j+1};x_j,t_j)}_{\substack{= G(x_{j+1},\, \underbrace{t_{j+1}-t_j}_{=\varepsilon};\, x_j,\, 0)\\ \text{short-time propagator}}}\right\}$$



all possible discrete paths on this time grid contribute

$t_a = t_0 \quad t_1 \quad t_2 \; - - - - - \; t_{N-1} \quad t_N = t_b$

8.4 Calculation of Short-Time Propagator:

$\longrightarrow$ initial value problem:

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t_{j+1}} G(x_{j+1}, t_{j+1}; x_j, t_j) = \overbrace{\hat{H}(x_{j+1})}^{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_{j+1}^2} + \underbrace{V(x_{j+1})}_{\approx V(x_j) + \dots}} G(x_{j+1}, t_{j+1}; x_j, t_j) \\ \lim\limits_{t_{j+1} \to t_j} G(x_{j+1}, t_{j+1}; x_j, t_j) = \delta(x_{j+1} - x_j) \end{cases}$$

correction terms due to Taylor expansion

$\Rightarrow$ This approximation makes $V(x_{j+1})$ constant

eigenfunctions of momentum operator

$$\left.\begin{aligned} &\hat{p}_{j+1}\, \psi_p(x_{j+1}) = p\, \psi_p(x_{j+1}), \quad \hat{p}_{j+1} = \frac{\hbar}{i}\frac{\partial}{\partial x_{j+1}} \\ &\int_{-\infty}^{+\infty} dx_{j+1}\, \psi_p^*(x_{j+1})\, \psi_{p'}(x_{j+1}) = \delta(p - p') \\ &\int_{-\infty}^{+\infty} dp\, \psi_p^*(x_{j+1})\, \psi_p(x'_{j+1}) = \delta(x_{j+1} - x'_{j+1}) \end{aligned}\right\} \psi_p(x_{j+1}) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{i}{\hbar} p x_{j+1}}$$

$$\hat{H}(x_{j+1})\, \psi_p(x_{j+1}) = E_p\, \psi_p(x_{j+1}), \quad E_p = \frac{p^2}{2m} + V(x_j)$$

Spectral decomposition:

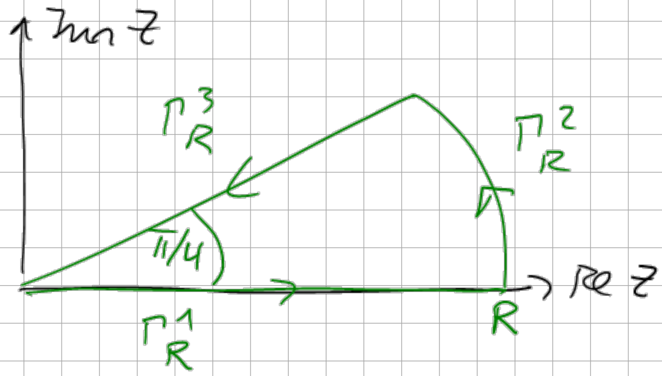
$$G(x_{j+1}, t_{j+1}; x_j, t_j) = \int_{-\infty}^{+\infty} dp\, \psi_p^*(x_j)\, e^{-\frac{i}{\hbar} E_p (t_{j+1} - t_j)}\, \psi_p(x_{j+1})$$

$$= \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar}\left[\frac{p^2}{2m} + V(x_j)\right] \overbrace{(t_{j+1} - t_j)}^{\varepsilon} + \frac{i}{\hbar} p\, (x_{j+1} - x_j) \right\} \quad \Leftarrow$$

$\Rightarrow$ Fresnel integral (due to i)

additional correction terms do not play a role as they lead to corrections of short-time propagator to higher than ε order

$$I_R = \int_{\Gamma_R} e^{-z^2}\, dz \overset{\text{residue theorem}}{=} 0$$



$\Gamma_R^1: \quad z = x \quad ; \quad 0 \le x \le R$

$\Gamma_R^2: \quad z = R e^{i\varphi} ; \quad 0 \le \varphi \le \frac{\pi}{4} \qquad dz = i\, d\varphi\, R e^{i\varphi}$

$\Gamma_R^3: \quad z = x\, e^{i\frac{\pi}{4}} \qquad R \ge x \ge 0$

$$0 = \underbrace{\int_0^\infty dx\, e^{-x^2}}_{\substack{\text{Gauß integral} \\ = \sqrt{\pi}}} + \lim_{R\to\infty} \int_0^{\pi/4} d\varphi\, iRe^{i\varphi} e^{-R^2 e^{2i\varphi}} + \underbrace{\int_\infty^0 dx\, e^{-ix^2} e^{i\pi/4}}_{\text{Fresnel}}$$

$$\left| \int_0^{\pi/4} d\varphi \ldots \right| \le \int_0^{\pi/4} d\varphi\, R\, e^{-R^2 \cos 2\varphi} \xrightarrow{R\to\infty} 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} dx\, e^{-ix^2} = \sqrt{\frac{\pi}{i}} , \quad e^{-i\pi/4} = \frac{1}{\sqrt{i}}$$

Note: $\displaystyle\int_{-\infty}^{+\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \qquad \operatorname{Re}\lambda \le 0$

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quadratic completion: $\displaystyle\int_{-\infty}^{+\infty} dx\, e^{-\lambda x^2 + i\mu x} = \sqrt{\frac{\pi}{\lambda}}\, e^{\frac{\mu^2}{4\lambda}} \qquad \operatorname{Re}\lambda \le 0$

$$G(x_{j+1}, t_{j+1}; x_j, t_j) = \sqrt{\frac{M}{2\pi i\hbar \underbrace{(t_{j+1}-t_j)}_{=\varepsilon}}}\, \exp\left\{ \frac{i}{\hbar} \underbrace{\left[\frac{M(x_{j+1}-x_j)^2}{2(t_{j+1}-t_j)} - V(x_j)(t_{j+1}-t_j) \right]}_{\text{discretized version of classical action}} \right\}$$

8.5 Long-Time Propagator as Path Integral:

$$G(x_b, t_b; x_a, t_a) = \lim_{\substack{\varepsilon\to 0 \\ N\to\infty \\ \varepsilon N = t_b - t_a}} \left\{ \prod_{j=1}^{N-1} \int_{-\infty}^{+\infty} dx_j \right\} \left(\frac{M}{2\pi i\hbar\varepsilon} \right)^{N/2} e^{\frac{i}{\hbar}\varepsilon \sum_{j=0}^{N-1} \left[\frac{M}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon} \right)^2 - V(x_j) \right]}$$

integration cure these divergencies — $\varepsilon^{-N/2}$ divergency

discretized version of action

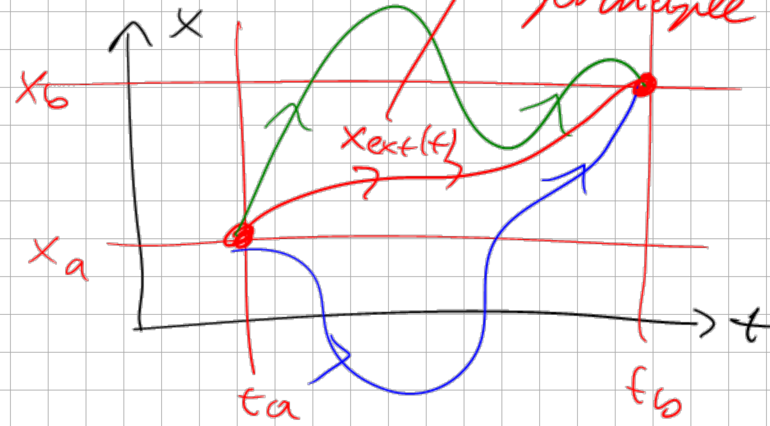
$$\mathcal{A}[x(\cdot)] = \int_{t_a}^{t_b} dt\; L(x(t), \dot{x}(t)), \quad L(x,\dot{x}) = \frac{M}{2}\dot{x}^2 - V(x)$$

discretize

$$\approx \varepsilon \sum_{j=0}^{N-1} \left[\frac{M}{2} \left(\frac{x_{j+1} - x_j}{\varepsilon} \right)^2 - V(x_j) \right]$$

$$\frac{\delta \mathcal{A}}{\delta x(t)} = 0$$

Hamilton principle



$$G(x_b, t_b; x_a, t_a) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\; \underbrace{e^{\frac{i}{\hbar}\mathcal{A}[x(\cdot)]}}_{\text{oscillating terms}}$$

path measure

8.6 Free Particle Propagator: $V(x) = 0$

$$G(x_b, t_b; x_a, t_a) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty \\ \varepsilon N = t_b - t_a}} \underbrace{\left\{ \prod_{j=1}^{N-1} \int_{-\infty}^{+\infty} dx_j \right\} \left(\frac{M}{2\pi i \hbar \varepsilon} \right)^{\frac{N}{2}} e^{\frac{iM}{2\hbar\varepsilon} \sum_{j=0}^{N-1} (x_{j+1} - x_j)^2}}_{K_{N-1}\, e^{\frac{iM}{2\hbar\varepsilon N}(x_N - x_0)^2}}$$

$N-1$ → Fresnel integrals

induction statement

check initial induction step $N = 1$

$$\sqrt{\frac{M}{2\pi i \varepsilon}}\; e^{\frac{iM}{2\hbar\varepsilon}(x_1 - x_0)^2} = K_0\, e^{\frac{iM}{2\hbar\varepsilon}(x_1 - x_0)^2} \Rightarrow K_0 = \sqrt{\frac{M}{2\pi i \varepsilon}}$$

starting value

major work: induction $N \to N+1$

$$K_N\, e^{\frac{iM}{2\hbar\varepsilon(N+1)}(x_{N+1}-x_0)^2} = \left\{\prod_{j=1}^{N}\int_{-\infty}^{+\infty} dx_j\right\}\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{\frac{N+1}{2}} e^{\frac{iM}{2\hbar\varepsilon}\sum_{j=0}^{N}(x_{j+1}-x_j)^2}$$

$$= \sqrt{\frac{M}{2\pi i\hbar\varepsilon}}\int_{-\infty}^{+\infty} dx_N\; e^{+\frac{iM}{2\hbar\varepsilon}(x_{N+1}-x_N)^2}\;\underbrace{\left\{\prod_{j=1}^{N-1}\int_{-\infty}^{+\infty} dx_j\right\}\left(\frac{M}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}} e^{\frac{iM}{2\hbar\varepsilon}\sum_{j=0}^{N-1}(x_{j+1}-x_j)^2}}_{\text{induction statement}\quad K_{N-1}\, e^{\frac{iM}{2\hbar\varepsilon N}(x_N-x_0)^2}}$$

$$= \sqrt{\frac{M}{2\pi i\hbar\varepsilon}}\; K_{N-1}\int_{-\infty}^{+\infty} dx_N\, \exp\left\{\frac{iM}{2\hbar\varepsilon}\left[(x_{N+1}-x_N)^2 + \frac{(x_N-x_0)^2}{N}\right]\right\}$$

$$= e^{\frac{iM}{2\hbar\varepsilon}\left(x_{N+1}^2+\frac{x_0^2}{N}\right)}\;\underbrace{\int_{-\infty}^{+\infty} dx_N\; e^{-\frac{(-i)M(N+1)}{2\hbar\varepsilon N}x_N^2 - \frac{iM}{\hbar\varepsilon}\left(x_{N+1}+\frac{x_0}{N}\right)x_N}}_{\sqrt{\frac{2\pi i\hbar\varepsilon N}{M(N+1)}}\,\exp\left\{-\left(\frac{M}{\hbar\varepsilon}\right)^2\left(x_{N+1}+\frac{x_0}{N}\right)^2\frac{2i\hbar\varepsilon N}{4M(N+1)}\right\}}$$

$= \cdots =$

$$= \sqrt{\frac{N}{N+1}}\; K_{N-1}\; \exp\left\{\frac{iM}{2\hbar\varepsilon(N+1)}(x_{N+1}-x_0)^2\right\}$$

recursion formula: $K_N = \sqrt{\frac{N}{N+1}}\, K_{N-1}$

successive $N-1$ iterations of recursion yield $\quad K_{N-1} = \sqrt{\frac{M}{2\pi i\hbar\underbrace{\varepsilon N}_{t_b-t_a}}}$

$$\Rightarrow G(x_b,t_b;x_a,t_a) = \sqrt{\frac{M}{2\pi i\hbar(t_b-t_a)}}\; e^{\frac{iM}{2\hbar(t_b-t_a)}(x_b-x_a)^2}$$

Short-time propagator = long-time propagator for free particle

Check all properties for a propagator in this case:

1) a) solves Schrödinger equation

b) $\lim_{t_b \to t_a} G(x_b, t_b; x_a, t_a) = \delta(t_b - t_a)$

$$G(p, t; 0, 0) = \int_{-\infty}^{+\infty} dx\, e^{-\frac{i}{\hbar} px}\, G(x, t; 0, 0) = \dots = e^{-\frac{it}{2\hbar m} p^2} \overset{t \to 0}{=} 1$$

2) spectral decomposition ✓

3) group property ✓

Goal: Schrödinger formulation → Feynman formulation